Theory of Group for Physics Applications
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Lecture - 44
Representation of Lorentz Group and Clifford Algebra - II

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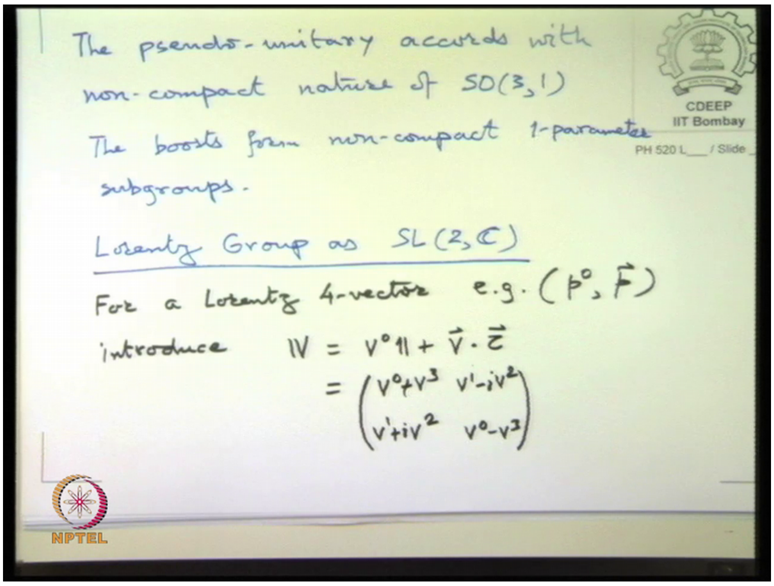

So, next what we will do is we will see too more things and that is the end of Lorentz group that we can do here. One thing is 1 more way of looking at Lorentz group. So, so far we saw that Lorentz group can be taught of as S O 3 1 with the real generators, L and K then you complex, if I do some (Refer Time: 00:39) then it looks like S U 2 cross S U 2.

We will now see a third way of representing Lorentz group and that is as S L 2 C. So, here the observation is that we introduce for any 3 D vector, for any Lorentz 4-vector like the energy. For example, p 0 p right the 4 momentum of a particle, introduce V matrix equal to V 0 times identity plus V dot sigma or V dot tau the physics notation, which in full form is simply equal to. So, V 0 times identity. So, the zeroth component you put here, but then there i V 3 times tau 3. So, there is minus V 3 sorry plus V 3 and minus V 3 and then there is V 1 minus i times V 2 and V 1 plus i times V 2.

So, the matrix looks like this. Now, the interesting thing is that if you take determinant of this matrix, then you actually get the Minkowski norm.

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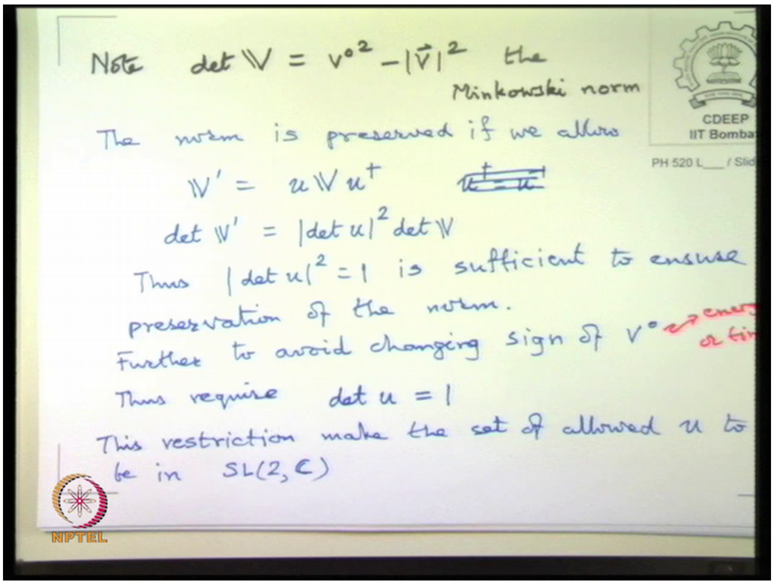

So, note det V is actually equal to take the determinant. So, you get this times this which is V 0 square minus V 3 square and this product is V 1 square plus V 2 square, but with minus sign. So, essentially it is equal to V 0 square minus the vector V mod square the Minkowski norm.

