Theory of Group for Physics Applications
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Lecture - 48
Fundamental Symmetries of Physics - II

Now, coming to the Hamiltonian version of this corresponding to generators of symmetries, there exist dynamical variables constructed from canonical variables p and q.

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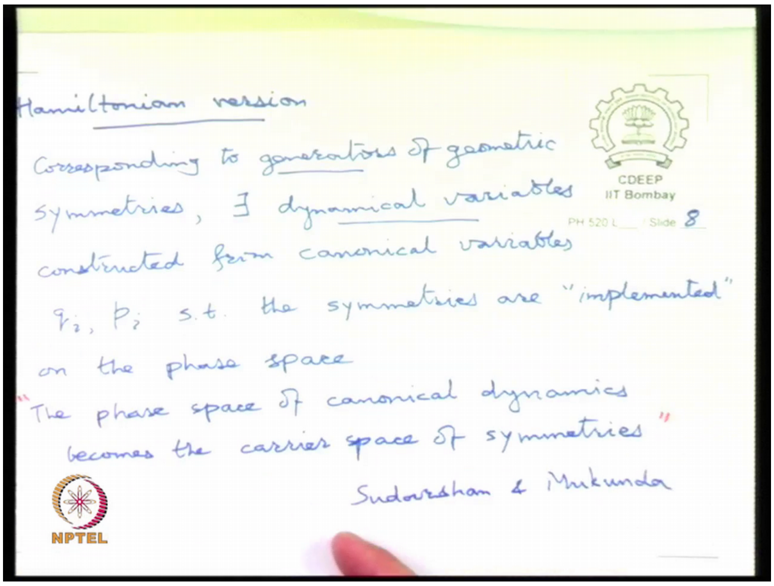

Such that, the symmetries are and we call it implemented on the phase space. So, effectively "the phase space of canonical mechanics becomes the career space of the symmetries". This is the main message of classical story of symmetries and physics and there is a book by Sudarshan and Mukunda, which is all about that?

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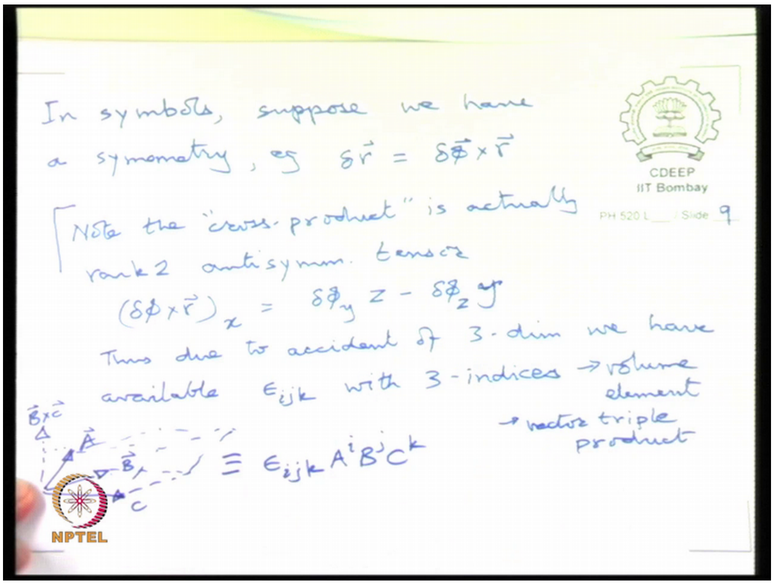

So, in symbols for example, let us work through examples, because that is the best delta r equal to delta phi cross r this is a small rotation infinitesimal rotation they. So, I have been meaning to tell this for quite some times. So, let me spend a few minutes now to do it here Note that the “cross product” is actually rank to anti symmetric tensor.

So, when we said delta phi cross r say x component it is supposed to be delta phi y times z minus delta phi z minus y times y component right. So, there are 2 there are 2 small rotation angles one which is y axis delta phi z axis delt a phi z. So, there are 2 parameters, but that is what it is?

