Theory of Group for Physics Applications
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Lecture – 40
Representation on Function Spaces – II

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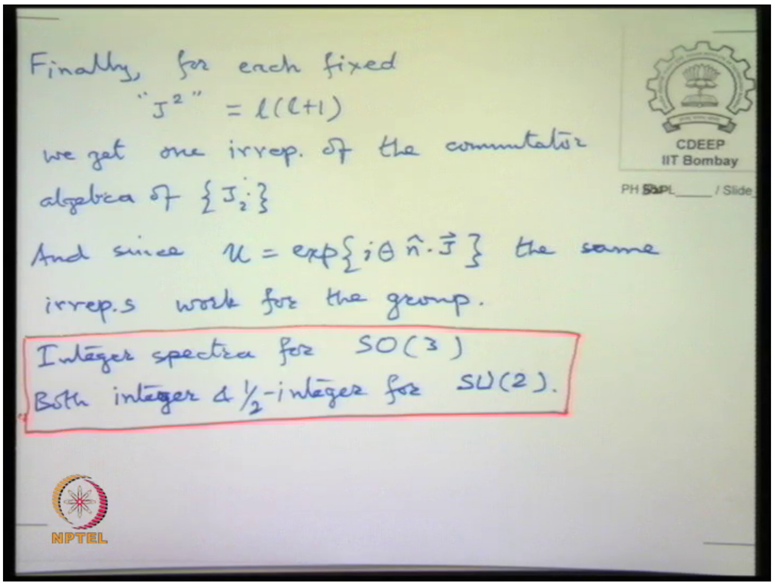

The next thing we wanted to do was the Lorentz group which I will do. So, we have two things to do either do the Lorentz group or do the realization of these groups on functions basis.

So, let us do the functions basis first because that is more that that is not directly connected to this relating to angular momentum and we will find some nice results for Lorentz group but that is later there ok.

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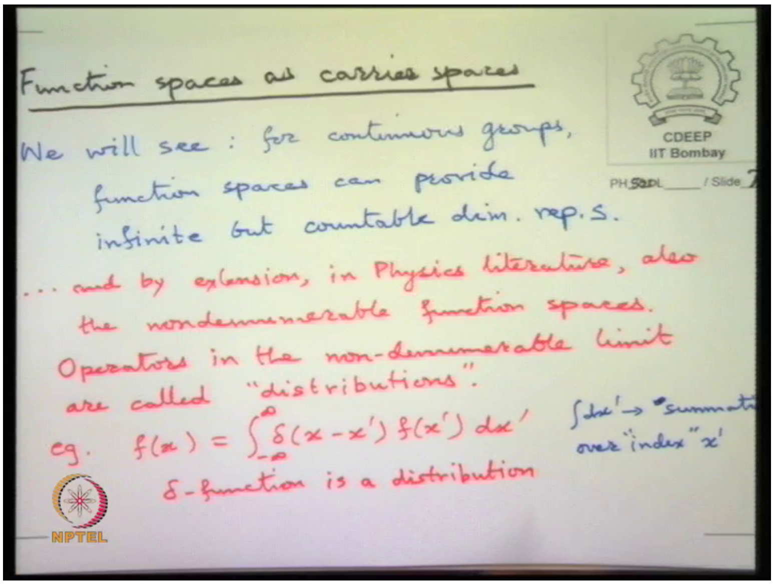

So, next we look at function spaces as carrier spaces. So, that is the big topic that so far we considered these finite dimensional representations or finite groups; but when we got go to continuous groups we go to the infinite dimensional representations, but they are still countable and by extension most physicists also assume that the uncountable ones also are representations. So, we will see it.

In physics literature only non denumerable function spaces. So, our favorite delta function actually resides in this non denumerable limit we will see that and so it is called distribution. So, delta function which we write as dealt of x minus x prime you can think of it as a matrix x 1 index x prime the other index and you can fold in one function f of x, do an integration which is like the index summing over x and obtain another vector f of x prime out of it.

So, and are called distributions. So, delta function is a distribution and this goes from minus infinity to infinity right. So, this is like a summation over x prime, since everything is in red to highlight I have to use blue, actually the summation over index x prime and delta. So, delta function is the distribution. So now, we go back to this we will see business what is meant by function space. So, let us consider functions over a definite interval.

