Theory of Group for Physics Applications
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Lecture – 42
Lorentz Boosts, SO(3,1) Algebra – II

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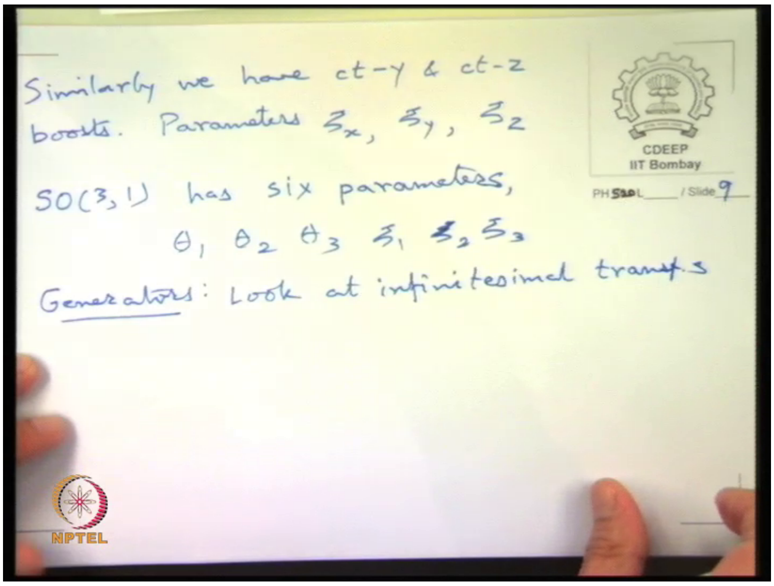

Now, next we go to generators. So, here we have to look at the infinitesimal transformations. The infinitesimal transformations are so all we have to do is look at those transformation matrix.

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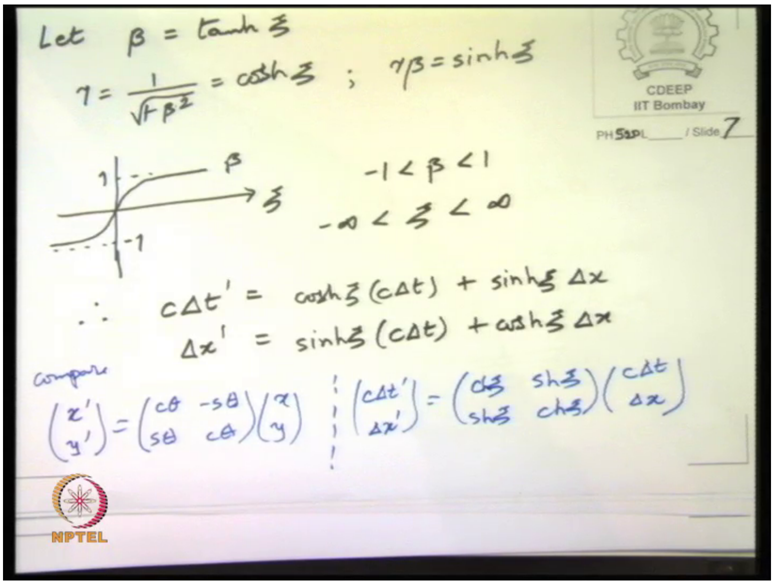

It is in the other language yeah this language is cosh, sinh, sinh, cosh they have just the same kind of expansion say sin and cosine except for the fluctuate, or the most important difference between the rotations. And these is aside from the fact that there are hyperbolic trigonometric functions, there is a minus sign here, and the plus sign here.

This sign these signs are always opposite for real rotations depending on the convention use the minus sign may be here or there, but they are oppositely signed, whereas, in boost these will always be the same signed. So, if it was boost in the opposite direction then both the sine hyperbolic's will change sign and you will get both minus signs.