So, actually what we do is the map due to accident of 3 dimensions, we have available the Levicivita tensor of with 3 indices. In any number of dimensions you can write the corresponding epsilon tensor, but it will have that many indices it is the completely anti symmetric tensor in that many number of dimensions. It is geometric interpretation is there it is the volume element. If you write epsilon i j k d x i d x j d x k that amounts to writing the volume element of the 3 dimension it is same as the triple product.

So, it gives the volume because it says A dot B cross C. So, when I do B cross C, I get something here, but then there is a here and if I do A dot B cross C, I basically get the volume of this parallel (Refer Time: 06:28) B cross C I am sorry B cross C will produce B C sign of the angle between them. So, that will anyway. So, this basic and that is exactly what is what this will be same as epsilon i j k A i B j C k, but now the point is that using the epsilon we can convert any anti symmetric tensor into a vector a pseudo vector ok.

So, so just to count note that in 3 dimensions we will have exactly 3 components would be anti-symmetric tensor, because that tensor is to is the 3 by 3 matrix, but it is an anti-symmetric. So, it has no diagonal elements and if you count the upper diagonal it is 3 elements remaining 2 and 1.

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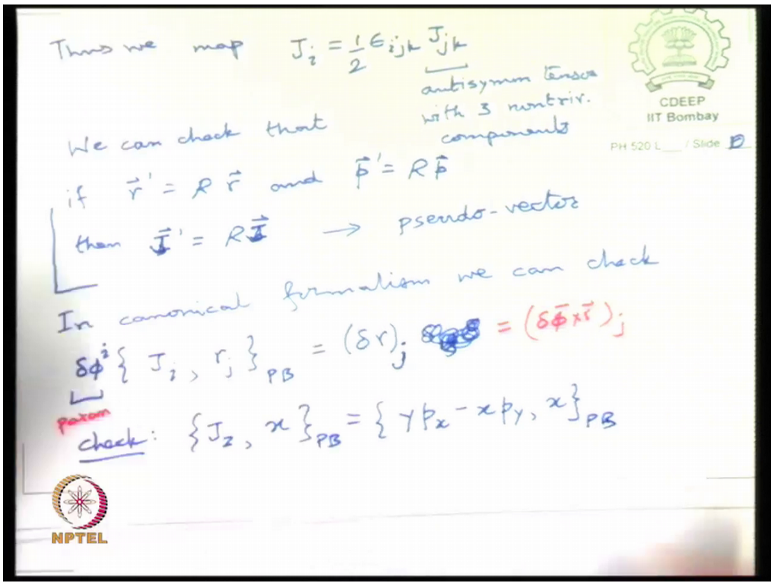

So, thus we can map an anti-symmetric tensor with 3 independent components, and it gets mapped into 3 vector like looking components. And, we put a factor half, because it will double count it j 1 will contain J 2 3 and then minus J 3 2, which because of the anti-symmetry again become J 2 3. So, we get it twice. So, we put a factor half.

And, we can check that if r transforms as than L J we have called it here J, J will transform exactly according to the R matrix. After you put the epsilons and so on you get exactly that this transforms like this. So, so hence the name pseudo vector because it will seen it will be seen to transform according to R, but if you flip the coordinates then because both R and p will change sign J will not change sign. So, it is called a pseudo-vector.

So, coming back corresponding to the 3 quantities, that the Poisson bracket of J so, coming back to this so, it covers also. So, if I have a geometric symmetry that r can be shifted to delta r shifted by delta r using some parameters delta phi, then the corresponding J that i construct will be such that it is Poisson brackets. So, J i on r k or j i on r j sorry for too many j s Poisson bracket will exactly reproduce delta r delta r i and times whatever the effect of this is going to be so, probably delta i j. It will rotate on so, times delta j that is enough right ok.

