Theory of Group for Physics Applications
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Lecture – 41
Lorentz Boosts, SO(3,1) Algebra – I

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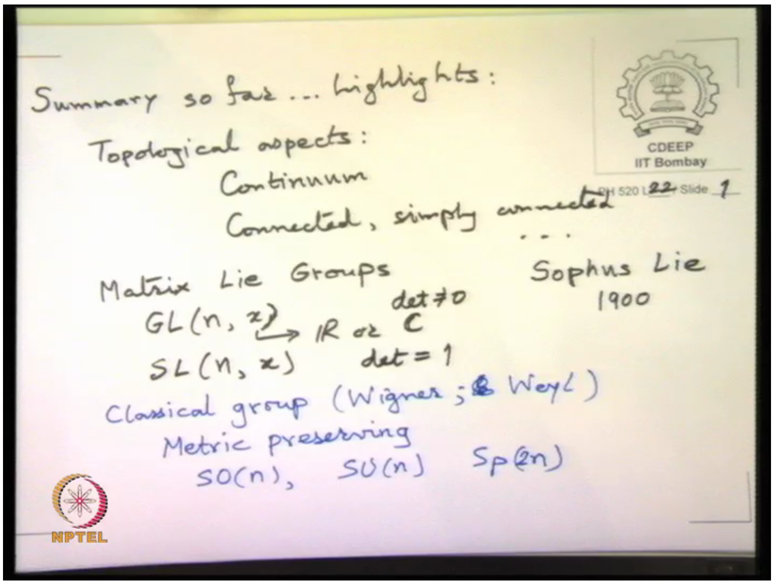

So, instead of writing chronologically I will just say some of the topological aspects of continuums like, what is the continuum and what is connectedness and all that and then we did matrix Lie groups. So, I do not know whether I said this so, far, but this is most people will read it as Lie in English language, but this is named after a mathematician called Sophus Lie who was Swedish I think he is from Scandinavia. So, this is the name and this work was done by him around year 1900 so much later than the development of the discrete group theory, which was about 50 years earlier than that ok.

So, matrix Lie groups and we saw that the big group there is G L n x where x is R or C n by n matrix is of non vanishing determinant, because your (Refer Time: 02:04) inverse and correspondingly there is also S L n, x which have determinant equal to 1. So, determinant here determinant is restricted not to 0, that is all you need for invertibility whereas, for the S L we restricted to be plus 1 ok.

And then so, I am just going to do this quickly because, we do not want to spend too much time. And then the so, called classical groups according to Wigner and Weyl;

Wigner and separately at two different, which is essentially metric preserving, which contain things like SO n and SU n and Sp n simplectic Sp 2 n typically.

So, this was our notation. So, there are matrix in each of the cases and they are preserved by these groups. So, this is what we did and then the latest thing we were doing was representations of. So, we got the angular momentum algebra what we meant was rotation group.

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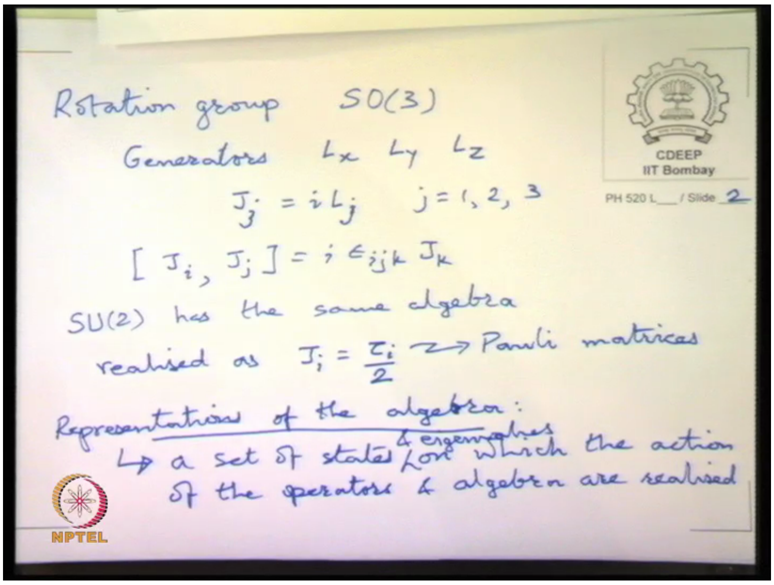

So, which happens to be SO 3, 3 by 3 orthogonal groups of determinant plus 1 and so, here we worked out the generators, L x L y L z, but then we said that we will define J i the complex notation to be i times these, sorry for the same i we can we can make this into j. So, j equal to 1 2 3 and then we found that the algebra is J i, J j equal to i epsilon i j k J k.