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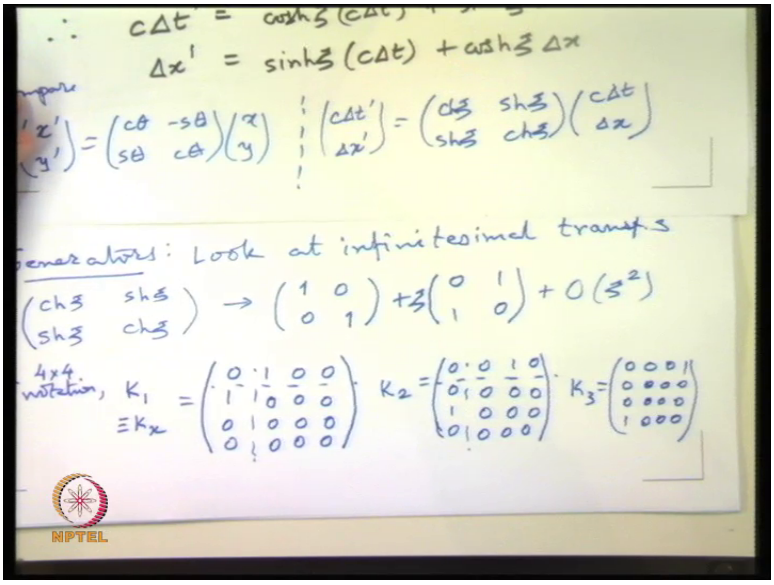

So, in the present case cosh psi, sinh psi if you expand out the cosh it is 1 to first order. And then there will be psi times 0 1 1 0. The sins series us just the odd powers with divided by the number factorial with no sign changes their both essentially becoming exponential for large.

So, there are no sign fluctuating signs and this is all we have and plus order psi squared which will come for the expansion of cosh, but we look at the linear term then this is all we have. So, in 4 d notation we have k 1 which is same as K x equal to so we call it K x 0 1 1 0 0 0 0 and the 3 by 3 part which is rotations will be a big 0. But as I said as per Steven Weinberg's notation I will fill it with 0's then k 2 which is same as k by would be equal to 0 0 1 and 0 and 0, 1 and 0.

So, the one just migrates along the top and the bottom and this 3 by 3 sub matrix of rotations remains null and does not require much imagination to, so you can fill it out 0 0 0 1 and 0 1 and all other 0's. Now, just as for the rotation matrices we would like to know the algebra of these.

So, we have to see what these things do under mutual commutation algebra of boost generators, K i. Now, here is where I run into lot of grief with signs and I hope that you will bear with me. So, I have some notes actually I will put these up, but I found that the notes has sin problem.

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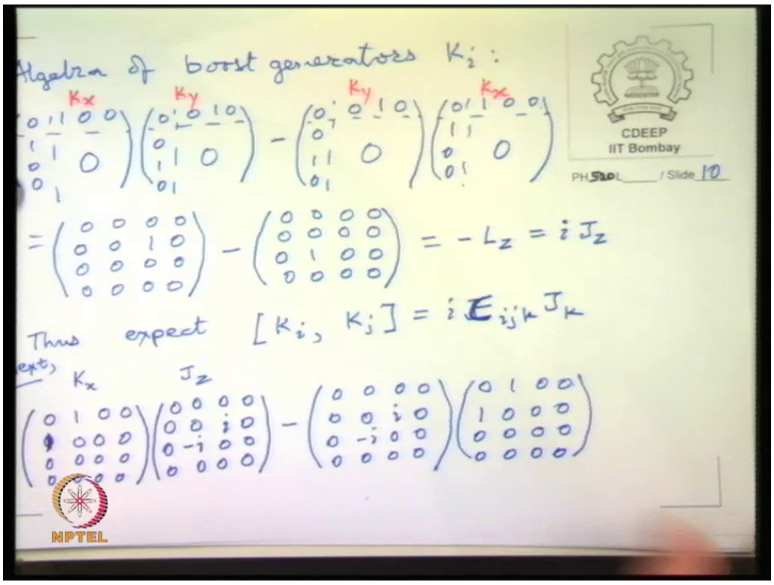

So, you should believe the results I tell you and work out the matrix multiplication yourself, but let us since there is time and since we never have any tutorial hour let us try to do K x K y and see what we get? A K x K y and commutator, so 0 1 0 0, 1 0 0 and if you do want to save time put 0 here and then 0 1 0 0 1 0 big 0 here.