So, if I take the so, here I have these are the parameters of the transformation, I hope this is it is now you are keeping track of this. So, I have the geometric transformation is delta r goes to is related to r proportional to r with parameters delta phi.

Now, I take the corresponding conserved quantity, which is the angular momentum and supply the same parameters. If I now act with this on r it will reproduce exactly delta r j equal to delta phi cross r j-th component that is the statement of it being implemented in the canonical sense.

So, we can check this let us try out one thing. So, suppose I take um, but I it is much easier to do in quantum mechanics, because you only have to compute commutators, but in any case here also it is ok. So, so suppose we take J z on let us say x, I keep writing this because the curly brackets have began to have connotations of spinner anti commutator in quantum mechanics. So, just to remind ourselves that we are with the classical canonical mechanics we write this. So, this is equal to J z is equal to Y p x minus x p y commutator Poisson bracket, x we have a so, we are trying to check this action of j z on x.

So, we put the generator j z, now we use just the properties of the Poisson bracket, because the bracket has essentially linearity and distributivity.

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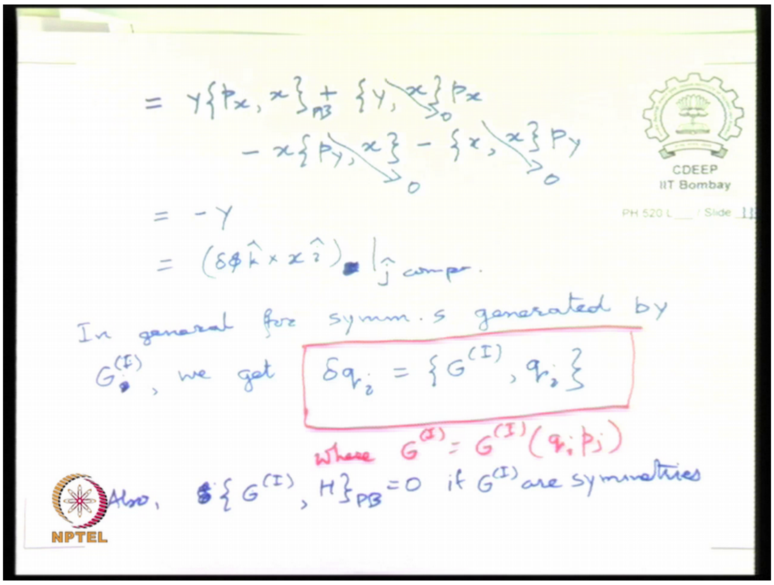

So, it is same as from the first term it would be y p x x, then plus y x p x plus. So, minus x p y x and then minus x x p y, but all of these will vanish because y x y x Poisson bracket x with a canonical variable not conjugate to it all of these as 0 and p x x is minus 1. So, this gives minus y x p x is plus 1. So, p x x is it is an anti-symmetric operation. So, it exactly gives minus y. So, this is exactly what you expect from the effect of a k cap times delta phi cross r ok, I cap z component.

So, this is the way it gets implemented in this particular way well i do not need to right this. So, it is times J capital come out it will come out proportional to j cap which is y.

Now, in general therefore, and let us list these with some other labels. So, some internal label these are these are not the canonical labels here they get mixed up because they are themselves space time symmetry, but they do not have to be space time symmetries. So, this is the course statement of what this in canonical language, where G itself is constructed out of out of the canonical variables. So, any symmetry operation is implemented like this and the G S all commute with the Hamiltonian.

So, I think the we basically ones you have this z x, because the internal index i became same as an external index; so this becomes as z x which is where we go y or that is why the j z j cap by j cap prime mean the z x component.

