Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 45
SU (3) and Lie's Classification – I

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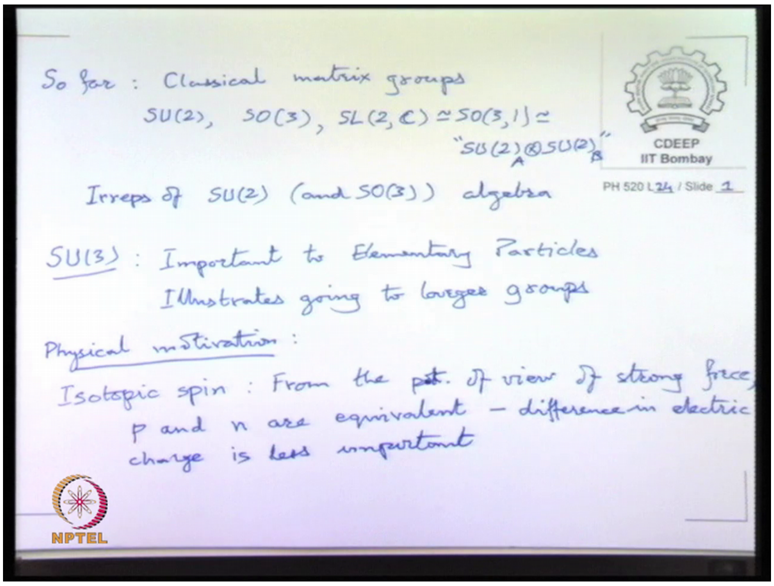

So, far we have seen the groups SU 2 and the generalities about groups and we saw their algebras for SU 2 and SO 3. So, far we saw classical matrix groups and we saw SU 2, SO 3 and SL 2 C as being equivalent to SO 3 1, as being equivalent to SU 2 a cross SU 2 B and this with a bit of inverted commas because there is a complexification involved which works at the level of algebra, but not at the level of the global group.

Now, what we are going to do next is to and we saw that representations of irreps of SU 2 and therefore, SO 3 which at algebra level isomorphic. So, we studied these and as the part of standard quantum mechanics you get the raising and lowering operators. So, now, we will what we will do is do SU 3 as an example of one higher level, but also as an illustration of how the general procedure works.

So, this group is important to elementary particles, that is also illustrates going to larger groups. So, let me give you first a physical motivation why these are important groups and this is really a one of the greatest discoveries of the past century which is so called isotopic spin. This was a observation of Eisenberg that it does not matter to nuclear forces whether there is a proton involved or neutron.

So, you know that the history of this nuclear physics bit strange because the neutrino which we might think to be a very unusual particle was discovered first and neutron the big thing was not known until 1931. But once neutron and proton were and that resolve lot of puzzles because the whole idea of why in the atomic weights 2 identical chemical elements at slightly different weights was not very clear, but that extra neutron.

So, isotopic spin you clarified lot of things. But at this point Eisenberg realize that the strong nuclear forces, which bind the nuclei are such that electromagnetism is almost a relevant there and even protons which are which should be just lying apart, if you try to pack 30 protons into nucleus, they are huge column repulsion, but they are they not have a flying away.

So, what it meant was that, the nuclear force was much stronger and electromagnetism was irrelevant to it. So, aside from electromagnetism then, there was still 2 species neutron and proton and the nuclei nuclear force treated them to be the same. So, you try to think of from the point of view of strong force, proton and neutron are on same footing, but the differ in charge in electric charge is irrelevant seems to be subdominant ok. But they would still be two independent kinds of particles because there other their spin is of course, half, but the magnetic moment or the properties would differ.

So, Eisenberg propose that actually they are like 2 projections, 2 spin projections of the same particle. So, we have up electron.

