Theory of Group for Physics Applications

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Lecture – 43
Representation of Lorentz Group and Clifford Algebra – I

Some, technical troubles we have been having last time about the commutation algebra of S of SO 3, 1.

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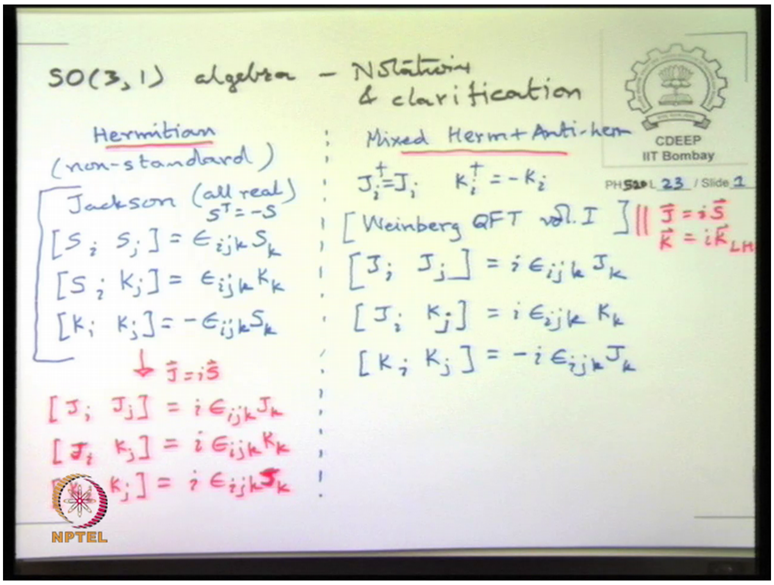

So, let me write down the algebra first because that is the most crucial thing, to worry about. So, our problems arose because we used a slightly unusual convention, and basically what we adopted was what you would call Hermitian convention, and this is non standard, in the quantum field theory. And, the other one is a mixed one, Hermitian and Anti-Hermitian. So, let me here quote Jackson, because that is the standard textbook and even if I have made various errors of sin I do not think there will be errors there. So, Jackson's convention is that, with everything real. S i S j equal to Epsilon everything is real here. So, there are no i's at all ok. And therefore, it is automatically well. So, it is yeah I am sorry. So, I have to correct this. So, first there is Jackson has this, which is all real and therefore, actually not Hermitian.

So, the rotation matrices are anti symmetric ok. And, so he uses S, S transpose is equal to minus S that is how is rotation matrices are and so, he has S k and then S j, S i S i K j are

Epsilon ijk K k and K i K j equal to minus sign, Epsilon ijk S k. Now, to recover our notation from this which was Hermitian. We need to do the following, we defined J to be equal to i times S. Basically, this is what we did. This makes everything Hermitian, because the K's are symmetric there were just 1 minus 1 on the in the 0th row and column. And, the rotation matrices were anti symmetric, but now if you multiply by i, then they become Hermitian. So, this whole notation becomes Hermitian. And in this case, I get so, it requires you to put supply i's everywhere. This we can see because, I put a minus sign on the left side here, and split that minus sign as i and i.

So, it will convert both of these into j, but now I have minus sign here I split it into again i and i leave one i in front and other i is absorb by this to become J. In the next line, I supply only an i. First, line I supply minus n, second line I supply an i, i times S will become J. And, here no change except that the i will sit in front. And finally, in the last line i supply nothing, but I split the minus sign into i and i and put one i into S. So, it will just become equal to K i K j, equal to i Epsilon ijk S k ok.

Now, if I did this I got into trouble. So, after this if I do J plus ik J minus ik it does not work. But, the thing that does work is the next Hermitian and Anti-Hermitian formalism followed in quantum field theory literature. Yes, what did I write? Oh, I have to write J. In the mixed Hermitian Anti-Hermitian formalism K's are actually Anti-Hermitian. So, we will not try to write any matrix representation, but I will just say that in this case, the J are Hermitian, but K are Anti-Hermitian. And, this you can find in Weinberg Quantum Field Theory volume 1 ok.