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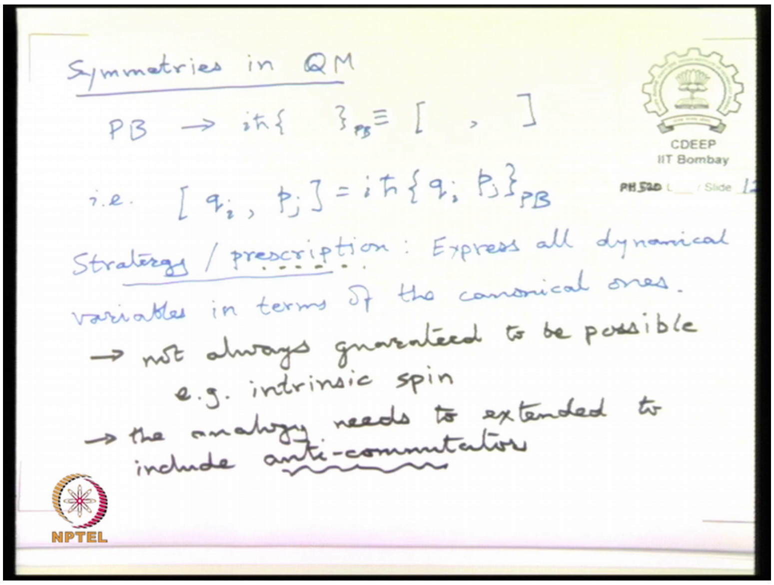

So, finally, we come to symmetries in quantum mechanics fortunately, because of the great observation of Dirac that the quantum mechanics amounts to carrying over Poisson brackets to commutators sorry. So, commutator goes to i h cross times this and that becomes equal to the quantum commutator i e, the corresponding Poisson bracket ok.

And, this is in general true for any variables, but the strategy is strategy come prescription of quantum mechanics express all dynamical variables in terms in terms of the canonical ones. This statement itself can be a stumbling block sometimes in quantum mechanics, because you may encounter new observables for which there is no canonical description that is why it is a prescription. And the prime example of this is spin intrinsic spin for which there is no classical analogue ok. And, similarly because of this fact when we go to fermions the fermionic quantization rules are completely out of the charts luckily, it is looks like just a clever change instead of commutators you do anti commutators.

But, why you have to do this is not clear we learnt it the hard way and it kind of extended the analogy ok. The anti commutators ensure poly exclusion principle and everything, but it is something osciallary that did not come out of the simple prescription, but the ore strategy to the extend this is possible.

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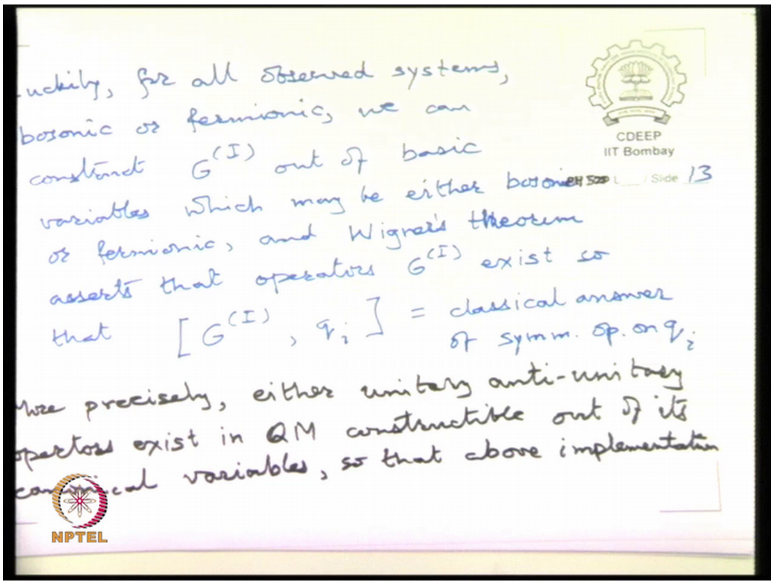

We, can bosonic or fermionic, we can construct G I out of the canonical variables, which may be either bosonic or fermionic, and the general statement that operators G exist.