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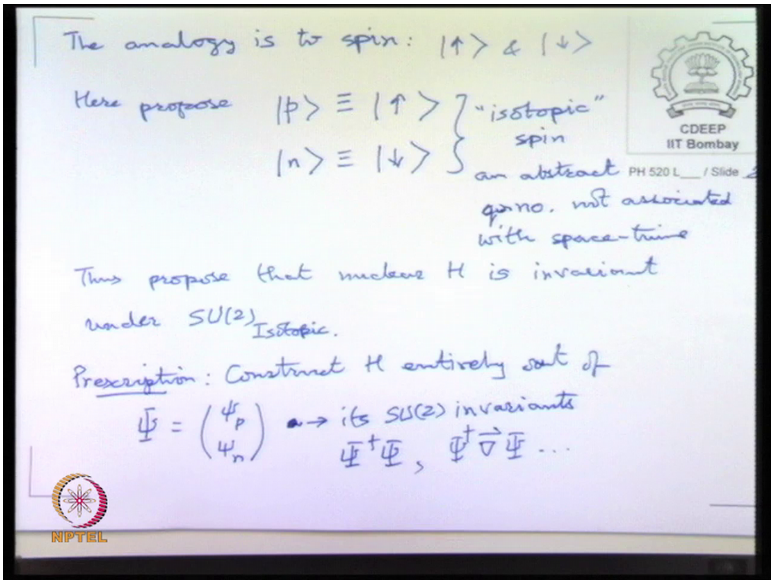

So, we remember that we had said earlier that the analogy is to spin because we had spin up and spin down states, and they can be flipped into each other by magnetic field, but otherwise you they would be just on their own. So, these mix by some interactions, but otherwise they are the same.

So, similarly here propose that, proton state is equivalent to some kind of spin up and neutron is spinned down. With respect to something called isotopic spin which is an abstract concept concepts are of course, abstract, but which is an abstract quantum number not associated with quantum number not associated with space time symmetries. So, thus was born the SU 2 of isotropic spin, propose that nuclear Hamiltonian is invariant under SU 2 of isotopic spin.

So, prescription is to construct the Hamiltonian entirely out of psi equal to wave function of proton, wave function of neutron put as a doublet as an SU 2 two dimensional vector and its SU 2 in variants like psi dagger psi right. So, that would be SU 2 in variant or things like psi dagger gradients psi etcetera ok. So, all these which leave rotation of proton into neutron Hamiltonian remains unchanged. So, all those kind of terms can be added to the Hamiltonian, but not proton and neutron wave functions individually.

So, this was a big change of course, that there is a small mass difference between the 2, then you can by hand add some quotes symmetry breaking terms; terms that break that SU 2 symmetry, but they would be small small corrections to the main Hamiltonian which would be isotopically invariant. Now, later on there were some historical accidents and should not say historical accidents, but sequence of discoveries which were which when they where first discovered where not their full significance was not realized, in elementary particles which were essentially began to be discovered in 1950s, 1950s and 60s after accelerators.

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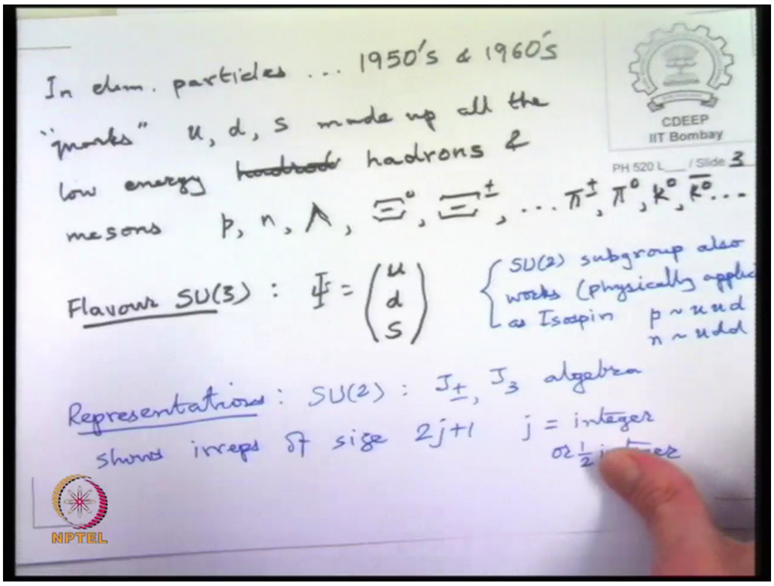