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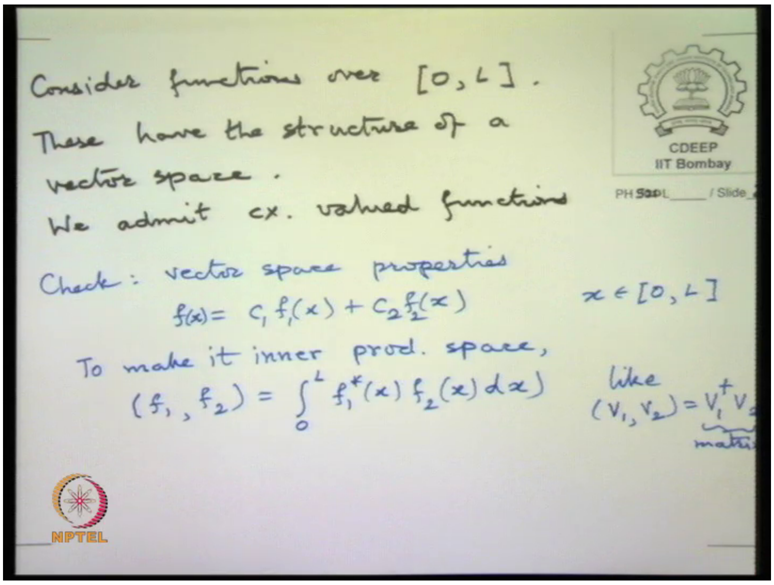

Say 0 to l; these can be thought of as a vector space. These have the structure of a vector space. And we can admit complex value functions or we can do it later but we admit complex value functions as well because we want to write e raise to y ix etcetera. So, we accepted although of domain is real values we take complex valued functions on that.

Now, what we need to check for it to be a vector space is very simple. If I have vector space properties is simply that f equal to c 1 f of x equal to c 1 f 1 x plus c 2 f 2 x where x belongs to this closed interval and there is no problem. So, we just symbolically we can immediately check that all the axioms of vector spaces are satisfying.

We can make this vector space also inner product space. We define f 1 comma f 2 the inner product to be integral 0 to l of f 1 star f 2. So, remember we needed this always. So, it is like our statement that V 1 V 2, it was equal to V 1 dagger V 2 in this is in matrix language. If you take V 2 to be a row vector a column vector then V 1 has to be made into row vector, but with Hermitian conjugate with as a Hermitian conjugate, so complexified as well.

So, the first term has to be first coefficient to be to be complexified, otherwise we will not have positive definiteness. So, we need this form. Now, the interesting point is that two things; there is a structure of a duel vector space which is slightly subtle, but the other thing that is familiar to us is are is differentiation. So, differentiation is a linear operator.

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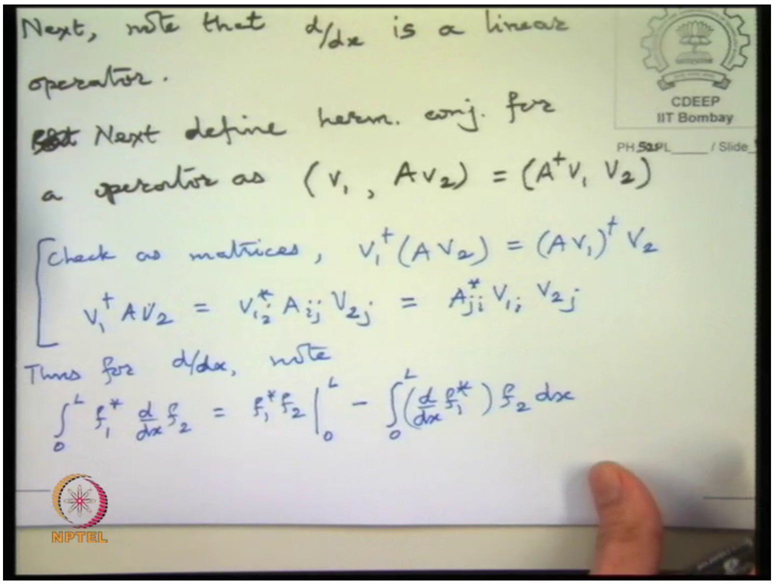

However, d by dx is not Hermitian. So, next define Hermitian conjugate for operators V 1 comma A V 2 equal to A dagger V 1 V 2. So, we can check this simply in matrix language. If I take V 1 dagger and act on A V 2 where V 2 was the column vector, so, A V 2 is also column vector, but this should be equal to A V 1 dagger acting on V 2 because V 1 dagger A V 2 is something like V 1 i star A i j V 2 V 2 j and that would be equal to A j i star V 1 i V 2 j. The definition of the Hermitian is that it is A j i star.