In this case, the commutation relations are the way I flip the sign suddenly last time, but you can be sure that if you use this convention, then you get this algebra, J i J j equal to i Epsilon ijk J k and J k. So, let us remember that we will always right in order to not get lost, rotation produces a rotation and then act which rotation on a boost that produces a boost. So, that is the second line here J k produces i Epsilon ijk K k. But, the third line which is K k or that is K i K j in this case turns out to be equal to minus i Epsilon ijk J k. Two boost leads to a rotation, but with a minus sign.

I believe that you can get this and this mixed Hermitian and Anti-Hermitian notation, if you multiply also the K by an i ok. So, J as here, J equal to i times Jackson's S. Jackson uses everything real so at least that is very simple to follow, and the real matrix is are multiplied out within nothing left imaginations. So, you can check everything, but also make the K nu here, i times K of left side ok.

So, if you do this then because the K bar as 0 1 1 0 1 1 type of things, if you multiply by overall i the become Anti-Hermitian they were symmetric to begin with. So, they become Anti-Hermitian. So, that then, tallies with this and I think this is the algebra you will get, and that you can also quickly see if you supply i here. So, supplying overall minus sign that will make it i and i and then put the minus sign here which will remove this minus sign, but then you need to put an i to make this into J which will required to put minus i in front ok.

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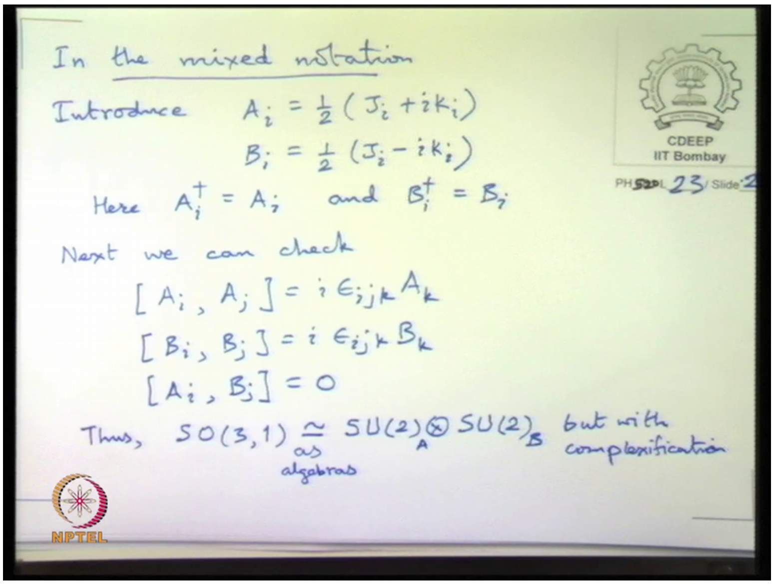

So, this and this tally and this is what the this is the notation we will follow. So, in this notation we then introduce, well I just want to realign my general argument. That if you have rotation in a plane and in another plane if the planes do not share any axis, then these rotations are independent they will commute, but if they share one axis, then that common axis goes away and you get a rotation in the only the third direction. So, if you do x y, y z then the y drops out and you get an x z rotation.

So, if you have 0 1 0 2 which are two boost, then you get a one two rotation. How to imagine it? Probably requires you to. I mean, firstly you can do it with S O and instead of S O 3, 1 and then try to, but that is the simplest way to think about it and that you can check algebraically by doing this or by taking some object and doing infinite decimal rotations, x z rotation, z y rotation will leave behind an x y rotation that is what the commutation does ok.

In the mixed notation which is what we will stick to because it is standard in quantum field theory. We now introduce, this A i equal to 1 half J i plus i K i. Now, this gets a little confusing in some sense, but let me just first write it. And introduce B i to be equal to 1 half times, J i times minus K i. We note that this way then the A and B are Hermitian. Because, K is Anti-Hermitian, but multiplied by an i and J is Hermitian to begin with. Now, if you work out that the A i will produce i Epsilon ijk A k, and we did this last time except that the only thing was last time I was writing K j commutator and not J k commutator. So, there is a sign change, but otherwise it is a same ok.