It was found that there was proton neutron and lambda. So, there were 3 particles, its slightly long story, but to cut long story short it was found that 3 kinds of quarks up, down and strange made up all the low energy by now we call it low energy in those days those high energy hadrons and mesons. So, this 3 quark states up down and strange were enough to explain all the articles like proton, neutron, lambda lambda capital lambda that this and then things called psi of different kinds you know psi 0 psi plus minus and then of course, pi mesons pi plus minus pi 0 and k 0, k 0 bar.

So, there are the long list of such particles, but they could all be found as if they were a spectrum excited spectrum of just 3 quarks up, down and strange. So, that was a big discovery and so, a flavour SU 3 we now call it flavour with in which psi was supposed to be up, down and strange in the order of increasing weight , but the electric charges are slightly different that is a different story, but. So, this is how you set it up and then you have to write all your Hamiltonian out of these that was the understanding.

Now, this is a really big break because until now you had things like chemistry in which you are excited states, but you could visualize it has some electron cloud and coming together of 2 atoms to form a larger body or high more excited state of a particular atom and so on, here you are actually saying that two different particles really work what rotatable into each other. The identity of a particle was not unique, that it was negotiable under some internal symmetry transformation, which were abstract.

So, this was the beginning of understanding of the use of unitary groups in quantum, quantum mechanics and in elementary particles, at the point when these things were realized SU 3 was not part of physics at all, SU 2 and SO 3 the rotation group had been all worked out in great detail clutch (Refer Time: 14:53) coefficients this that, but nobody knew about SU 3. So, Gelman worked and what was the representation of SU 3.

So, the question is now of representations. For SU 2 we had the raising and lowering operators J 3 this algebra is enough to shows clearly irreps of dimension 2 j plus 1 right and j equal to integer or half integer. So, if we have a 3 by 3 matrix, we may assume that a triplet like this is a representation, but what about higher dimensional representations.

So, this was not clear at that point and it had to be essentially workout by hand and we will go through a particular version of this.

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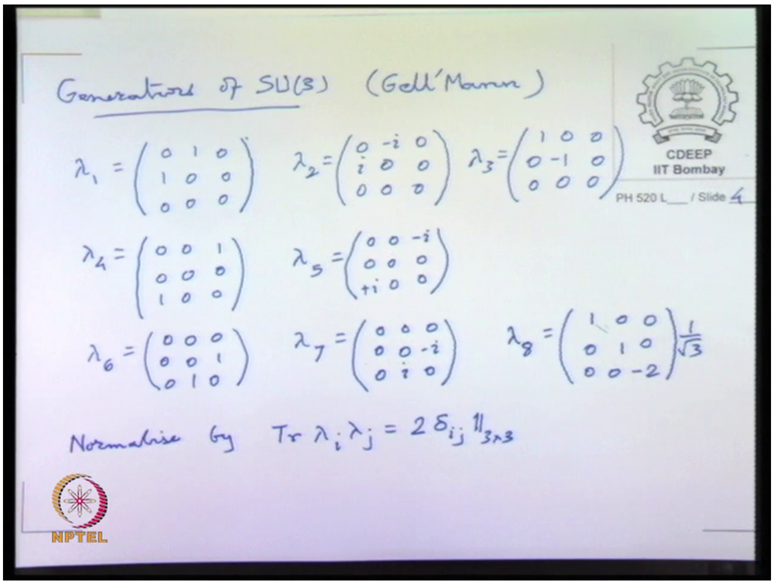

So, far SU 3 first we need to guess the generators and what we are writing down is, what is Gell Manns notation and ideas like this if there is SU 2, then sorry if there is SU 3 then it clearly has a subgroup SU 2 ok. So, suppose we start with a matrix lambda 1, which is 3 by 3, but such that its upper corner is essentially tau 1 plyometrics and I fill the remaining with 0's.