So, that now G I on your variables q i now a commutator equal to classical answer ok. It reproduce on q. Wigner also so, the same thing that we wrote about j j acting on this now using commutators, but because of the analogy with Poisson bracket the calculation exactly remains the same and then you get the same answer, but the exception is that more precisely either unitary or anti unitary operators exist in Q M, which can be constructed out of the it is canonical variables.

So, that above implementation works; implementation works through the quantum brackets.

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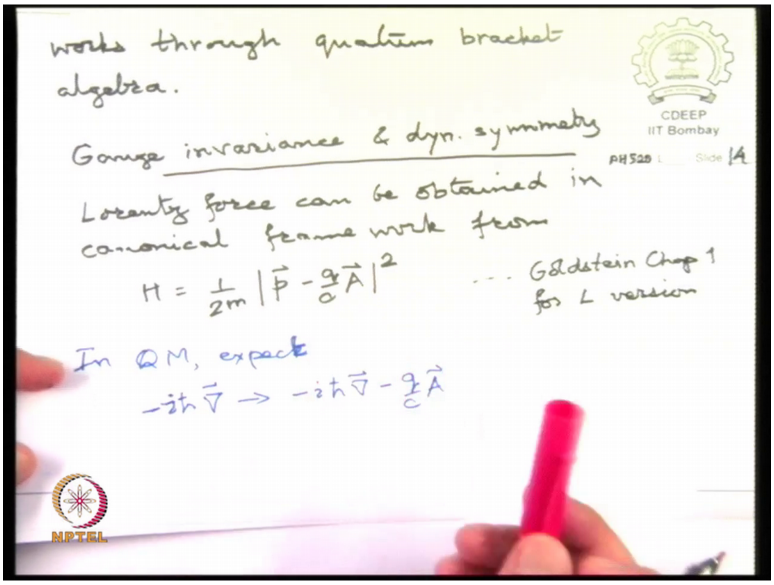

So, the quantum variables now become the carrier space of the quantum symmetries, we need not write that there are lot of intricacies that arise when we go further with this, for the time being I will leave at this. That it works like this in quantum mechanics almost exactly the same way it works for the classical system, provided you keep track of the canonical variables. Every once in a while you will not be able to construct a quantity out of canonical variables, then you supplement it by whatever new is there, again determine what there Poisson basic brackets have to be, but then again anything else you construct out of it you can continue.

The last comments I want to make is that there are a symmetry alone determines the classical wave equations ok, but let us just go to the issue of gauge invariance and dynamical symmetry. So, the story begins by observing that Lorentz force can be obtained in canonical framework from H equal to 1 over 2 m p minus q over c times A.

So, if you do this then it will automatically generate the Lorentz force you shift in other words you shift p by a. So, this is given in gold times chapter 1, well gold's chapter one gives the Lagrangian version, but you will see it same thing it is q dot minus q over c A, then you find the canonical quantities they became like this. So, in quantum mechanics expect that expect that minus I H cross gradient, which is what stands for p which is same as shifting p.

Now, if you write this for a particle. So, this form absorbs the gauge invariance of electro magnetism.

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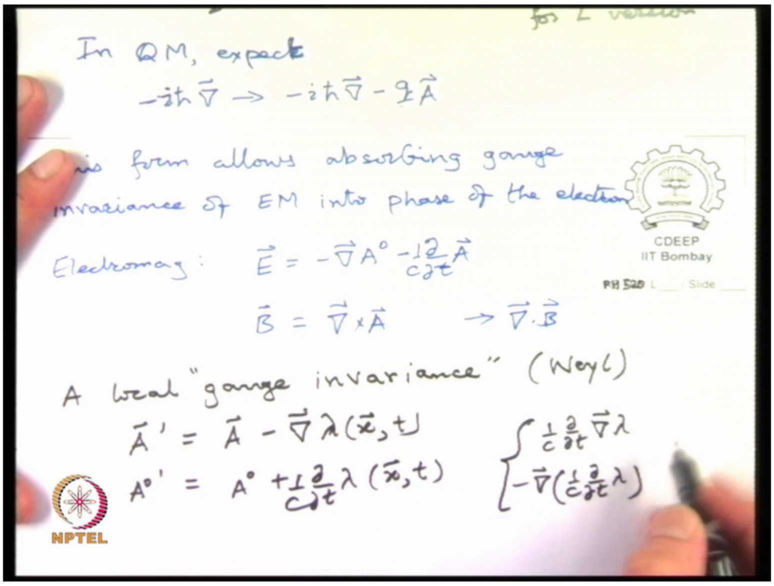