So, from this we can get. So, then we found that SU 2 has the same algebra. And in fact, realized as the specific matrix is J i equal to tau i by 2 which are Pauli matrices, they satisfy exactly the same algebra. And then we could work out the representations, in other words we found so, what does representation means, we found the set of states that completely realize the algebra. There are set of states set of states and Eigen values, on which the operators and there are algebra are realized.

So, that is what we mean by knowing the representation because, we can then manipulate we can handle any computation that deals with this algebra.

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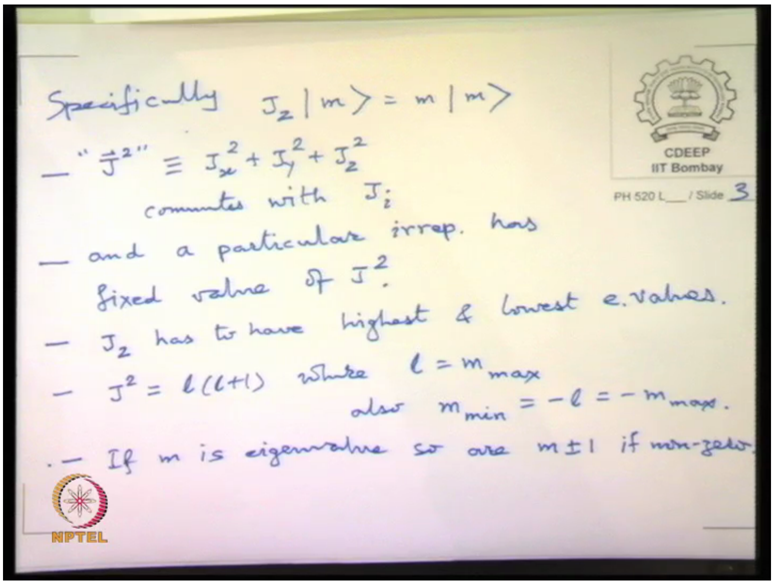

So, specifically we had we introduced J z on m equal to m times m in quantum mechanics notation, but m is any vector. So, you could write in vector algebra term treating this as a matrix and this as a vector and Eigen value. But then the interesting things was that we could realize that J squared and this in inverted commas, you treat it as composite symbol J squared it is definition is J x squared plus J y squared plus J z square yes. So, this has same Eigen value on all the end.

So, this commutes with each of the J i and a particular rep particular representation irreducible representation has fixed value of J squared. So, we can say something from bulleted things. So, J squared equal to this commutes with these and the particularly irreps has fix values of J squared, also J z has to have highest and lowest Eigen values. So, that the values of m are bounded and then we could also prove that J squared is equal to l into l plus 1, where l equal to m max. Also m min equal to minus l.

And finally, of course, if m is an Eigen value then m plus minus 1 also are that should actually have been the very first treatment, m plus minus 1 provided not 0, at some point that too become 0. So, this is sort of the first one actually. Basically these are the facts which we proved in the last two terms and, this gives as a complete control on a particular presentation. Any particular presentation is given by the highest value l and then given l. So, we have a spectrum this is the word people use.