Then similarly we put lambda 2 as been 0 minus i i 0 and then 0 0 0 0 0. So, this should certainly be in the list of and we put a lambda 3 matrix, which is 1 minus 1 0 0 0 0 0 0 and 0. Now, there should be similar matrices in which these things migrate to other locations. If you go to higher dimension so, lambda 4 and this is I should probably write down here from here good. So, lambda 4 is where we make this one migrate to this corner, lambda 5 is and you can get and you can guess that lambda 6 and lambda seven are found in similar proposed in similar way, where the one is then migrate to the lower box.

But now you know that from counting that SU 3 has to have 8 generators. So, we need an 8 generator, but we have exhausted all the sigma metric structures; a sigma 1, sigma 2 structure has been all exhausted, we should also remember that you have to have a Hermitian and traceless matrices to get the unitary determinant equal to 1, because if the if that trace is non zero then the determinant will come out come out bigger than 1.

So, you need to propose something that is Hermitian traceless and would be diagonal it will commute with all the others. So, the only way to make this is to write 1, 1 and it has to be linearly independent from that. So, the only traceless matrix you can write which is also not interfering with your SU 2 subgroup that top line is the SU 2 subgroup and the new diagonal matrix you are proposing should not disturb it.

So, it should be diagonal in the upper to the upper corner, that only leaves you to put something nontrivial in the bottom, which you put to make the trace equal to 0. And then you need to normalize this. So, if you normalize the this by trace of lambda i lambda j equal to 2 times delta i j times the identity matrix. So, this is because you remember polymetric says if you square the polymetrics you get 1 identity and if you trace it therefore, you will get 2 as the answer.

Gell Mann made the choice 2, which I do not think is universal the correct thing should have been to put n for size n, but that does not matter that is the choice made and use by particle physics is everywhere.

So, all of these of course, have this property their squares is essentially the traces 2. To make the trace of this equal to 2 you need to divide by a 1 over square root 3, because it will be 6 you know min 4 plus 2 plus 2 6 the square and remover 2 that leaves 3. So, you normalize by 1 over square root 3. So, this was the choice made and then all you have to do is work out the algebra ok.

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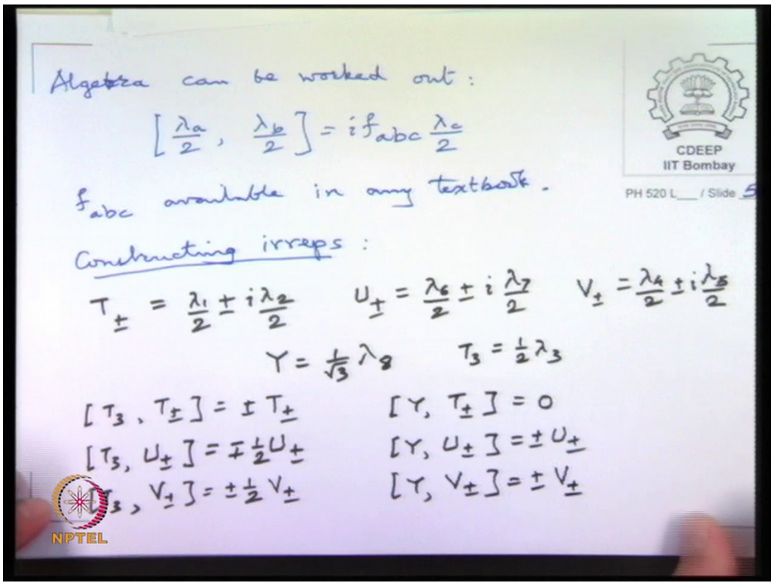

We are not going to be interested in the details of the algebra, but let me just write down what it looks like. So, the algebra is algebra of computation, algebra can be worked out simply by working through all the products and one writes it in the form lambda a by 2 comma lambda b by 2 equal to i f a b c times c by 2.