So, this we have check last time provided the, we adopted this convention make defining A to be J plus i K and then I suddenly switch to writing this algebra. So, then it works out correctly ok. So, we can check this and similarly that the B's do the same thing. And, the most beautiful part of it is that they mutually do not interfere, A with B is 0 ok. So, as a result the SO 3, 1 is equivalent to SU 2 cross SU 2, but with complexification, so, as algebras ok. So, you can see what has happened, the J is equal to i times the real rotations, S of Jackson. So, this is imaginary; the K was i times the real case, but we put i times that. So, we retain essentially a real piece here and that complex imaginary piece here.

So, this is intrinsically a complex algebra created and that is when it resembles this SU 2 algebra. So, it is a sort of a pseudo trick it is not really an isomorphism, in the full group sense. But certainly has algebras they are isomorphic, it looks the same. Once it looks the same we can deploy all the tricks that we use for finding the representations of this, which is as you know if the raising and lowering operators.

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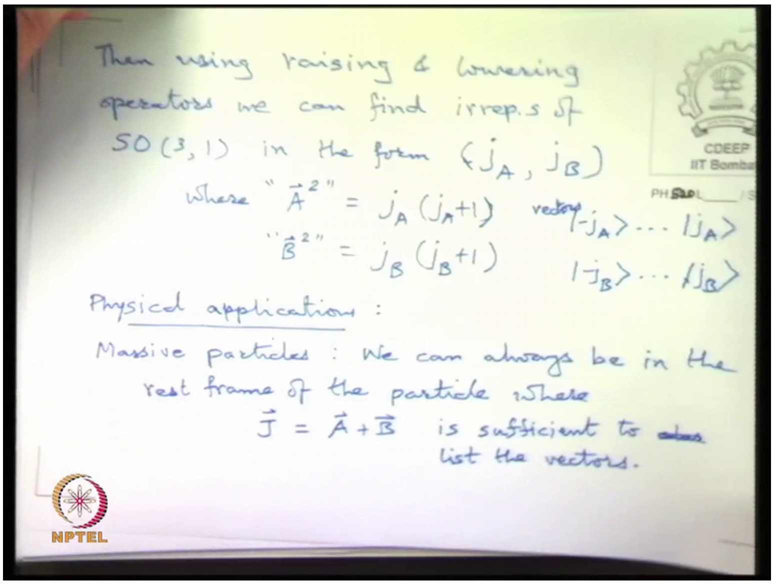

SO 3, 1 in the form, j A comma j B where A vector square will be equal to j A into j A plus 1 and B vector square will be j B into j B plus 1 etcetera. Because, each one just became A and B individually is like looks resembles and SU 2 algebra. So, these raising-lowering trick will work and you can apply this same arguments and you will find representations that go from, minus j to plus j. So, with vectors; now, from the point of your physics, the representation although it is classified by this what really we observe we can for any massive particle we can always go to its rest frame, we can boost ourselves into its rest frame. So, for massive particles we only need to worry about its rotations. So, physical application in the rest frame of the particle, in which case it will be the J really, which happens to be equal to A plus B right because we defined A and B like this. So, if you add then you get A plus B, then you get J back. So, J is equal to A plus B is sufficient to, classify the representations, to list the representation vectors. So, rotations alone are enough to give you the required basis vectors and then you can find the wave function in any other frame of reference simply by boosting by applying the boost which will be exponential of this A plus B.

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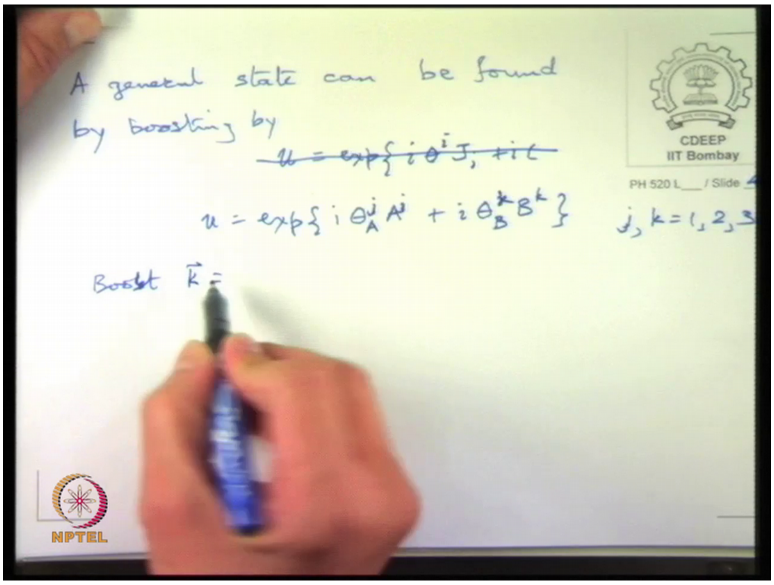