Now, yeah I am trying to put K y first and times K x. So, what do we get? So, the only non-zero term here is the one that comes from second row and third column, so second row and minus third row and second column.

This is equal to our old friend L z all right yeah L z had so this is minus of L z, this is L z because it is in the one two location. So, this is giving x y rotation basically with 1 here, and a minus 1 here. And our convention the way I remember we introduced the active rotations was that this is equal to minus of L z and therefore, is equal to i times J z we had set J z equal to i times L z; J z equal to i times L z.

So, L z equal to minus i times J z, so minus L is equal to i times J z. So, we expect that K i K j to produce i times j epsilon i j k J k if I take K x and K y right this is K x and this is K y and it produce J z. So, we expect that the algebra is K i K j to produce i times J epsilon i j k this. The next thing to check is what happens between j and k? And this is where I had some problem with.

So, you will see where the sin crops up in a most important why most of this that we are writing it does not really matter, but anyway let me in next try to do K x and j z. So, can you do K x and J z in your book and better fill out everything whereas, J z is equal to i times L z and L z has 1 here, so that is i and minus i ok.

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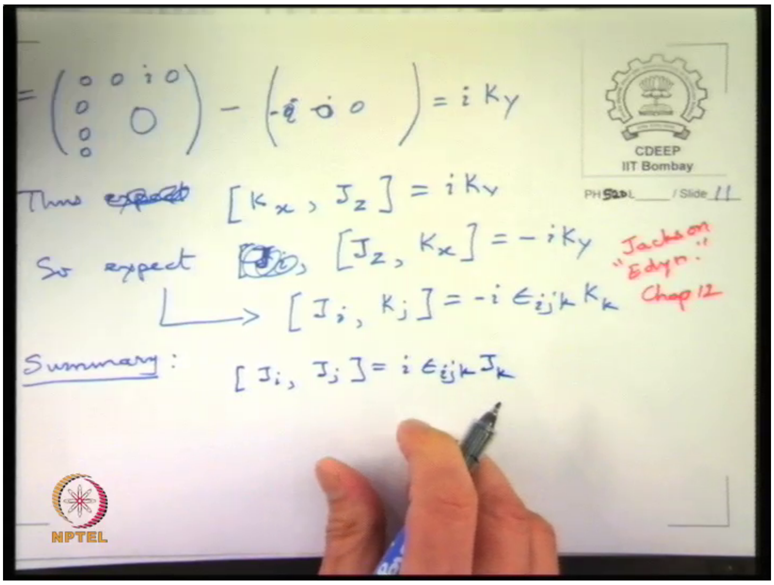

So, next consider this all the other things are going to be 0, and minus third row into second column which is going to be equal to minus i sorry so minus i. So, what do we get? We get i times K x is not it K y so, thus we expect that K i.

So, let us first write K x J z was equal to i times K y. So, what do we expect? K i, but this is in reverse order so, we should such that x would be more correct would be minus. So, better write it as J i very sorry. So, this is same as J z K x equal to minus i K y, or J i K j equal to minus i epsilon i j k K k. So, now, we have this algebra all this looks pretty boring, but now something interesting happens.

So, summary by the way this is what out in Jackson Jacksons Electrodynamics book chapter 13 has all this written out in great detail. He does not use complex notation, but I did try to check the sign, but there are all these binary choices do you use active rotations or passive rotations do use active boos or passive boos you write j first or k first.

So, the signs have to be worked out, but it is all there in Jackson’s book electrodynamics book it is there chapter 13 I think especially relativity. So, we are not doing something all the terribly exotic so far. But now let us see what is the summary? Firstly, we had the good old J i J j equal to i epsilon i j k J k.