So, the gauge invariance of electro magnetism is E is equal to minus gradient A 0 sorry minus minus gradient a 0 and minus d by d t 1 over c d by d t of vector potential A and B is equal to curl of A. So, equal to minus grad phis of course, electro static statement in electro dynamics you need to add that d by d t tom to take care of the amperes law correctly. Once you introduce B equal to this is always possible because the B equal to 0. And once you do this you can then do this.

Now, this has a strange local invariance term given by Herman Weyl, which is that if I change A to hold A minus the gradient of sum lambda of x n t, and change A 0 plus the time derivative of then we can make these 2 get compensated here ok.

So, 1 over c d by d t of grad lambda can be compensated by minus grad of so, we need a one over c here one over c d by d t of lambda. So, E changes by this and then therefore, we get 0. So, E and B do not change.

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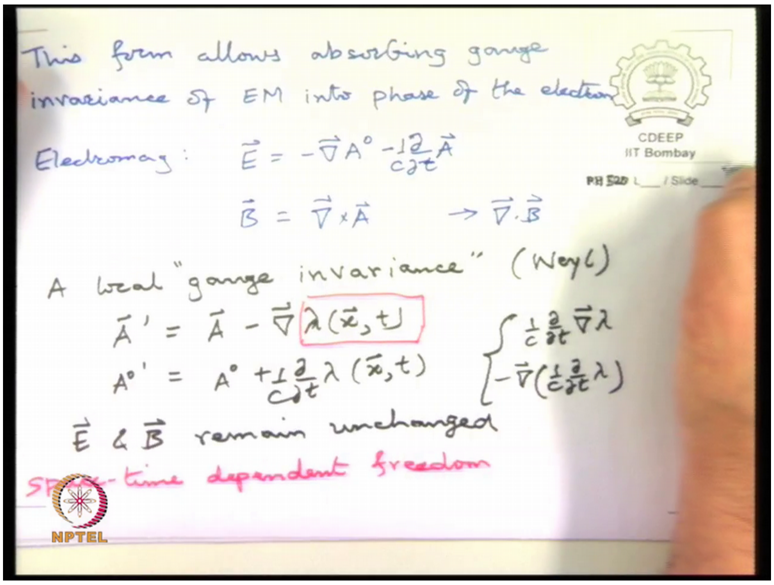

This is why Dirac called while (Refer Time: 34:33), so as for so, there is a simpler way of looking at it, B is curl of A. It is obvious that a is going to remain ambiguous because to A I can add gradient of any scalar and because curl of a grad is always 0, that gradient can always be added.

So, that much is somewhat simpler, but that there is this combine gauge in variance where you also shift E by time derivative of A and shift the scalar potential by this makes for a larger gauge invariance where, there is a space time dependent degree of freedom. So, this is essentially an infinite dimensional freedom available, if you count in terms of group theory freedoms.

It is a huge amount of degree of freedom, but the plot gets thicker now because when you now combine this with this particular form of how electro magnetism couples to a point charge? You can then modulate the phase of so, let us look at this.