And of course, that lambda 3 and lambda 8 commute, I will not. So, this f a b c I think I will not spend time writing them here, but these can be worked out ok. What we want to now see is how we repeat the raising and lowering operator business. So, here we see that there are going to be two kinds of raising and lowering operators or rather 3 kinds; in the case of SU 2 we had simply taken lambda 1 lambda 2 combine then into a lambda plus and lambda minus, but here we also have this pair and we have this pair.

So, there are actually 3 directions in which you can raise and lower, and this is done like this. So, you define T plus minus to be equal to lambda 1 by 2 plus minus i times lambda 2 by 2. So, these are our usual old isospine. So, that is the isospin subgroup of the if; there is a very cute accident that this flavour SU 2, flavour SU 3, its SU 2 subgroup actually works as the isospin isospin group although the flavour we apply to this fundamental particles quarks u d and s, and isotropic spin is propose for proton and neutron.

But it turns out that the way proton and neutron are composed of quarks the isotopic spin just applied there without without too much change also works as I mean physically applicable as isospin, because proton and neutron only contain this. So, proton is u u d and neutron is u d d.

So, they differ only in a in this middle entry u d. So, because this u and d u and d are same, essentially the u d algebra works for proton and neutron which is actually subgroup of this. So, that is a nice accident, I mean that is how we eventually it was discovered. So, I should not call it accident, but that is how it works, but. So, coming back here that for this T plus minus are they iso spin raising and lowering operators of before of previous discussion and then from 6 and 7 we call it u.

And finally, v equal to lambda 4 and lambda 5 and we introduced y equal to 1 over square root 3 times lambda 8 ok. If you do this then you can prove the commutation relations which look like this now. T 3 and of course, T 3 is same as half of lambda 3. So, T 3 comma T plus minus is equal to plus minus T plus minus, then y, T plus minus equal to 0 this is expected because y was that its generator, which does not interfere with the SU 2. Then T 3 u plus minus is equal to minus or plus a half u plus minus and y u plus minus is plus minus u plus minus.

And finally, T 3 sorry yeah T 3 v plus. So, what we have written is, the diagonal generator T 3 diagonal generator Y and its commutations with the raising and lowering operators T T U and V. So, what does this mean, we have now all these extra generators and things are being raised and lower in different directions. So, the solution is of course, to draw a two dimensional diagram.

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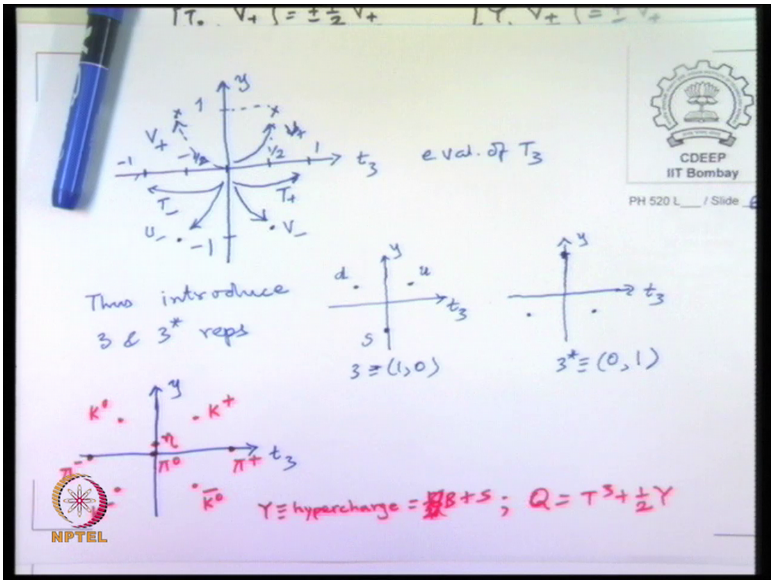