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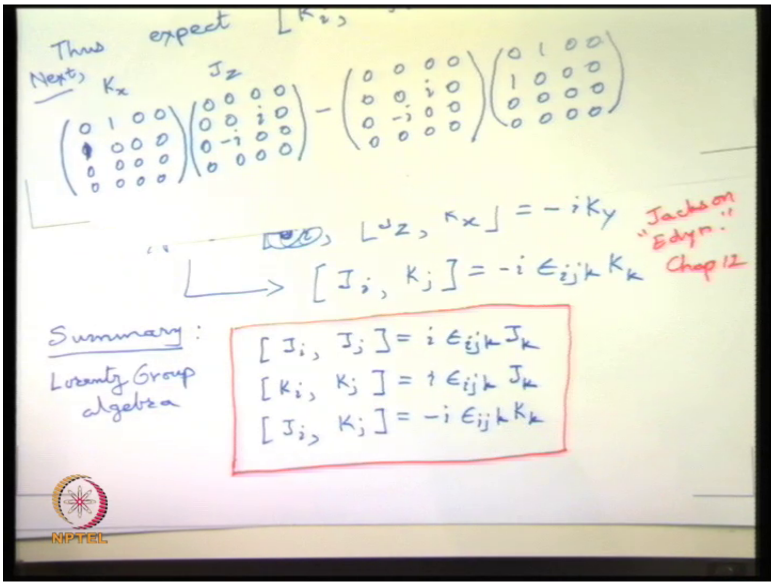

Then we calculated K i K j equal to i epsilon i j k J k. And finally, we had J i K j which is equal to minus i epsilon i j k K k this is the total SO 3 1 algebra, or the Lorentz group algebra.

What is interesting to note is that if you have two boos if you do if you come you two successive boos you actually get a pure rotation. So, in atomic physics this was discovered because some factor half was not coming out correctly it is called Larmor precession if I am not mistaken.

So, Larmor physically worked out the electron orbiting the atom and then it is here and then it is there. So, there are two successive boos those successive boos amount to a rotation of the electron. So, and we will have where I will tell you a little more about it quickly soon.

So, two sets two boos commute to give a rotation, but then a rotation and a boos mix things produces a boost. Now let us try to understand physically what this why it is like this? So this is the boxable result and I hope that sings in it are all correct.

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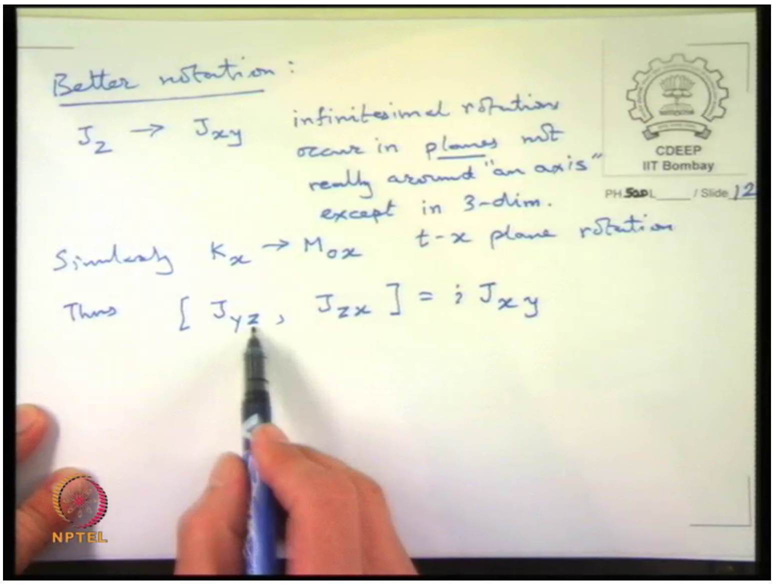

So, let us understand and write an easier way of remembering this. So, the better notation actually is to remember that the z rotation is actually x y plane rotation. The when we say that I rotate around z axis, this is a convenience of 3 dimensions, because actually the rotation is happening in x y plane and to say that it is around z axis, there is a unique third axis in only in 3 dimensions in many dimensions are would be many many axis that are orthogonal to x y.