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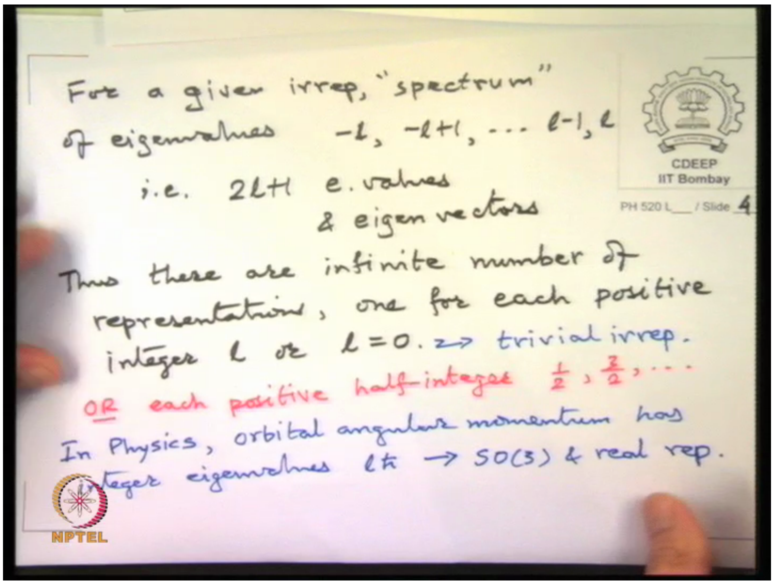

For a given irrep spectrum of Eigen values minus l minus l plus 1 upto l minus 1 and l. So, i.e. 2 l plus 1 Eigen values and that many eigenvectors. So, the point is that the algebra will only mix these, you can construct the raising and lowering operators j plus j minus and they will convert one vector into the other. And this forms an irreducible representation, but there are so, there are infinite number of irreducible representations, one for each positive integer or 0 or l equal to 0, l equal to 0 is an angular momentum 0 Eigen state and I mean it is a trivial representation which would be something that is completely isotropic.

So, it shows no; it is the trivial presentation that l equal to 0 is the trivial representation, trivial irrep as we know trivial irreps are important in counting everything. So, this was the summary about the angular momentum algebra we write and of course, we went through topology of SU 2, SO 3 and all that right. The last comment of course, is that l integer so, this is where I made a mistake.

So, for each positive integer l equal to 0 or each positive half integer half 3 half etcetera. So, the of course, when half 3 half etcetera are included; so, in physics we get the so, called orbital angular momentum, as integer Eigen values l h cross. While spin has the half integer Eigen value, spin can have either integer or half integer and so, because this is related to SO 3.

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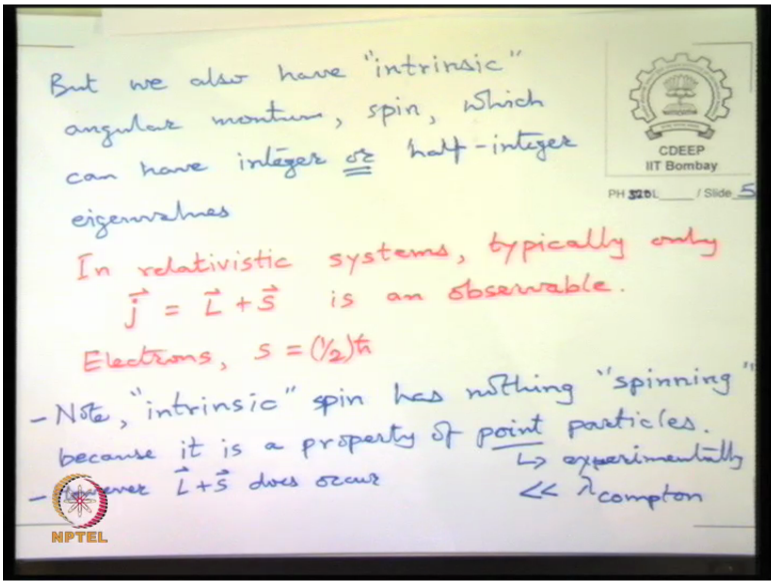

Whereas, we also discover spin, intrinsic instead of orbital, integer or half integer Eigen values, but one of the two in any case because, anyway or it can have the given a particular system, it has either integer or half integer Eigen values. It is also find that in relativistic systems, you cannot actually distinguish between so, called intrinsic and orbital and only the combinations j appears ok.

So this can be either integer or half integers depending on whether S is integer or half integer. In the case of electron therefore, it is always half integer because L is integer and intrinsic spin is half S equal to half h cross. So, now what we want to do next is to go to Lorentz boosts, or the special relativistic transformations yes.

I just wanted to make a comment here, because spin is a very puzzling property in relativistic field theory and in the very high energy experiments, we never find any structure to the electron. So, there is nothing rotating there, there is nothing no dimension to the object. So, what spin means it is somewhat mysterious. So, note that intrinsic spin has nothing really spinning there, because we have a point particle, no structure has been discovered to the electron, you know electron rest masses half an MBB.