So, and here is where the (Refer Time: 19:50) u enters, you can write theta A or theta i J i and plus i am sorry u equal to exponent i times, first let us write theta A i A i right plus i theta B i times B i, or call this something else we have been writing so, call it j and put this k.

So, we have six parameters theta A 1, 2, 3 theta B 1, 2, 3 j and k equal to 1, 2, 3 and boost means, boost K happens to be equal to A minus B divided by i.

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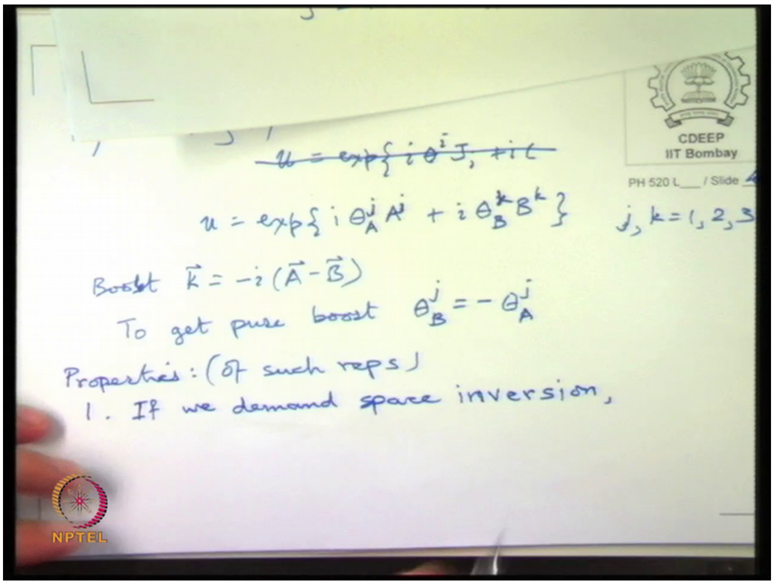

So, if you want pure boost, then you adjust you want A minus B. So, theta B's equal to minus of theta A j, you choose the parameters to exactly equal then you will be carrying out a boost. So, that is one thing. The two important comments now are that; one is about I hope I do not forget telling them. One is about non hermitcity of this, the other is more importantly that, so, properties of such reps, one is that we do want space inversion to be part of our symmetries. If you want space inversion to be part of our symmetries, then we can see the tenders space inversion J goes to itself J is a pseudo vector remember. So, it is like x y rotation is, but if you flip sign of both x and y then it will not change. So, space inversion means that x go to minus x.

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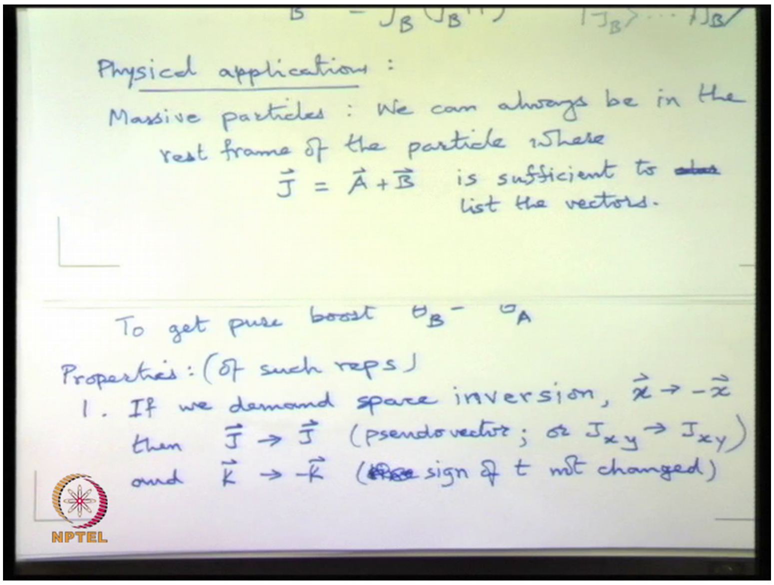

Then J goes to J, sometimes because it is called pseudo vector; or note that J x y will remain J x y under both x and y changing signs. The, if you flip both the plains then it remains the same the both the axis. But K will go to minus K, because it is a space time boost and time axis you are not changing.