So, they are calling it as z rotation is actually just a luxury of 3 D really speaking J z is actually J x y ok. So, the rotation infinitesimally rotations are in planes occur in planes not really around an axis except in 3 D.

In fact, this is the reason I can now tell you if you have suffered with this notion all along. Why is it that I have to take when I take cross product I have to erect it perpendicular to the plane in which I take the cross product ok, and I have to use right hand screw rule.

What happens if I use left hand screw rule? Well they say it is matter of convention well if it is convention it can be very physical right. So, if there is actually no perpendicular vector sticking out of the planes it is actually within the plane and it is it is basic the rotation is actually a second rank tensor, and not a vector. So, the correct way may so maybe we will come to that at some other point even we come to representations, for the time being note that actually it is not the axis of rotation. So, it is a two index object not really one index object.

So, similarly the K x is basically let us call it M 0 x, there is time and space ok, so the K i are basically t x plane rotations. If you now think like this then we know J remember what we use to write x y z. So, x is actually y z and y is z x is equal to i times J. So, x y would have given J z, but z is equal to x y this is what the commutation algebra is. And what does this say? It says that if there is a y z rotations let us similarly write down say this K i K j.

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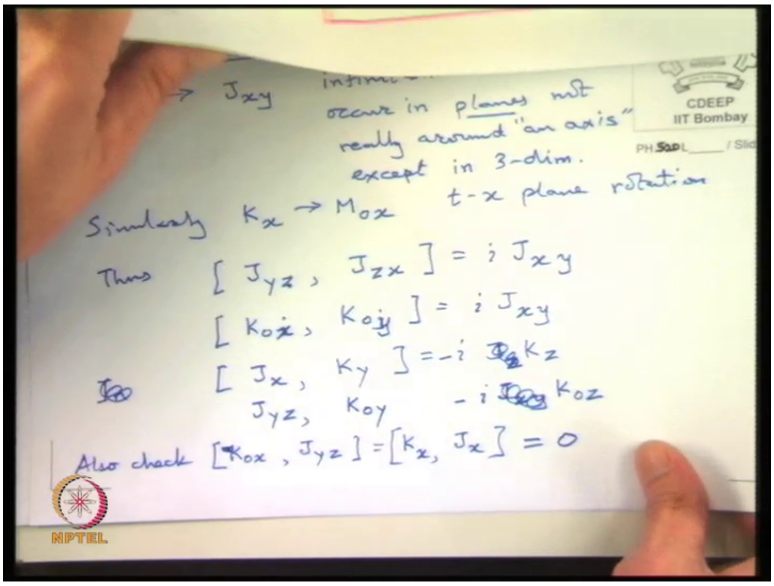

So the K 0 i K 0 j becomes equal to i times J. So, let us be specific so make it 0 x, and 0 y then from this algebra we know that this is J z, but z is equal to x y. Now, what do we see common between these what is happening is that if I have a y z rotation followed by z x rotation not really followed by commutation then does that drops out and I am left with an x y plane rotation x y y z becomes x z directly.

If I have 0 x, and 0 y then I the 0 drops out and I basically get an x y rotation. If I have larger number of dimensions, if I rotate 1, 2 plane and 7, 9 plane and later I rotate 7, 9 first and 1, 2 they are going to be independent, they will all commute. So, those rotations that share one axis are equivalent to the rotation which is the two un where there shared axis drops out. So, if it is z and z then there is no z here, 0 and 0 there is no 0 here.

So, I said from sin conventions that is what is happening ok. And in general we will have. So, we can also quickly check the third thing where if I take J x and K y I should get i times. So, which is sorry so which is same as J x is equal to y z and K y is k 0 y.

What do I get? I get from this J x x y is equal to minus i J z becomes equal to z is equal to minus x y; x is equal to yeah sorry this is k sorry thank you that is why I was going wrong. So, it becomes K 0 z, that again we see the same rule if it is y z and 0 y then y is going to drop out and I will get a 0 z rotation. So, this is basically how the algebra operates and therefore if you go to larger dimensions then planes that are independent that do not share any access they will just commute.