So, this is what we call the Hermitian conjugate of sorry so, I do not have to put any equality. We want to see what acts as the Hermitian conjugate of A yeah so, I think this is correct. So, in any case that is the definition of the Hermitian operator. So, for d by dx note that integral 0 to l f 1 star d by dx f 2 we can use integration by parts and say this is f 1 star f 2 0 to l and minus integral d by dx of f 1 star f 2.

Now, to make this into a vector space we have to put some a periodic boundary conditions or some reasonable conditions. So, we can said if we set periodic boundary conditions on f’s then this will be 0, then the then A will be its own Hermitian conjugate right. So, now, I realized what I did in a hurry. So firstly, we define Hermitian conjugate and then next we say an operator is Hermitian, if it is equal to it is own Hermitian conjugate.

So, this equality actually already insisted that it is self Hermitian self adjoint but the definition of adjoint is whatever that produces, so, this is Hermitian conjugate is yeah that is right. So, this was correct and what we say is if this is same as A, then it is self adjoint; for x to be self adjoint now, we should somehow have this also give the same answer as this and this happens if we choose the if you put i d by dx instead of d by dx then when we then we can write this as equal to star of id by dx and still retain this sign on f 1.

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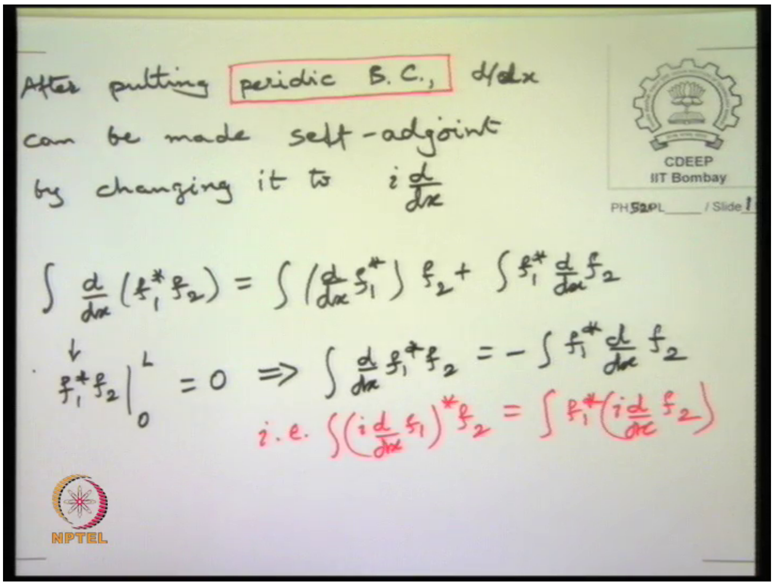

So, we can start it also by; so, this because it is a total derivative it is just equal to the n point values and this we set equal to 0 would implied that integral d by dx f 1 star f 2 is equal to minus integral f 1 star d by dx of f 2 or so the operator id by dx x acts as a Hermitian operator, provided we put this boundary we chose we restrict ourselves to functions that have periodic boundary conditions; 0 and l they have same value.

So, that is an important condition. Now, the point is that this has a also a group theory meaning; the operation d by dx has a group theory meaning and that is why it is an important operator.

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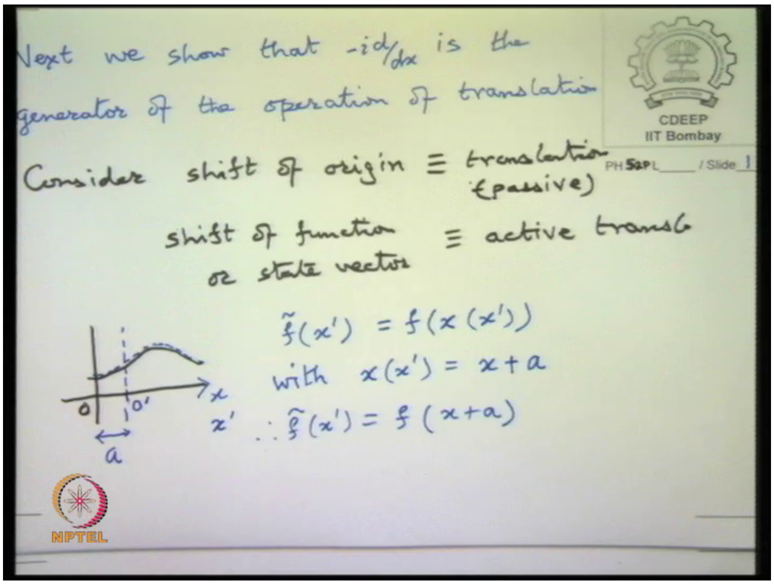

Next we show that id by dx is so, I should put minus i because in physics momentum is put I mean Schrodinger chose a sign once. So, we have to put this minus here. So, we show that and it is a matter of convention but that is the physics convention. Next we show that minus id by dx is the generator of the operation of translation. What does this mean?