So, only the space axis space part changes and what this means is that you are force to take a j A equal to j B. So, if you want this to happen under this A and B get exchanged right. Because, A is j plus k B is j minus i k I does not even matter. The point is that J parts remain same, but K parts change sign. So, under space inversion A and B get exchanged.

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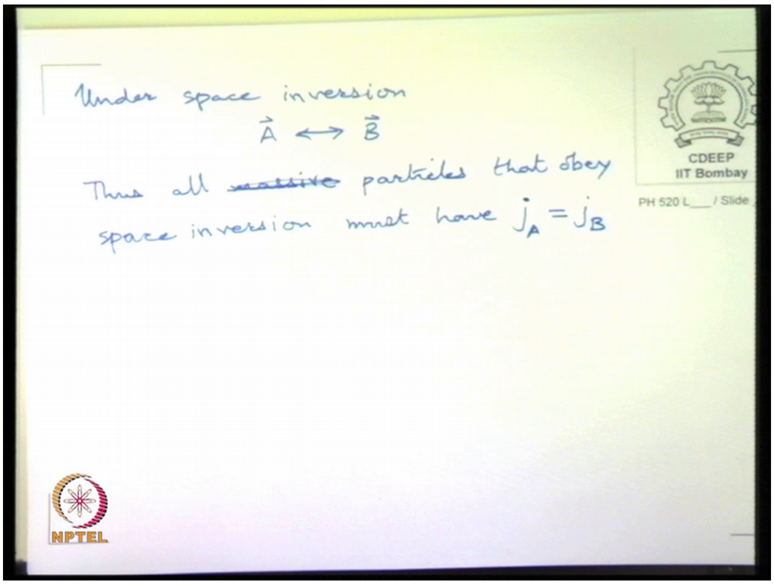

Thus, any realistic particle massive particles where we can demand, demand this. Well, actually we need not say. So, all particles that obey space inversion, must have j A equal to j B, the highest weight has to be same right. Because otherwise the representation will remain imbalanced, where had be written some kind of j A j B yeah this thing. A representation like j A j B where one of them runs over let us say minus 5 to plus 5, but other one only minus 3 to plus 3 under space flip they will get exchange. So, that cannot happen unless j A is equal to j B.

So, you are force to had j A equal to j B. Here, the interesting historical fact is that the neutrino in 1956, was realize to be purely left handed and completely massless. So, for massless particle actually you can get away with using only j A or only j B and not the other one.

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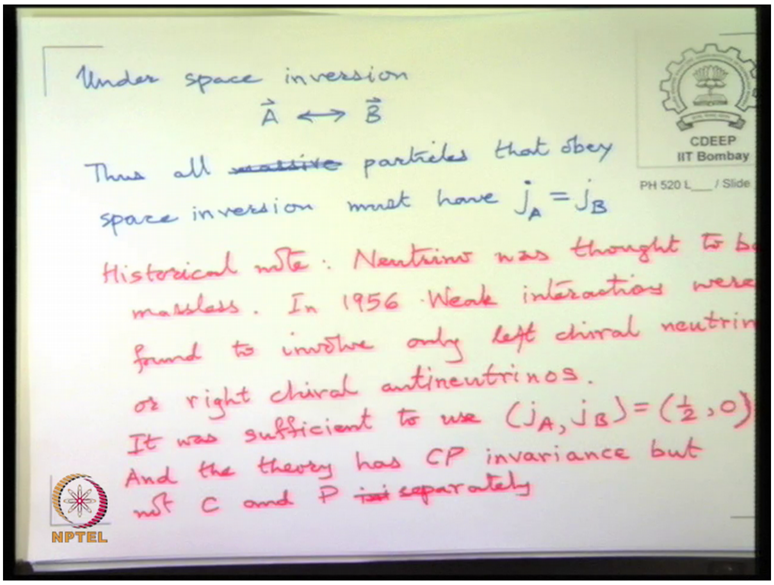