So here too you can think of 0 x, and y z. So, also check this should be equal to 0, so y z is equal to J x, but according to this algebra if the two indices are because there is a epsilon tensor here if these two are equal this is 0, so this is indeed 0. So, if you have completely independent planes 0 x plane and y z plane then the commute that is the moral of this whole exercise and that is it is basic geometry it captures the geometry of rotations.

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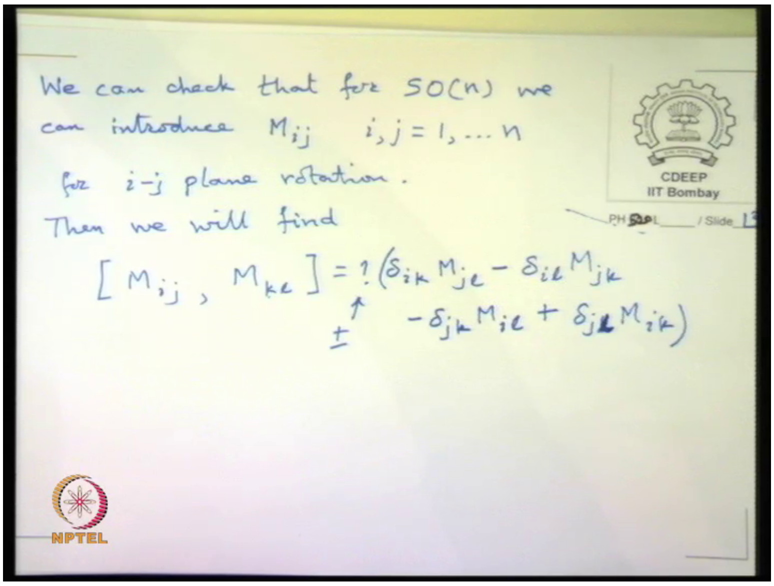

We can check that for SO n we can introduce well first let us just say M i j with i j running over 1 to n for i j plane real rotation. Then we will find. What do you expect? M i j; M k l to be equal to what?. So, I am guessing if i equals k then I will drop out and J k j l will remain.

So, delta i j i k M j l then there will be a so next if i equals l then I should put delta i l and i should be left with M j k except that because I am matching first sign with a second sign there will be a relative minus sign.

Similarly a relative minus sign and delta j k M i l and finally, j equal to k M i k and the overall sign I do not know it is plus or minus 1 that one has to check by checking one of them correctly in detail. But this is what we expected to be if the top infinitesimal rotations share any one axis then we get an only if the share one x axis then we get a non-zero answer. And if neither I equals k nor, if i does not equal k or l and j does not equal k or l then we will get 0 that is this.

So, S 1 algebra will be like this we worked it out you know no need to be afraid of higher number of dimensions yes ok. So, that is see first one I put as delta i k and j l in the end I should put delta j l and i k right. So, aside from an overall sign which you can fix so when string theories tell you we have to living 10 dimensions you have to sum prepared I can rotate internet. So, this is algebra of SO n.

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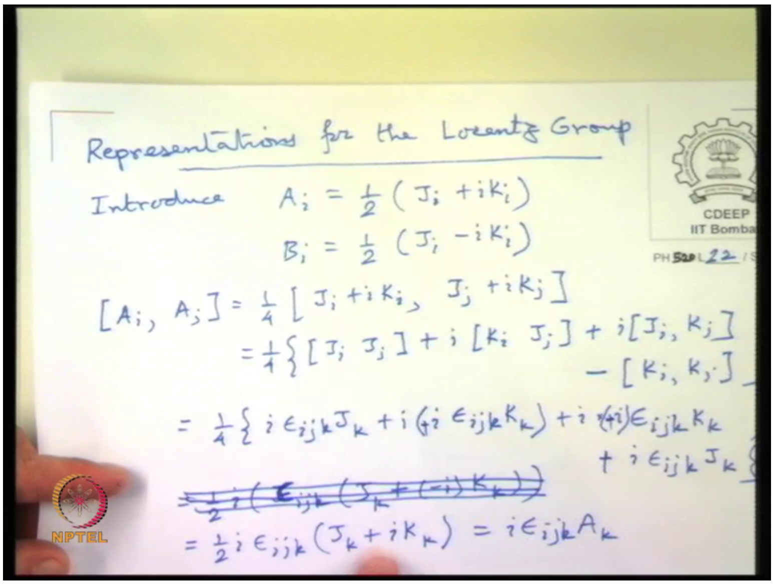