We draw something like this the T 3 and Y directions. So, let me first draw here T 3 which is the eigen value of T 3 and remember that this, these eigen values because they are SU 2 eigen values they are half integer and so on. So, plus minus half plus minus; So, put a 0 here, 0 was also of course, allowed and plus minus half then plus minus 1. So, these are the possible things for t 3, but now and you could raise and lower using t plus t minus ok. So, t plus takes you here and t minus takes you here raises the eigen value by unit ok

So, this is minus 1, this is 1 and these are half. So, the action of t plus and t minus changes the eigen value of t 3 by 1 unit. This we knew for SU 2, but now we get eigen values of y as well and the y eigen values are also raised and lowered by u plus and u minus by 1 and minus 1. So, let us put a 1 here and minus 1 here and somewhat similar scale, but the action of U is also such that it reduces the value of T 3 by a half unit and action of v is to increase the value of T 3 by half unit.

So, what happens is that, we have new eigen values here such that, the action of V takes us to this right. V because we want to increase both here the sign is minus yeah. So, this should be U. So, this is U, where as this point is reached by action of a V.

Similarly we can see that we will get these points minus half and minus 1, the action of U minus because if I have U minus then it reduces the value of Y and it also reduces the it increases the value of t sorry what did we get I. So, Y, Y values is increased by U and T value is reduced by U.

So, these 2 are yes that is right. So, both of these which increase the value of y by half are plus are in this and minus are in the lower quadrants and here this lowering is done correctly by v, and the lowering this is the reduction is done by u. So, this is u minus this is what I have drawn at least u minus and v minus this is correct. So, what one does is to in actually introduce first the simpler representations, which are just triplet. This just shows the direction of increase and decrease and its anyway got slightly messed up.

So, we start with 3 and 3 star, which are drawn as. So, these 2 are separated by this is half and plus and minus half and this is plus 1 again of course, these are t 3 and y eigen values and we call this the 3 or 1 0 and this 3 star or 0 1 representation of the 2 generators y and t 3. So, these are the triplets the u d as that we wrote, we are now going to represent it like this and since u and d where for me and SU 2 doublet, that is what are these 2. This is essentially u, I do not want to clutter of the diagram, but anyways just to indicate u and d are here separated by t 3 value of 1 and then we got the new state here s and then these are the stared ones.

So, the old SU 2 iso spin will continue to be there on the horizontal axis and then plus we get the extra thing. Now, the trick is that we can make higher dimensional representation I will not try to give you the graphic methods because I will I actually want to do algebraic 1 which is more reliable, but to quickly show the historically what was important was that, it was possible to now find tensor representations of these. So, products of these.

So, just tell if you are vector v i you can make a v i v j a higher dimensional higher rank tensor out of it. You can make higher rank tenses out of 3 3 times 3 three times 3 star and so on. So, 3 times 3 star contains a representation which is 8. So, it splits into an irrep 8 and a 9th which is SU 3 singlet 3 cross 3 produces 9 states and as physical applications, I am actually saving space which is cramping the thing a bit, but its.

So, as it worked out the pi meson fell on an SU 3 diagrams of. So, of course, Gell Mann had to identify what physically y meant and that y generator was hypercharge, T 3 equal to T 3 plus sorry equal to b plus 1 b plus s. So, electric charge is that is it. So, this hypercharge is Baryon number plus strangeness and factor half I has to do with normalization, but I think it is or no there is no half kept here and Q electric charge would turn out to be equal to T 3 plus a half y yeah half goes there.

So, with making suitable charge assignments, one finds that this is pi minus pi 0 and pi plus they form a isospin SU 2 triplet just like angular momentum l equal to one state minus 1, 0, plus 1. But then there were other mesons which fit here and these are called k 0 and k 0, k plus, k minus and k 0 bar. So, although k 0 was a neutral meson the k 0 bar was not same as k.

So in fact, it is one contains strange quark one contain s bar quark. So, they are opposite. So, u s bar and u bar s something like this where as the k plus k minus are charge mesons. So, these 8 fit here and you of course, have a ninth particle if you square take 3 times 3, that one is usually indicated nearby and that is called an Eta mesons. And this eta mesons frankly till date build which particular particle state in those excited states that the. So, k k k and k 0 bar are reasonably long livelihood mesons.