So, the Compton wavelength is inverse of h over m C. So, it is determined by that half MBB energy scale, but we have now explored the electron to 100 g g e b. So, that is like 10 raised to 5 slight million times the mass of the electron. And it has been probe to such scales much smaller than it is Compton wavelength and, it does not reveal any structure to it. So, electron has never revealed any structure experimentally, much smaller than lambda compton. However, this mysterious intrinsic angular momentum does add to the orbital angular momentum in the case of hydrogen.

So, the spectrum of hydrogen gets classified by j and not by L ok. So, that is the big mystery. Sometimes people say that if you so, the in history of quantum mechanics Dirac came up, with this slightly strange looking equation called Dirac equation, which I hope to derive if we can because it is only group theory actually there is not much physics real, as far as the free Dirac equation is concerned. It is the lowest dimensional representation of SU 2. And lot of people think that if you do special relativity, then because Dirac equation was intrinsically relativistic and head spin bundled in it. The first spin is an intrinsic relativistic entity which is of course true it is intrinsically relativistic because everything in the world is intrinsically relativistic.

But that it necessarily follows from special relativity is not true because, you can have special relativistic spin 0 particles as well. So, special relativity does not imply half integer spin ok.

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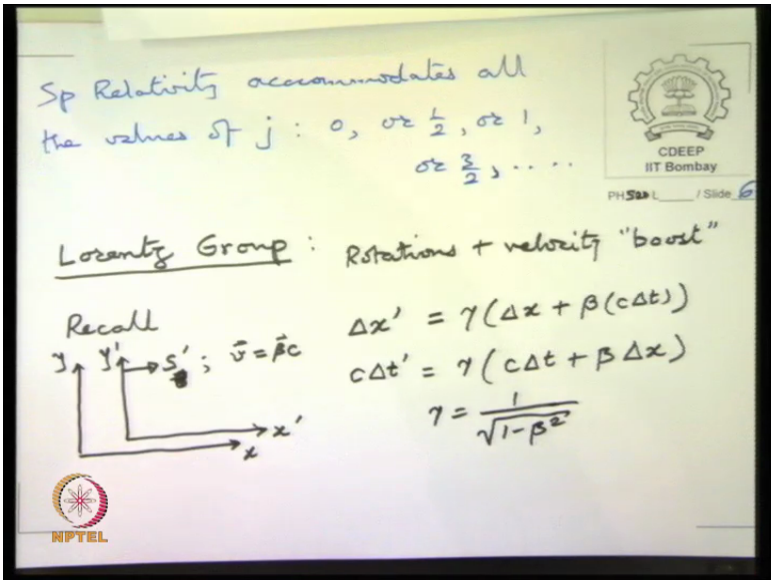

So accommodates all values of spin this much statement I can make of j, 0 or half or 3 or 1 or 3 half etcetera. So, this 3 half and higher things usually occurring nuclear physics, or nuclei and nuclei have some total spin which can be large also, it can be 13 by 2 it can be I do not know how large it gets, but it can be large, but half integers, but those are composite objects you know they are composite. And so, there you can actually try to think that something is spinning, but as you know in the in the intrinsic objects do not really code spins. So, we do not know what it is. We had if you remember talk also couple of weeks ago on where is the spin of the proton coming from and so on.

So, the mystery sort of remain but it is not true that relativity implies that there are spin of particles relativity allows all of them and does not select between them. So, now, we want to go to the relativistic symmetry algebra and this is called the Lorentz group, Lorentz group include both space like rotations as well as boost velocity boost, I do not know who invented the term boost, but it is very popular.

It is it is sort of to remember that you are not actually accelerating the frame to the other velocity, you are only transforming to other value of the velocity. So, it is just instantaneously it is there is nothing happening in time, it is just change of frames to another frame which has a different velocity. Let us try to recall special relativity, I hope that you have done this and I have already sick of drawing this diagram, but here it is.

So, we draw this y although it is not really needed in this picture, x and y, x prime y prime and this is a frame S prime moving with respect to S at some velocity v. So S prime velocity v. In this case we say that the transformations are and note that I am going to write differences of coordinates. It sometimes confusing if you write x and t, it is more correct to write delta x prime and c delta t prime. So, the transformations are on differences of coordinates origin should not actually matter.