Now, we come back to the algebra of SU SO 3, 1 and there are some interesting things here. So, representations of the Lorentz group for now when we had the 3D rotations, we at this clever J plus J minus construction. But now we have 6 generators corresponding to the 6 parameters. So, by the way I hope you know how to count these are the they have to be orthogonal. So, only the upper triangle matters the lower triangle is minus of those that are above and the upper triangle in 4 by 4 matrices is 6, so there are 6 parameters.

And how do we generate the representations? So, here there was this clever trick by Herman Weyl it says the following; so by the way Dirac was considered a wizard by most people of his generation and later once we have radius papers and Dirac was famous for not answering anything in more than one word at a time monosyllabic yes no maybe or keep quiet say nothing.

So, there if you Google interview of Paul Dirac, that I think it was university of Wisconsin or Minnesota one of these northern university. So, interview of Paul Dirac you will find this one webpage of an interview taken by a response in journalist and it is an American journalist. So, he goes and says Professor Dirac can I interview you and all this.

So, and Dirac is sitting there yes no no; so he is asking you are you the greatest genius has no, so and it goes on. So, eve eventually Dirac gets bored. So, I just kind of gets up and starts living and then this man says wait wait, but tell me whom do you consider clever who are you scared of or something like this, who would you think is yours superior and Dirac stops and says while and leaves the room.

So, that Herman Weyl suggested that what we should do is to define A i equal to and I hope I pull this off correctly I feel like Harry Potter movie and try to do some magic and should work.

There is an i introduced, so and b i are of course, with the opposite sign minus i K i. So, there are two independent linear combinations introduced and now let us see what we get with their commutation. So, if I take A i, A j so here what we did was there is i times K i J j, but I have here a J k commutator.

So, to reverse order of J and K I get a minus sign on the commutator which I put here, but now the order of J i has got reversed. So, if I change order of J i and the epsilon I get a minus sign. So, there is so firstly, became plus i, but then a minus because the epsilon sign change, so I get a minus i this is straight forward.

So, I get both plus signs which looks very good except that J i J j K i K j should also have given i times, so these two should have added. So, I claim that this is equal to one half times i times epsilon i j k times J k plus. So, if I pull out and I then I will get a everything is here is wrong I get a minus i going to change the sign here and put this sign here. So, these signs have to be reversed ok.

How would we did the whole matrix multiplication? But I do not know how I got it wrong, but if this is minus, this is plus, then this will become plus, this will become plus, and I will have a minus sign and the K k commutator as a minus sign. So, that will cancel this and so I get totally one half times i times epsilon i j k times J k plus i because this is this becomes now a minus epsilon i j k K k, so it becomes this. So, equal to i times epsilon i j k a k.

So, we need to fix these signs K i K j equal to minus i epsilon i j k j k let me just try to think if I were not to do this anyway right now there is there are binary choices so that I told you. And if you flip some signs and some signs will change I do know that this is all this also in some convention this is the algebra and so let me just write down the book name.