Thought to be massless in 1956, weak interaction were found to be involve only, left chiral neutrinos or right chiral antineutrinos. So, you could only use either j A or j B description without invoking both because you did not need the some j there, you did not have this requirement of A and B getting exchanged. And it was sufficient to, use j A comma j B, equal to 1 half, 0 representation of the group. Firstly, the do to need to be equal and one of them you just said 0, one of them is enough. Because, there are only two spin states either rotating either. So, its spin and momentum gets correlated if it is going this way then it has to be counter clock wise it is coming towards you then it has to be clockwise.

So, either its going away from you or towards you and there are only two spin states you cannot have antineutrino going being left chiral. So, it was sufficient to use this. This has, now changed as of year 2000 or so, we now know that neutrino there were tiny little mass. So, this description is still mostly valid, but the chances are that there is the other half presence 0 plus half. And also the possibility that the neutrino is coat a mayorano particle, which is a real representations. So, I hope to do it separately, but this is just to say that when you impose parity, then A B get exchanged then you are force to a J equal to j B, but if there is no parity there is actually for some reason in nature parity violation which was I use shock to everyone that weak interactions had no right handed neutrinos, no right (Refer Time: 30:20) and which is true till date. At least the weak forces do not see the right chiral neutrinos. We do sees slight mass to the neutrino which probably means there is a right handed component, but that right handed component does not enter the weak force that much is certainly true.

So, it was sufficient to use this and the theory was has only C P invariance, but no individual P separately. So, if you perform a P transformation you will get a right chiral neutrino, but then you should do a charge conjugation as well and make it into antineutrino, then the physics would be find that is what is observed in nature. So, it was link that we I mean this is still true till date it is true although the neutrino the massive. The weak interactions are only C P invariant and not P or C invariant. Now, the other comment, so, one was about this the other is about I said property somewhere write heres. So, one property is this, the second property is also interesting is that we only get the representations are not unitary.

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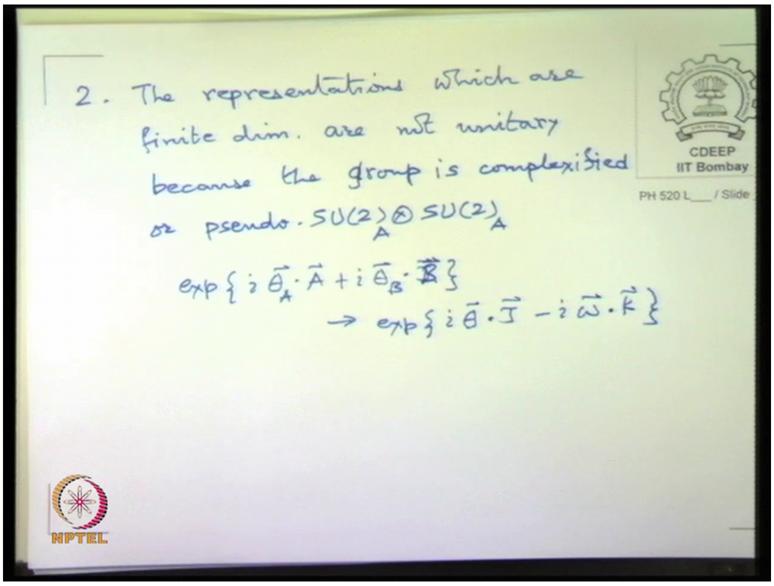

Which are finite dimensional, are not unitary. Because, the group is pseudo S U 2 cross S U 2. If you had a strictly compact group S U 2, then it would have been the representations would be finite dimensional and unitary like we proved in the using Schur's lemma, but in this case exponential i times, theta A let me write this notation theta A dot j A plus i theta B dot j B dot B. Which can be go into something equivalently into i times some theta dot J and minus omega dot K right that is what it would become, because if you put A equal to sorry so, let us just take you.

So, so, if you put A equal to j plus, so there is no i here. So, A will have j plus i k. So, that i times this i will make it only K. So, there will be no i here, it will be like this with theta and omega both real ok, but K and T Hermitian. So, here lets workout.