So, far we consider the rotation group which is actually the more complicated things. The simpler actions in physics are just shifting the origin which is called translation in technical language and just as we had the infinitesimal generators of rotation group. We might then ask what is the infinitesimal generation of translation group and we will find that id by dx works as the generator. So, to see this first consider active translations in this is where I get lost sometimes but the signs plus and minus signs consider.

So, let us draw some pictures. So, consider translations shift of origin that is the meaning of translation and again as we did for the rotations either you have active rotations where you rotate your vector or you shift the origin which is the passive of the action ok. So, here, so if you shift origin this is translation but passive and shift of the function or shift of the state, the quantum mechanical state vector is active translation.

So, suppose we have a function f and what which sign do I want in the end? So, suppose which this was the 0 and we shift it to a new 0; 0 prime and this is x axis.

So, we say that f of; so, I have some new function f tilde of x prime system where f tilde is I can draw it in blue it is the very same curve but thought of in terms of the x prime system that starts with o prime system. Then f at some value x prime is same sorry f tilde function at a value x prime is same as the function f at the corresponding value x corresponding to x prime ok, this is where it is slightly slippery and but it will the only damage it will do is change the sign overall sign but let me repeat.

So, we want to say I have some functional form f tilde which is to be read in terms of the blue coordinate system. This functional form in terms its own arguments, its numerical value is same as what the other function has as a function of its argument x which should correspond to the point x prime ok. So, the way it is written the domains of both are same because I have the functional form here is f it is argument is x but x is in the range of the transformation x to x prime. So, we have made a transformation x to x prime and this is what it is.

And the x prime coordinates are such that when I have. So, x minus a where this is a. So, when x prime is zero I should get x equal to plus a yeah so, plus a correct we can see it here; the x x to x prime transformation is such that when x prime is value is 0 the value of the x coordinate is a. So, this is the transformation. So, therefore, f tilde of x prime is equal to f of x plus a.

So, this is the operation; this operation could be a symmetry operation but right now we are not saying it is a symmetry operation for some physical system, but nevertheless we can see that set of all such operations forms a group shift by a 1, a 2, a 3 any real number right. So, if we let the domain be entire real line is we have now switch back from finite intervals to entire real line for the time being.

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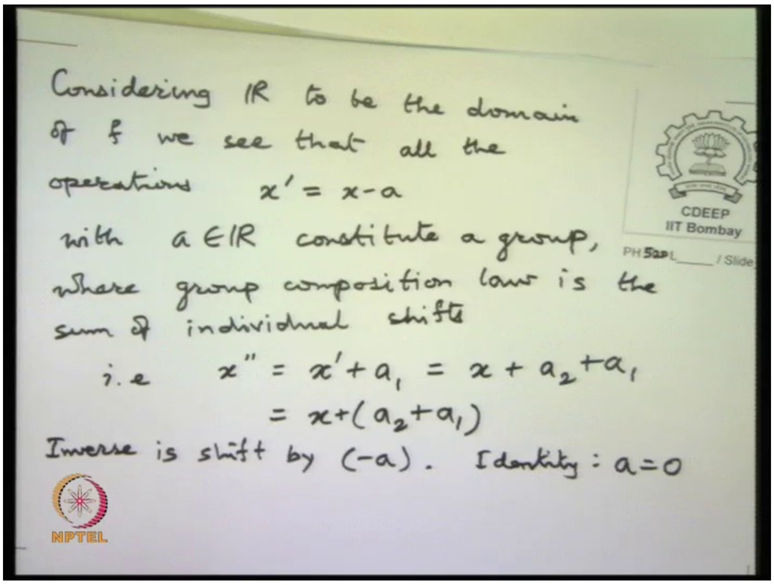

So, all the operations x x prime equal to x minus a right because if x is equal to so, right we; so, we did check that x at x prime equal to 0 is equal to the so, there is a prime here this is x prime right correct that is the confusion. So, this is x prime plus a and this is also x prime plus a yeah.