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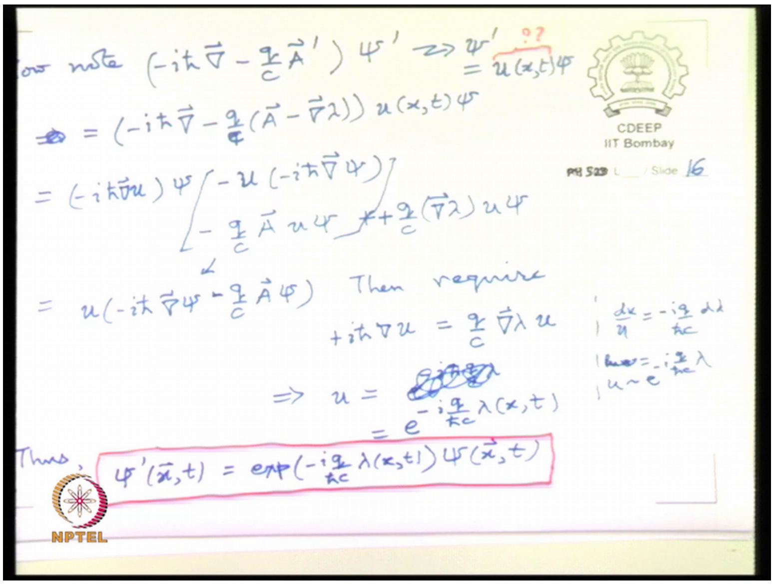

So, now note minus i h cross grad minus q over c A prime suppose I have this acting on psi prime. So, that would become equal to. So, I am looking for what should be psi prime. And, let us say it is going to be u of x t psi when I shift the A q over c into A minus gradient lambda u of x t psi, that is what it will become.

So, this we do not know what you to put there? If I do this gauge transformation on A feels what should I do to psi prime. So, that my Hamiltonian does not change remember that the physically electro magnetism is not changing when I change a like this. So, I have to somehow compensate it by changing identify what change in psi to do to do this. And, the answer is that this gradient u times i and then minus minus u times minus i h cross grad psi right just letting this term act on this product this. And, this n times minus q over c A u psi and plus q over c grad lambda times u psi all of this as to remain the same.

Now, it has to look like the older version of the lagrangian. This works provided so, I gather these 2 things together, because they are my old form they are equal to u times minus i h cross grad psi and these are can be taken out u is some overall factor here there is no derivative inwards. So, I can take this u out and it remains inside minus q over c A psi right, but then these 2 I should get them to cancels. So, there were plus sign here I can get them to cancel provided require that, there was gradient u's right this was gradient u i h cross grad acting on u first and then grad acting on psi.

So, this means that minus i h cross grad u should be equal to for plus this equal to q over c taking this on the other side q over c grad lambda, it times u. This can be achieved if you notice that u should be equal to e raised to so, you can put this u below here and think of this as log.

So, then it becomes equal to e raised to minus i h cross q over c times lambda will solve this I take times. So, u equal to this will solve this differential equation. So, here is plus sign or minus sign plus. So, we made a mistake. So, we need to do a little (Refer Time: 41:28). So, d u very roughly speaking d u over u equal to minus i q over h cross c right times d lambda.

So, l n u equal to this times lambda. So, u equal to e raised to minus i q over h cross c times lambda good. So, so this was correct except that the sign was and h placing of h cross was not correct. So, i pick a u of t psi prime of x t equal to exp. So, i pi play this game of shifting the electromagnetic potentials by gradients of lambda, I can compensate it by shifting the phase of psi in a space time dependent way this is the big thing. So, note that. So, this is where we will end forms the group u 1 that is unitary matrices of one by one size.

But, now here we find electro magnetism has a space time dependent this we call local gauge invariance. The point of local gauge invariance is that it does not give as any new conserved quantities it is a symmetry gives no new conserved quantities there is only still the electronic charge, but determines the form of the Hamiltonian.