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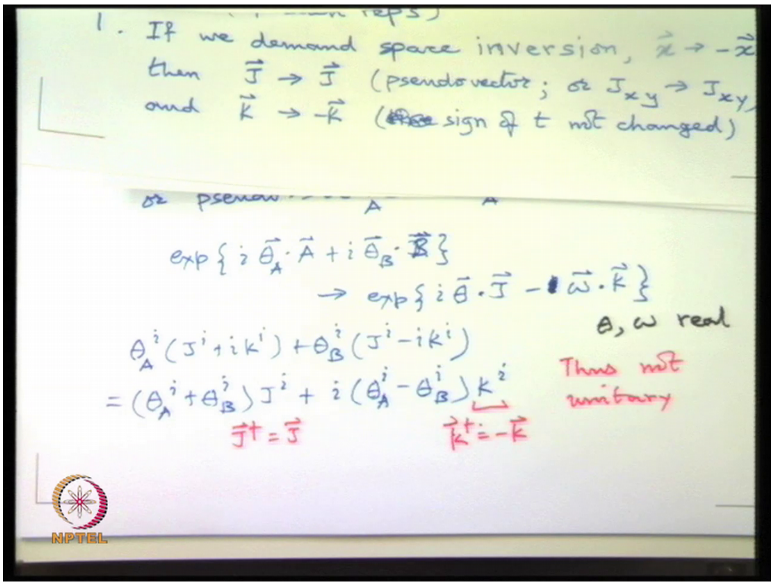

So, if we have theta A i times J i plus i K i, plus theta B i into J i minus i K i, then we get theta A i plus theta B i times J i and plus i times, yeah theta A i minus theta B i times K i. So, if you had boosted with i in front, then this i will drop out and this will be a real parameter theta and omega are real parameters. But, this is an anti-Hermitian operator. So, what we find is that this is no longer a unitary matrix actually, and that is why you will not get. So, it is not actually unitary the pseudo unitary and K dagger equal to K the minus K. So, the representations are finite dimensional, but intrinsically complex.

Now, why this was important was because the complexity of description of electron having. So, you remember the Pauli matrices are intrinsically complex you can have 1, 1, 1 minus 1, but you have to have minus i i. You cannot escape the complexification of the representation, because of the fact ultimately there it has to do with the Lorentz group and you are trying to get a finite dimensional representation of a, otherwise non compact group.

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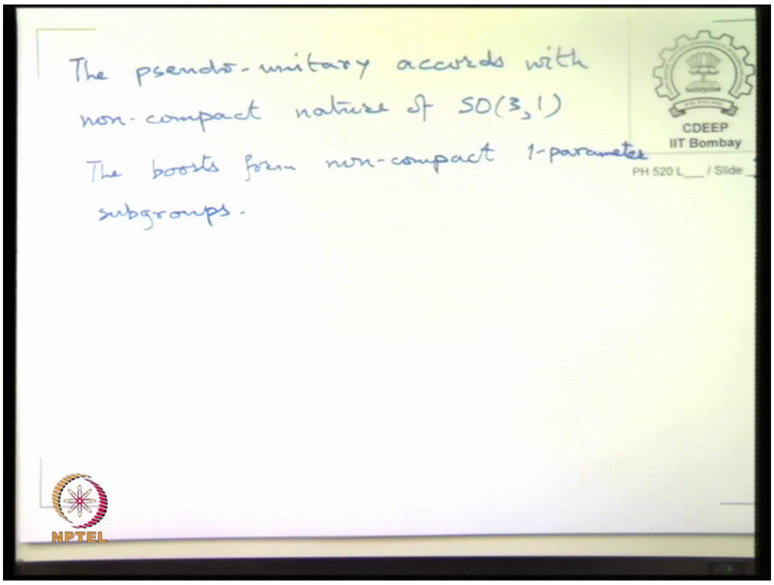

So, this is no surprise because, actually accords with non-compact nature of S O 3, 1 right. Because, the boost are not compact you can have you can boost arbitrarily close to speed of light, but you cannot actually hit the speed of light. So, it remains a non compact the boosts are form non-compact 1 parameter subgroups.