So, all the operations x prime equal to x minus a, so because x is equal to x prime plus a x prime is equal to x minus a; all the operations x prime equal to x minus a with a again belonging to R constitute a group where composition is simply the sum of individual shifts i.e. a 1 followed by 2 is what should we call this things x double prime equal to x prime plus a 1 equal to x plus a 2 plus 1 is equal to x plus a 2 plus a 1. So, if we do two successive operations then the equivalent operation is just a 1 plus a 2. So, this is just ordinaries addition on the real numbers and subtraction. So, the inverse is shift by minus a and identity is no shift.

So, this forms a group and we want representations of this group on the space of functions. So, now, we go back to our statement.

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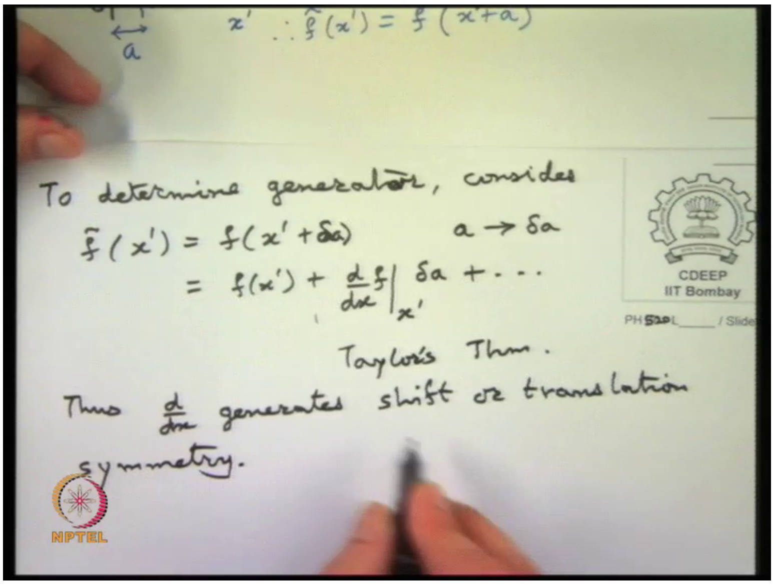

So, to obtain the infinitesimal version; so to determine generators consider f tilde of x prime equal to f of x prime plus a but put a delta a symbolically. So, that is equal to f of x prime plus df by dx at x prime times delta a plus, so using Taylors theorem. So, it is clear that d by dx is the generator of this shift operators; shift or translation symmetry.

Shift is sometimes reserved for strictly those things that are discrete transformations but we are going to consider all we have continuum. So, d by dx is clearly the generator and for convenience in quantum mechanics to make it Hermitian with respect to that particular inner product we chose minus id by dx.

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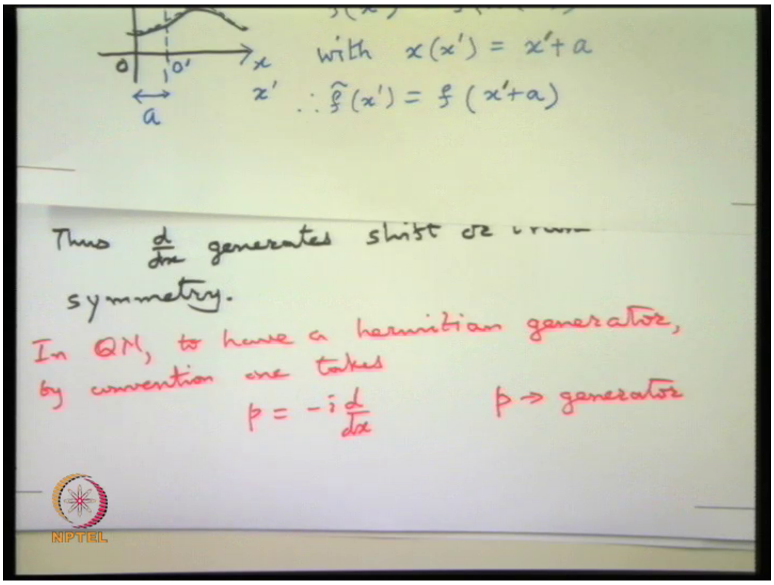

With p the generator and we can see that any finite shift can be obtained as exponentiation because the exponentiation is nothing but the full Taylor series along with its 1 over n factorials; exponentiation has the 1 over n factorials.