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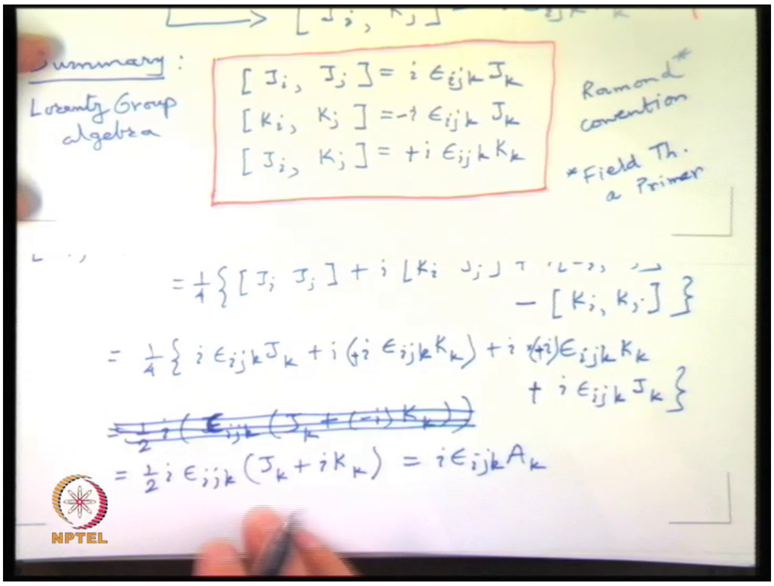

This is it field theory a primer, modern primer. So, if you use this conventions and then introduce these A's. Then the A i satisfy exactly as if it is SU 2 algebra. More importantly A and B mutually commute.

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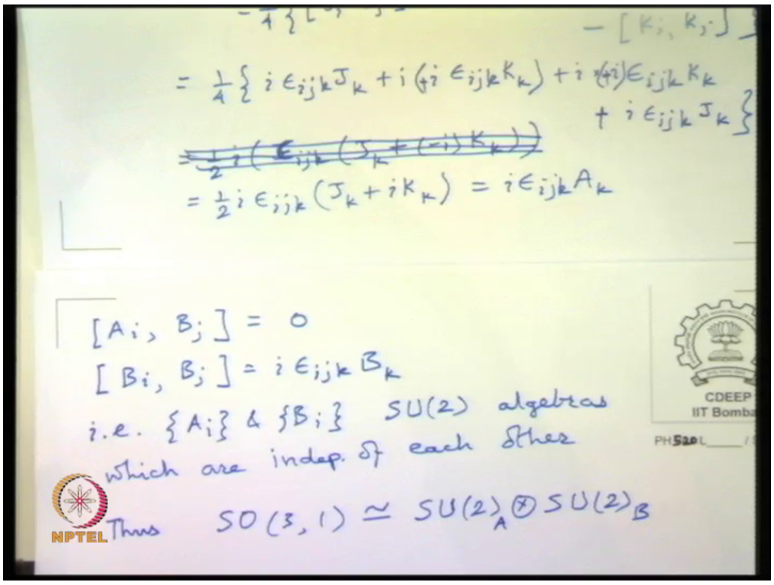

So, this we can check by just running through this algebra because if you want to calculate A B commutator all you have to do is replace this by minus sign, but you can see that changing that sign does all the mischief. Because this first term will remain the same this term will change sign because of this and this, but K j will not change sign and K k will change sign.

So, the relative signs of the things that was supposed to add actually become opposite and A and B then commute and finally, we can work out B i B j this turns out to be not surprisingly i epsilon i j k B k because once you here change both of these signs then everything will remain as it is except for the overall sign here and you will get a j k minus i K k.

So, the summary is that A and B SU 2 algebras which are mutually commuting independent of each other right the A is satisfy A i A j equal to i epsilon i j k A k, B satisfy the same and the A B mutually commute. So, we saying group theory a SO 3, 1 is this is the special sign equivalence SU 2 A cross SU 2 B it boils down to some SU 2's. There is there are not genuine SU 2 there pseudo SU 2's because of the introduction of the I in this convention, but then nor is this is a full orthogonal group.

So, system SO 3, 1, but this trick allows us to reduce at least the representations of Lorentz group to 2 SU 2 groups, it is just like two different spins. So, the A can take any value minus L to plus L and B can L A 2 plus minus L A 2 plus L A L as independent quantum number going from minus L B to plus L B. So, we will see more about it next time.