So, we can see that this norm would be preserved provided we transform this by a unitary transformation. So, right because than determinant of u dagger would be equal with u unitary, then this would be fine or another way of looking at it is if it is not unitary then you would have found that it would become that u squared. So, so, you can drop this further time being we will actually derive it. So, suppose you propose this then you will find that determinant u is product of the determinants of the other side which is determinant of this (Refer Time: 05:19) remain u dagger, but determinant of u dagger is just going to be star of the determinant of u, because it just start transpose.

So, it is mod squared of determinant u, thus that u squared equal to 1 this equal to 1 is sufficient to ensure preservation of the norm, but further to avoid getting changing sign of V 0 and so on, V 0 the zeroth component in 4 vector is important, because it is usually that the energy or the time or something.

So, you do not want to change it is sign. So, you require determinant you to be plus 1, it needs a little bit of calculation to check why this (Refer Time: 07:03) is, but I am skipping it further time being. Thus require that determinant of u equal to 1; this is what we actually call. So, always started which was most general matrix u 2 by 2 complex matrix u and we transformed our representation V by u times dagger of that u we dint restrict any u that point up to that point. And, the only restriction we have finally, put is that determinant u is equal to plus 1, this is called the special linear group.

The subgroup of g l to c makes the set of allowed u to be in S L 2 C, I mean that is the definition, if you put know the restriction and 2 by 2 complex matrices of G L and C. So, determinant should be nonzero, but the only restriction you put is you said the that u to plus 1, that is called S L 2 C and therefore, S L 2 C is 1 more realization of the Lorentz group as a group theory.

Now, there is something a little interesting as well if you just, if you just change this sign V 0 1 plus V dot tau if you put minus V dot tau also everything works ok, because it will only change signs of V. So, determinant will still come out the norm and so on. So, we get 2 inequivalent S u S L 2 C representations of so, I will just make in case you have a read mod literature on this and nowadays that the (Refer Time: 09:14) particles are become important to condense meta physics; if you ever read to component formalism, but let us not mention it.

So, this is 1 more way of thinking about the Lorentz group.

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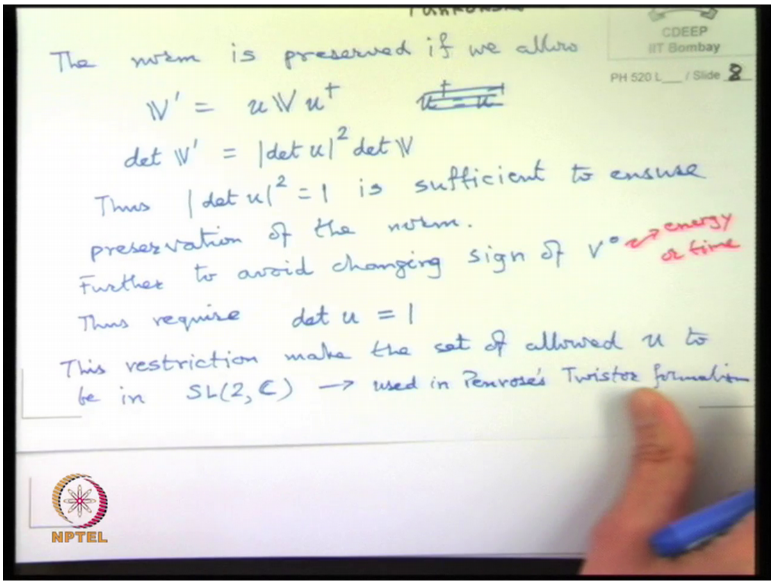

Essentially means a spinner, but spinner with some direction or something like that ok. So, that is 1 remark about an additional way of looking at Lorentz group. And finally, I will end with talking about what is called as Clifford algebra ok.

This is algebra of gamma matrices of the Dirac equation. So, the thing is that Dirac invented this; this is 1 of the major applications of group theory to physics. So, the last bit gets the most exciting, because it was a pulse meant to a whole generation of physics is; so in 1929 Dirac formalism and Clifford algebra.