Now, we know that the way this transform is gamma times delta x and here I have to make a choice. So, we will say plus v by c. So, we call that parameter beta time c delta t. And c delta t transforms into gamma times c delta t plus beta times delta x, where beta is v by c, gamma is 1 over square root of 1 minus v squared, or c squared 1 minus beta square. So, we have restricted to x t transformations to keep thing simple.

Now, what is nice about this transformation is that did we do this a bit earlier we introduce that cautions inch notation for this? No? Ok. So, now, we can there is look rather mysterious, but actually they are just rotations of they are rotations with a graduate degree ok, slightly more sophisticated looking rotations, or slightly more upscale rotations.

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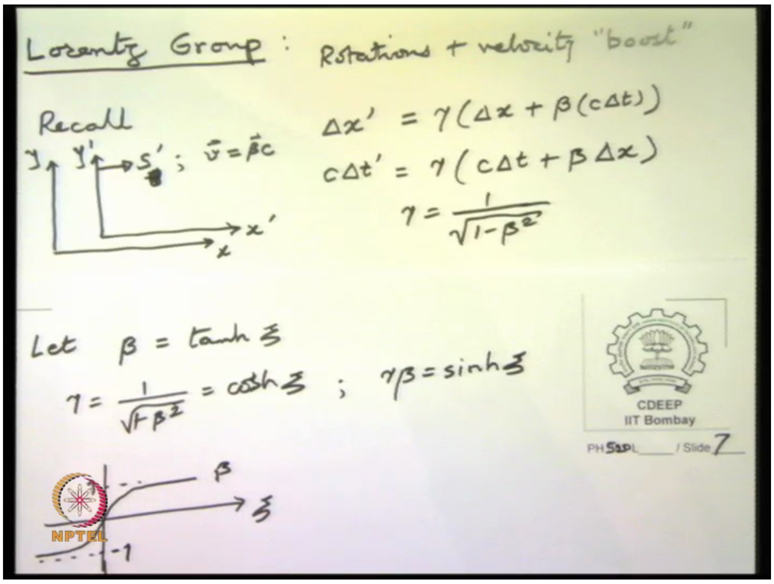

So, we define, so, let this factor beta b equal to tan hyperbolic of some object some symbol xi. So, the point is that we are looking here at 2 quantities gamma and beta typically in the transformation it will be gamma times this gamma beta times this, but gamma is nothing, but again related to beta. So, we want there is only 1 parameter. So, what we are suggesting is that better parameter to use something called xi which is tan inverse of this.

Such that then gamma factor which is equal to 1 over square root of 1 minus beta squared becomes equal to cosh xi because 1 over tan squared. So, you will put 1 minus sinhs this is sinhs squared over cosh squared so, cosh will come in the numerator 1 minus sinhs squared is cosh squared. So, we will get square root and so, this becomes equal to cosh, sinh xi is product of the 2. So, gamma beta is equal to sinh hyperbolic xi.

So, just to draw a little picture what we have said is that I have a parametric xi, as a function of it this beta parameter will be tan hyperbolic. So, it will be saturated between minus 1 and plus 1. It is, the beta will look like that because anyway beta parameter is v by c. So, it cannot exceed plus 1 or minus 1.

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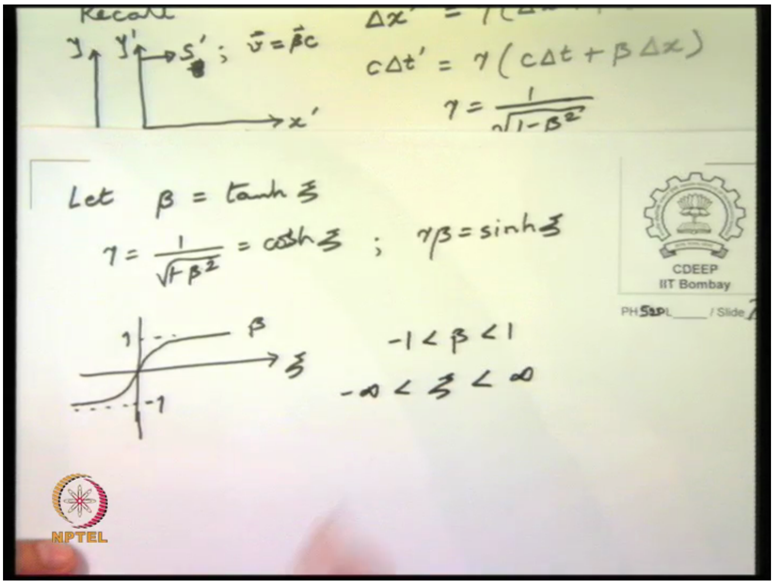