So, they fly through your detector and you can see them splitting later and it is easy to identify all and mesons of the pi mesons of course, live long tracks in the emulsions. But all other particles that were discovered were all much short lived. So, there only seen as resonances and the width of the resonance basically tells you the mass of the particle the time scale for which it leaves is inverse of the mass of the particle.

And so, this is a physical application there was another well. So, I meant to drop the other diagram let me draw it anyway because the historically that was the most important thing discovered.

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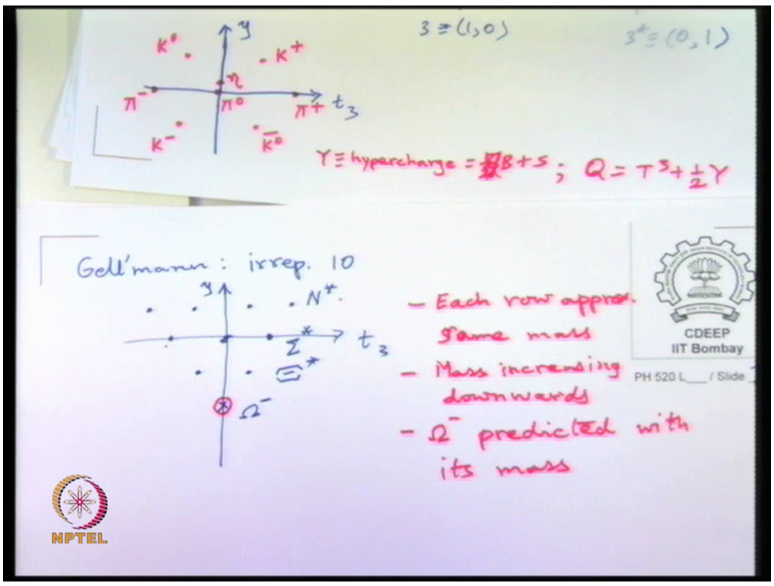

Gell Mann went on to propose that there is a representation 10. 10 dimensional irreducible representation which now looks like this t 3 y on which here we have these 4 states the jump is by 1. So, these are half integer representation of t 3 then there is integer 1 0 minus 1 and then another plus minus half.

Now, these particles have been discovered and these were called N star and their various charge states which I forgot to write sigma star and psi. So, it called psi hyperons. So, they fall on this, if we think a little bit we can you and this is this is y and if we apply this formula c 3 plus half y we can work out the charges. So, for example, this has to be minus 1 0 1 and this may start with 2 1 0 minus 1 something like this.

But so, these are all separated by shifting T 3 which would change Q by the same amount. So, they differ in charge by 1 unit. The interesting point was that these were known and Gell Mann using this formula let them out like this, and then you realize that that did not complete a representation of SU 3. So, you needed something here to make up the full representation and so, I proposed that there should be a particle here and this has actually omega minus.

So, it is omega of charge minus 1, that is what fitted with rest of the multiplet. So, this decuplet was more or less by matching. So, it was like spectroscopy of the hadrons and from it with some educated guess work, he guessed that there was this representation 10 which we can work out next and that this particle should exist. Not only that this splitting along the direction y of this multiplet is proportionate.

So, if you know this distance the mass difference. So, row was all roughly same mass, each row has approximately same mass. So, knowing the mass difference mass grows as you go down, these are simply observations by putting things on it and you see that, this is how they fall. In fact, you would put them in same row because they all have same approximately same masses and there are isospin differences can be found because their d k modes would contain pi plus or pi minus.

So, that gives the I 3 difference to be correctly plus or minus 1 and so, finally, this had to exist and so, omega minus can be predicted along with its mass. So, this was one of the striking discoveries of 1960s ok, but that launched this whole idea of an internal symmetry of particles, which is not just something spectroscopic, which is not just some excited state of something, but fundamentally distinct species actually have an internal symmetry that connects them. So, this is one of the biggest discoveries of the past century.