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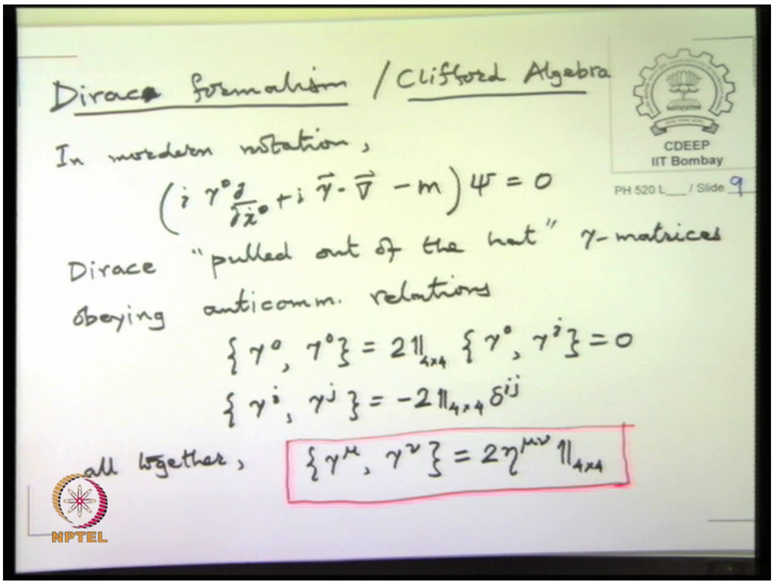

So, in 1929 Dirac proposed in modern form x 0 is c times t and i gamma dot grad minus m psi equal to 0, this is Dirac equation. Where, Dirac proposed these the matrices. So, that their anti-commutation relations or should I write it like this you have (Refer Time: 12:33) ones. So, there is of course, identity there are 4 by 4 matrices times delta i j.

So, if you wrote out in detail gamma 0 square basically if this is an anti commutator. So, it is gamma 0 gamma 0 plus gamma 0 gamma 0. So, that is factor 2 any way it says gamma 0 square equal to 1, this is that each of the gamma i squared is minus 1 minus the identity. And, if you take gamma i with gamma j so, gamma 1 gamma 2 plus gamma 2 gamma 1 will be 0.

Similarly, gamma 0 gamma 1 plus gamma 1 gamma 0 will be 0. So, altogether 1 writes this elegant notation gamma mu gamma nu anti commutator equal to 2 times eta matrix mu nu times the 4 by 4 identity matrix ok. So, Dirac use slightly different notation, but this is what it is? So, what is the utility of all this the utility of this was that if you squared this operator if you applied this operator another time, then you got a relativistic wave equation d'Alembertian times the m square term plus the m square term.

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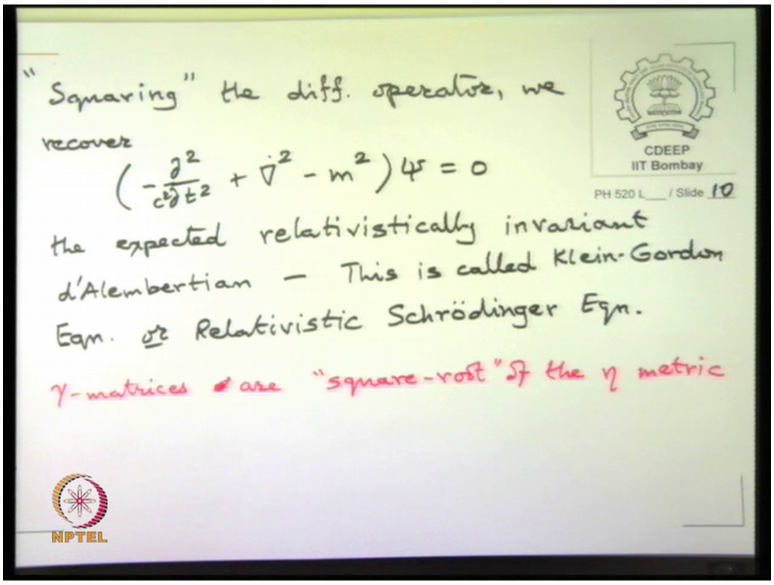

So, now, what this says is that if this matrix is your 4 by 4 then this is a 4 component object the size itself a 4 component object. Here, it says that this is times an identity matrix. So, each component psi obeys this wave equation. So, this is sometimes called Klein Gordon equation, but it should mark correctly be called relativistic Schrodinger equation. So, you might wonder why Schrodinger put i d by d t equal to minus grad square, the Schrodinger was not done this is 1927 it was 20 years after special relativity.