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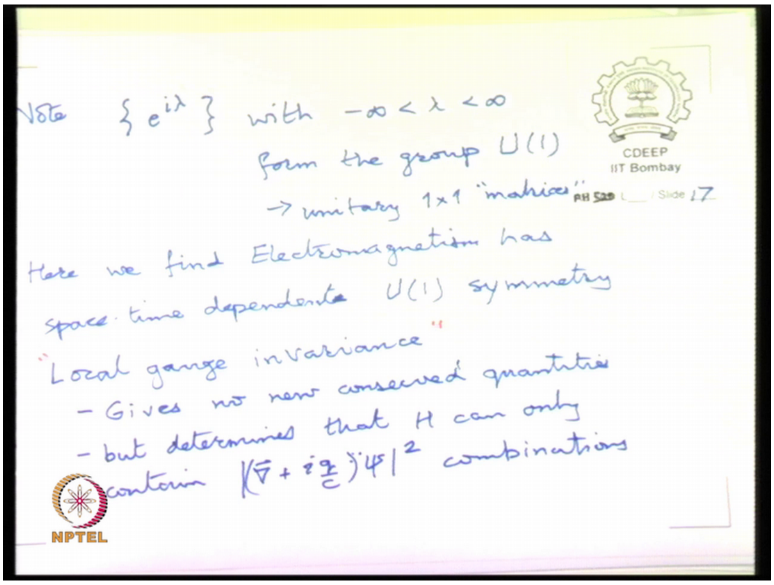

Grad (Refer Time: 44:13) you have to always construct a quantity in which the factor, you drops out you cannot construct it an any other way. So, it fixes the interaction of the electron with the electromagnetic field.

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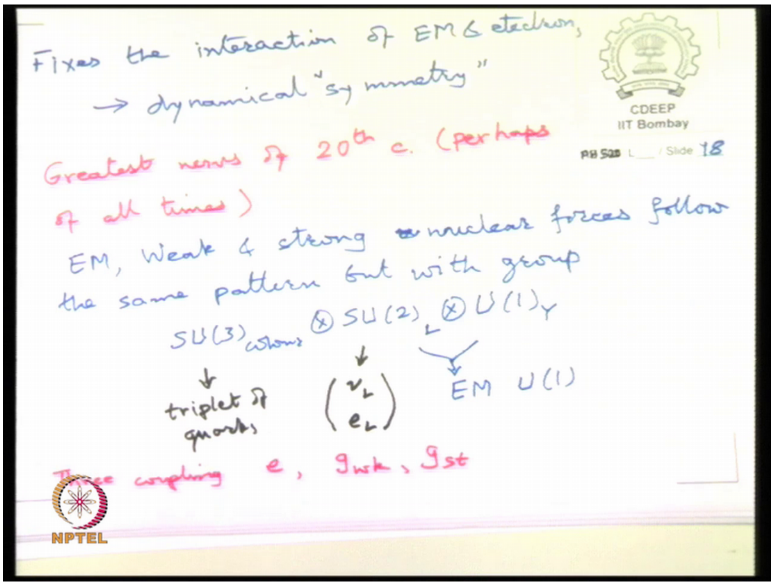

And therefore, called dynamical symmetry, and the greatest news of twentieth century is that the weak force strong force all follow the same rule, or perhaps of all times if there only 4 forces of nature we have finish this story now, but with groups S U 3 color. So, then there are 8 photons there are 8 gauge potentials cross. So, called S U 2 left cross a U 1 y.

So, electro magnetism comes out of linear combination of the tool the U 1 of electro magnetism is linear combination of those tool, but this S U 2 acts on the doublet of electron and neutrino and S U 3 acts on the quads triplet of quads. And this acts on left handed neutrino and the left part of electron left projection of electron it does not affect right projection of the electron.

So, it is slightly strange theory, but this is how it is ok, but the dynamics is completely fix by these and with 3 couplings e u can say g weak and g strong. So, just those 3 numbers fix or everything, that you have in terms of interactions all the embarrassment comes in the masses because you do not know where the masses comes from, but that is where the hex mechanism comes and so on, but the core of the dynamics of the theory is fix by this.

So, that is the end of this is the end of the course.