But then xi parameter is free to go between minus infinity to plus infinity. So, minus 1 less than beta less than 1 where as xi refer goes from infinity to minus infinity.

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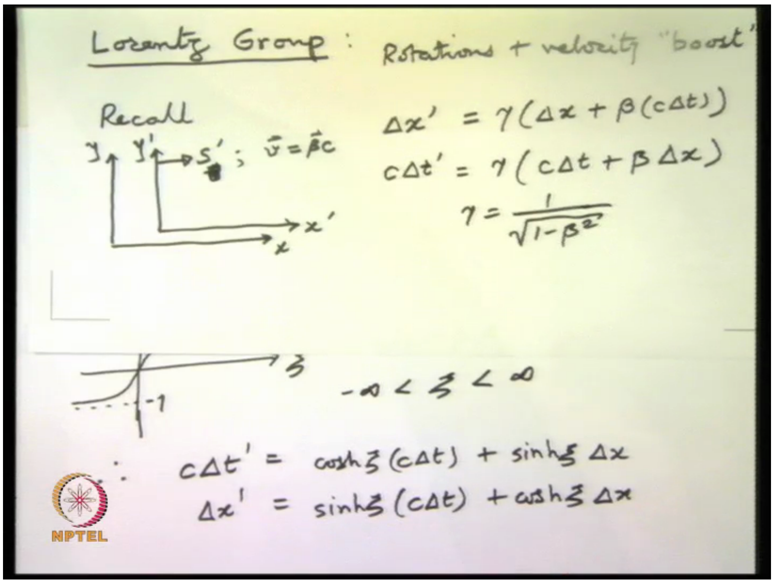

Therefore, if we now come back to these we can recast these as c delta t prime equal to gamma times this becomes cosh psi times c delta t, plus gamma beta is sinh xi times delta x and delta x prime becomes equal to sinh xi times c delta t plus cosh xi times delta x ok. Now, this is just like rotating x y into x prime y prime.

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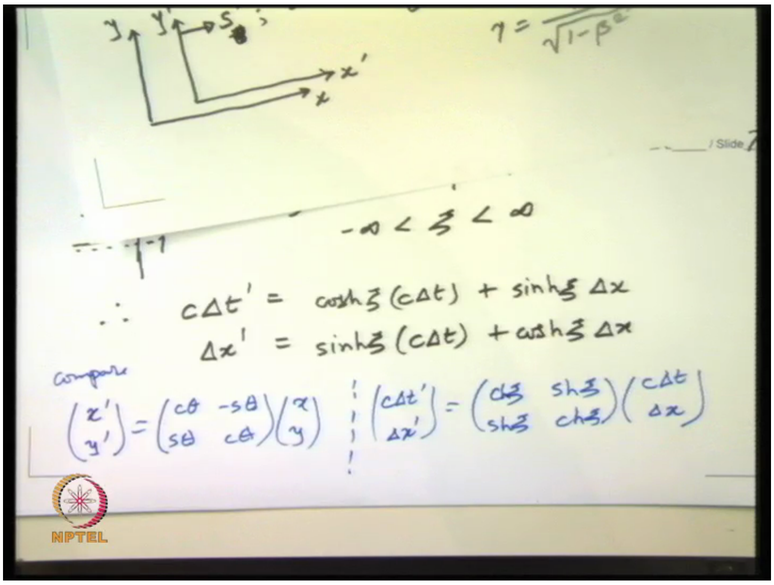

So, we can make a comparison whereas, here we have c delta t prime delta x prime equal to cosh xi sinh xi sinh xi cosh xi times c delta t and delta x. So, it is essentially a hyperbolic rotation of time into space.