But, when you did it he did not get the hydrogen's lines correctly. So, he had to fudge something. So, he said [FL] he then applied the non-relativistic equation got the hydrogen spectrum perfectly by using the Laguerre polynomials and all this end of the story, but in the end of that paper Schrodinger says that ideally. However, we should be using the relativistic equation and then leaves it at that. Later Klein and Gordon got on to the job and the realized lot of problems; first was that once you have this you do not have conserved probability of Schrodinger equation, you get you do not get a psi dagger psi conserved current ok.

In the Schrodinger non relativistic theory you take side psi dagger psi and normalize it, but that is because psi dagger psi is the converse density. Here, you will not get any conversed density like that. So, that is 1 problem of the wave equation, the other problem which reason why Schrodinger missed using it is, because he was getting this j into j plus 1 and he kept thinking here to put integer. But actually it was not integer it was half integer ok, which he did not know at that time. So, if he had put half integers you would have got it right.

So, anyway this is called Klein Gordon equation given Klein and Gordon basically got very puzzled by this equation and what it means and so on, because it was giving indefinite probability. So, Mister Dirac said 1 day and said how do I have an equation? That is both relativistic and gives positive definite psi dagger psi. So, he said I remember that if I had Schrodinger equation with first order time derivative, I do get conserved current density conversed density. So, it as to be first order in time, but then by relativistic invariance it should also be first order in space. And, so put some coefficients here and then try to see how I recover the Klein Gordon equation.

And, if you do it then on squaring you have the anti-commute this to get the cross terms to drop out ok. So, that is what Dirac did and it. So, mystified everyone that nobody understood, it for couple of decades at least and Dirac got from it both positive energies and negative energies, that happened that you cannot avoid you know the property that this had. So, Dirac was bold enough to this is what you have to gave to Dirac? He said the negative energy states are all filled come for ever and by then for me statistics was known so; he said each state can be occupied only once.

So, if the states all the negative energy more there fully filled then nothing can go into it. So, we are seeing only positive energy electrons. However, every ones in a while 1 of the negative energy states may get kicked up to a positive energy state and then you will be left with a whole there in this filled up C. So, the whole then he argued would act like an oppositely charged particle of same mass.

So, you would have to have positrons, but he did not know of any positrons existing, but a new proton existed. So, I said well if you allow me a factor 2 thousand mass difference between positive and negative charges, then I have a relativistic theory of electrons and protons ok.

So, this he proposed in 1929 and it was. So, surprising that Niels Bohr said that this was a good way for catching elephants in Africa, because you write Dirac's theory on a board put it put the water hole of the elephants, when the read this theory they will be. So, stand that you can put them in your truck and take them home so, etcetera.

And for several decades people wondered what the hell this was? Well it turns out it is just a representation of the Lorentz group, but couched in a different language. And the point was that this kind of a so, one other way of thinking that people caught on to was that this gammas wearing some sense the square root of this differential operator ok. That this differential operator, which was first order in the derivatives, was like a square root of this second order differential operator and so, the gammas were somehow square root of the space time metric ok.

Because, this involves minus d t square plus mod d x squared and that can be seen here because we are saying that this I put a time mu nu where remember our notation was a eta 0 0 S plus it is that Minkowski 1 minus 1 minus 1. So, it l is correctly with this if you put eta here, but now think of this as the metric, but the left hand side is quadratic in gamma's.

So, the gammas are somehow taking a square root of the eta metric. So, that was the main hint people had, then the mathematicians when they heard about it they said, but physicist are a little slower and they because almost 50 years earlier in mathematics, there was a man called Clifford who had identified this algebra which would take square root of the Pythagorean metric.

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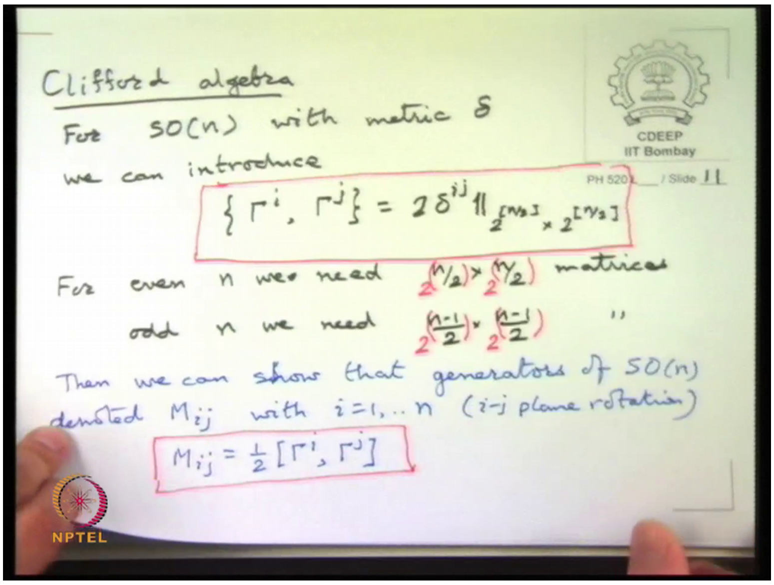