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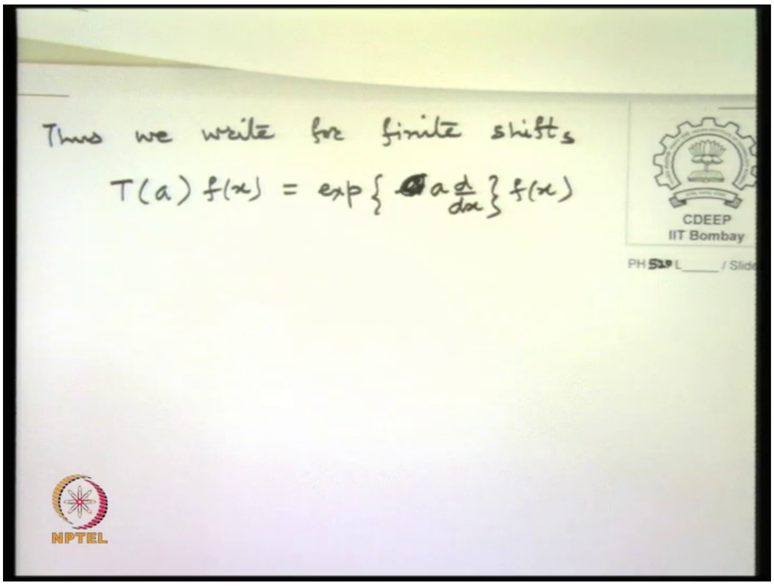

So, we write but of course, eventually want the it to produce the real values I mean the for the classical wave function classical functions the usual functions on the function space while we were using Taylor theorem, we do want d by dx back. So, we can just write over here for the time being we will just write for the finite shifts we can write like this exponentiation of a d by dx but in quantum mechanics or complex wave functions we use this as the operate generator and so, in complex notation ok.

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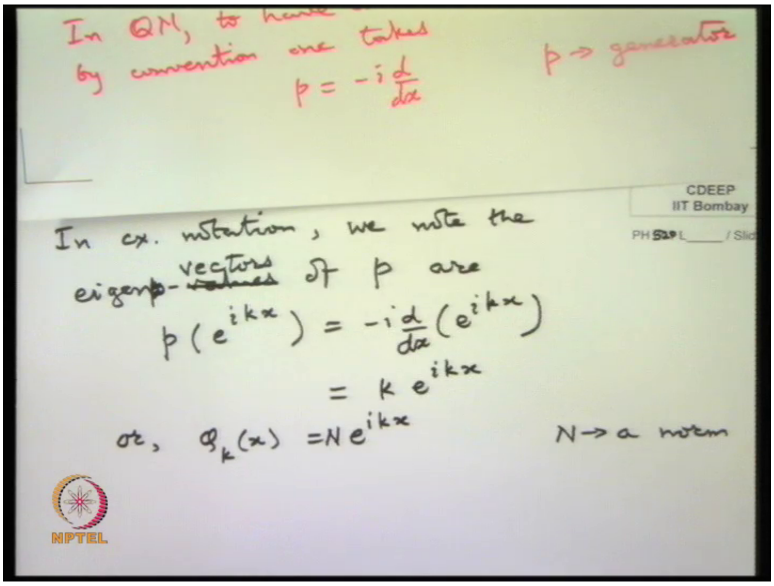

So, in complex notation, we note the eigen-values of p are so, eigenvectors. Eigenvectors of p are the e to the i k dot x and we have to normalize it to by something. So, put it N for the time being.

So, the structure of the functions on the on the real line can work for these free translations and the corresponding eigenvectors in the sense of functions on the in the function space will be the functions e to the i k dot x in order to have that inner product we introduced f 1 star f 2 and the generalization to 3 dimensions is kind of trivial; well one non trivial thing is there.

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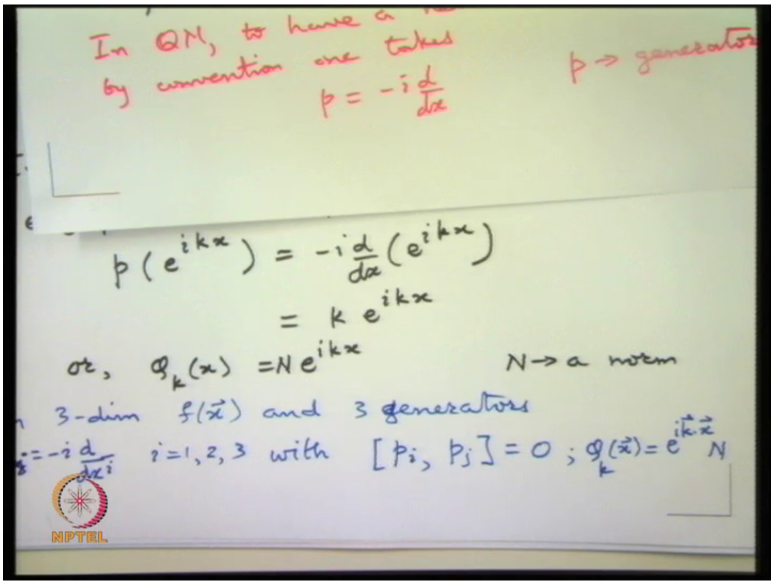

So, in 3-D we have f of x and 3 generators p x equal to minus id by dx or pi equal to minus id by dx i, i equal to 1, 2, 3 with commutation relation that they all commute; the translations in independent directions all commute and everything else remains the same.