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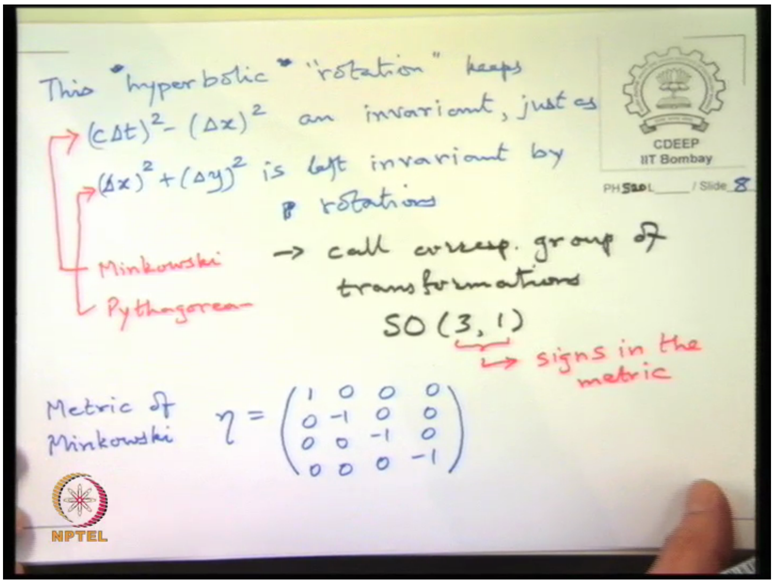

Because after all what you have to do was keeps c delta t primes squared minus delta x prime squared invariant. So, the hyperbolic rotation well the inverted commas are unrotation. So, this hyperbolic rotation keeps c delta t squared minus delta x squared an invariant, just as delta x squared plus delta y squared is left invariant by Pythagorean rotations, we will call then Pythagorean ok. This Pythagorean metric is by rotations.

So, this is Minkowski metric and this is called Pythagorean metric that is the only difference. Again to remind you we had introduced the rotation group as some metric preserving group set of matrices and that metric was the Pythagorean metric. Now, we have this metric. So, we call this group SO 3 1, so we call the corresponding group of transformations, which can leave this invariant SO; this is special orthogonal, but 3 comma 1 because the matrix signs are like that.

So, we introduced eta many people quickly write a big 0, but I had professor who already insisted that 1 has to fill it with 0's. So, this is the Minkowski metric and that is left in variant by this hyperbolic transformations. We have written of course, only the t x, but likewise also t by t, z and t t by t z transformation.

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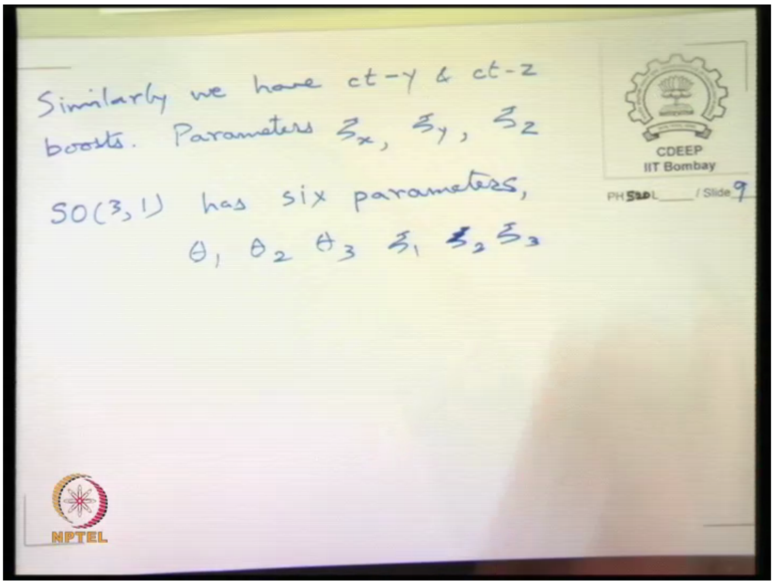

We can put c t y and c t z boosts, parameters would be then xi x xi y xi z independently for each of those boosts. So, SO 3 1 has totally 6 parameters. So, we could have called them xi 1, xi 2, xi 3 also, theta 1, theta 2, theta 3 the 3 rotations and the 3 boosts.