For S O n with metric delta write the delta g, we can introduce gamma i gamma j equal to 2 times delta i j n times identity matrix of size 2 to the power n 2 2 2 to the power n by 2 ok, which is biggest integer smaller than.

So, for odd for even n we need n by 2 cross n by 2 matrices and for sorry. So, of course, in 4 dimensions we want 4 dimensional matrices goes as powers of 2 2 2 to the 1 4. So, that is correct 2 to the power 2 and for odd n we need n minus 1 by 2 times n minus 1 by 2 size matrices.

The main connection between this and the group S O n group please leaves the delta matrix invariant is that the rotation generators of s o n denoted M i j with i equal to 1 to n. So, remember this is rotation in the i j plane. We can show that in fact, M i j are equal to 1 half gamma i gamma j commute anti commutator.

So, how do we show this well the statement is that you know the basic algebra of the gammas you assemble the M i j out of these as products. So, this commutator make sure that you do not any get any i i you only have 0 1 0 2 0 3 you do not have 0 zero this commutator will obliviate that. So, there are exactly as many M i j s as many commutators as there M i j'S and you can compute the algebra of M i j you will recover the algebra that I told you last time with M i j M k l it should be delta i j delta times M k l. So, the algebra of S O n can be recovered by from this by using the Clifford algebra of the gammas themselves ok.

So, let me check if we have 3 dimensions then we still need 2 by 2 matrices 2 to the power 1 so, this is 2 to the power that is what I will missing was in 2 2 to the power I wrote it here correctly (Refer Time: 28:42). So, 2 to the power 1 which is for dimension 3 a 2 as well as 3. So, sigma matrix is will work probably matrix is will work in dimension 2 and dimension 3, the gamma matrix is become 2 to the power 2 because 2 to the power 4 by 2. So, 4 by 4 matrix is are needed in 4 dimensions, but they will also work in 5 dimensions and so on. And, what one can show is that the M i j exactly satisfy the S O n algebra satisfy the Clifford algebra.

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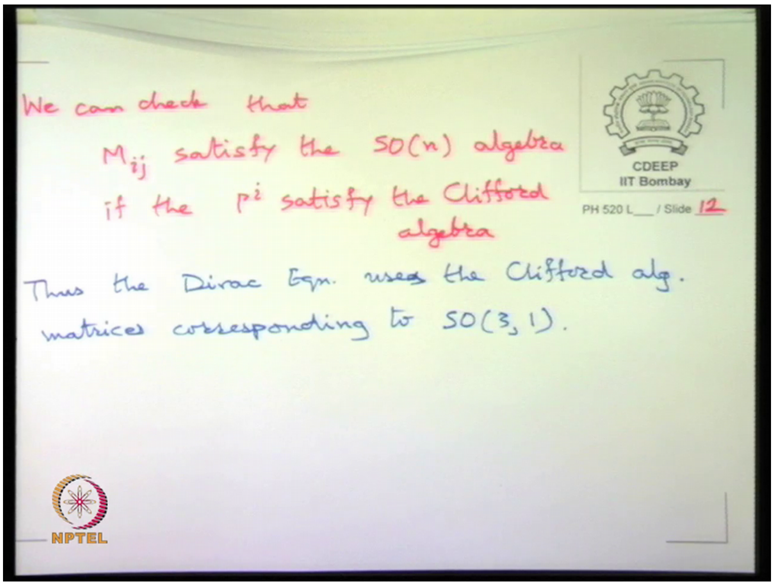

So, what Dirac had stumbled on was the Clifford algebra of the Lorentz group of course, with the big change that it was not delta i j, but eta i j. So, there is quite a few changes of there as relative sign minus sign and so on, but this is what essentially what the Dirac equation is, ok.

So, I think we will stop with Lorentz group there.