This is a square root 2 pi. So, some N times e to the i k dot x and nothing else much changes. So, now, we can quickly link these things to what we just did earlier SO 3 and SU 2. So, the translations was a warm up to see how symmetry operations work on the functions basis.

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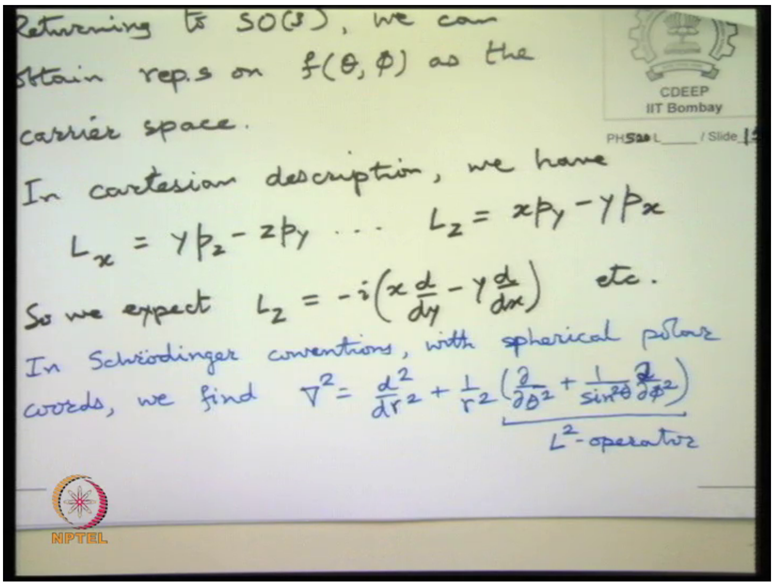

So, returning to SO 3, we can obtain representations on functions of theta phi as the carrier space and as everyone. So, let us start with the Cartesian description and L z equal to x p y minus y p x. So, we expect yeah I did not bring out the importance of the shift operator p as the momentum.

But anyway we are not doing physics right now. So, whatever the operator p is I now claim to you that whatever the translation operator p is to obtain this Taylor series correctly. What we need to do is construct the L's as products of such operations where y x multiplicatively and p is the shift operator for z and so on ok.

So, we expect that L z will be equal to minus i x d by dy minus y d by dx etcetera and we do not want to enter all the detail here. If you go to spherical polar coordinates then you obtain the y Ln is the Eigen functions, I am not writing the operator here. So, we have the Laplacian operator which can be broken up into d d square dr square plus ok.

So, we find 1 over r square times and let me symbol well we know what it is and this acts as J squared or L squared and the y L m which purely from differential equation theory and Sturm Liouville theory, you will get the Legendre polynomials and e to the i m plus minus phi as solutions of this.

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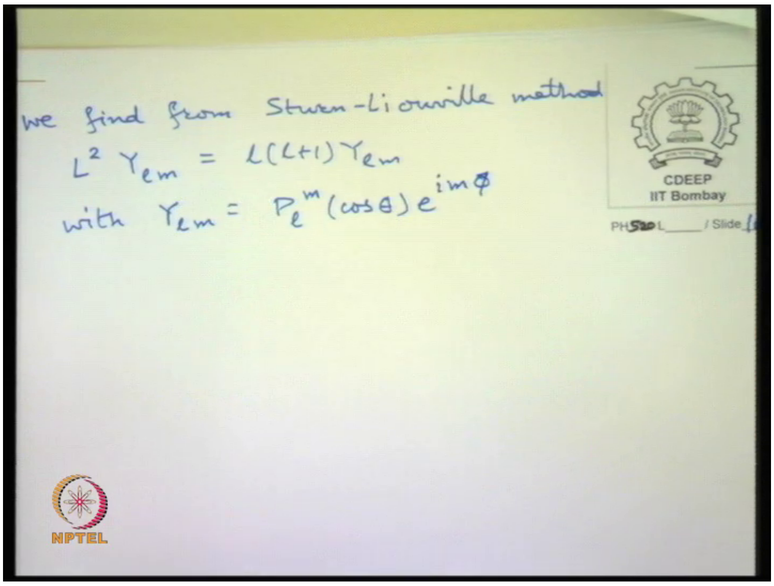

So, this we are not as if proving in any great detail but what we want to bring out is the fact that as operations as operations which constitute a group rotations which constitute a group, we saw firstly, it is abstract. First we derived its matrix algebra and then we check that that matrix algebra gives a particular commutation relations for the; so, let me write it over here this. We connect back to matrix group matrix group SO 3.

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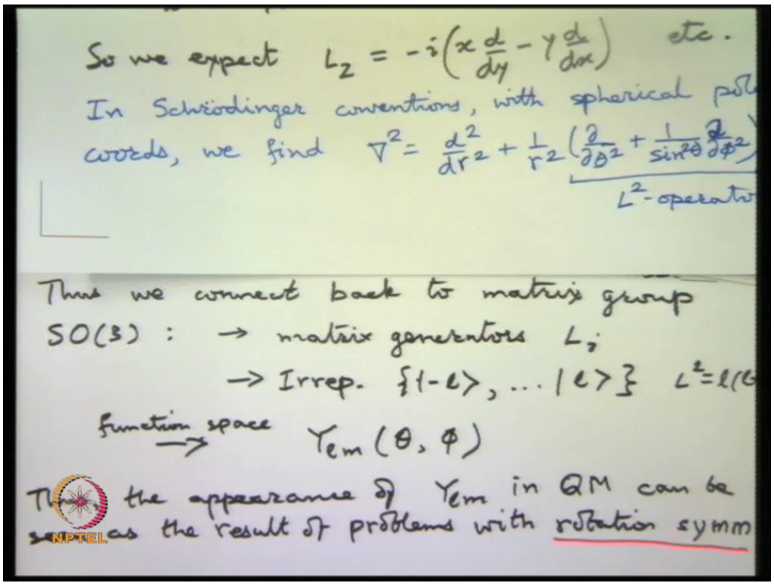

So, the matrix generators L i allowed representations and we saw that they were irreps because you cannot escape the those m representations with L squared equal to l into l plus 1.

And these exactly correspond to the Y l m Of course this is useful when the potential in Schrodinger equation is separable into r theta and phi. So, but most spherically symmetric potentials will give this; So, the Y l m we use are essentially representations of the rotation group and when there is spherical symmetry it is possible to decompose the answer for the wave function in terms of the eigenvalues of the symmetry operations that is what the summary is.

So, that is the connection of group theory to quantum mechanics. These things which were first discovered in the context of just trying to solve Schrodinger equation eventually we understand them in terms of group theory. So, result of rotation symmetry even if it is not symmetric it is at least represented by also applicable to so long as r can be separated from the partial differential equation because then you will be dealing only with this part of the Laplacian and the solutions will all be in terms of the Y l m which being irreducible representations constitute a vector space. They constitute a basis for representing everything that is in theta and that is function of theta and phi.

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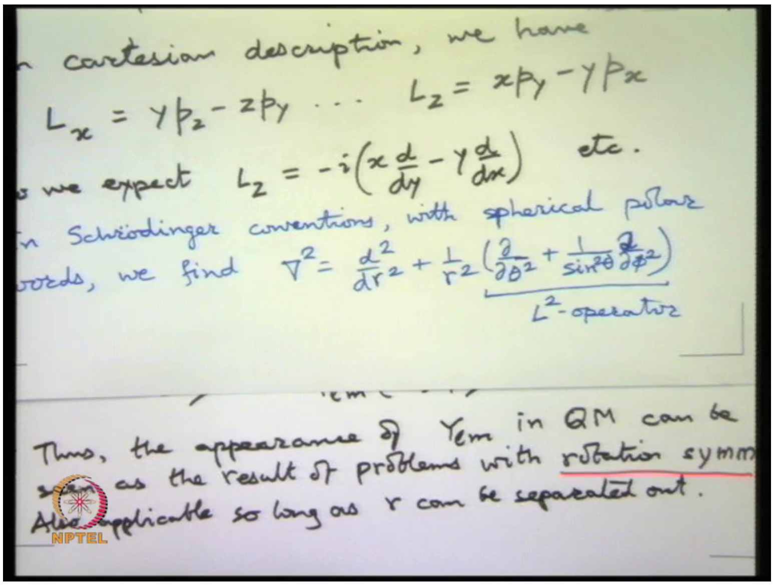

So, that is the use of group theory that is the group view of what we do in quantum mechanics.

So, I think we can stop with this here.