Theory of Group for Physics Applications
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Lecture – 04
Basic Group Concepts & Low Order groups – II

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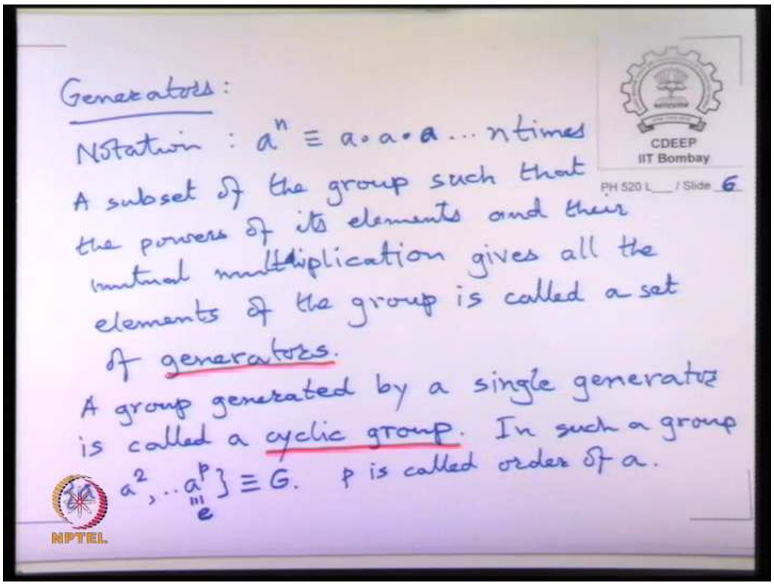

Now, we see one other example of a group that so, just to complete this part.

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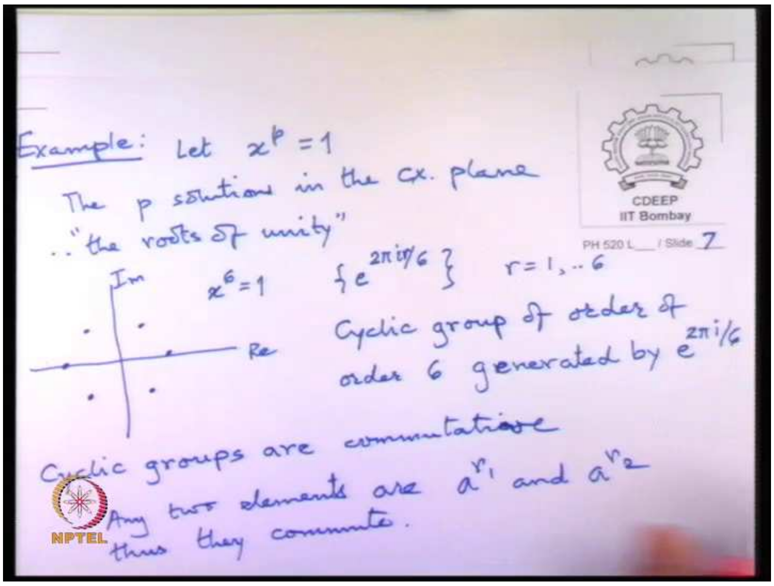

We can see very common example of cyclic groups, p-th group of unity you know the p-th root of unity you know the root of unity. So, you say $x^p = 1$.The p solutions in the complex plane, which are called roots of unity.

These p solutions form a cyclic group. So, we can draw a little picture here yeah. So, draw the complex plane real and imaginary axis and let say 6th order so, hexagon is easy to draw right. So, I find the solutions of the equation $x^6 = 1$, they are the 6 points you know what they are so, the group is{ $e^{2\pi i/6}$ }. This list of elements constitutes this group and clearly, this is a cyclic group $e^{2\pi ir/6}$ with r =1 ,2….,6.

So, this set this list of elements is group it is cyclic group of order 6, order 6 generated by $e^{2\pi i/3}$ clearly, you can have any number of for any p for any integer p any natural number p you have a corresponding cyclic group. Such groups are always commutative because, they are all made up of powers of only one element commutative because, any element any two elements are a^{r_1} and a^{r_2} right, you can always write them like this.

So, clearly it is a commutative group it is. There is another interesting fact which is that, if you have well so, will come to it later. So, you can when p is a prime number you get some other interesting facts, there are no sub cycles and so on ok. So, I think this is good for the time being we are building up the group concepts by example.

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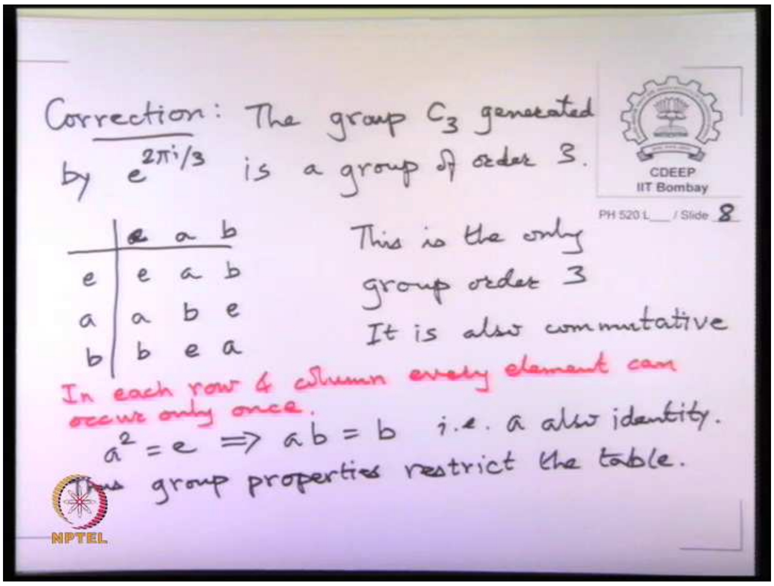

The next group, we want to look at is called the 4 group.The group C_3 generated by $e^{2\pi i/3}$ is a group of order 3 and, this is the only group of order 3 ok, there is no other group and it is Abelian.

So, we what we are claiming is that this is the only group of order 3 and it is also commutative. So, why are we saying this is the only group of order 3, well let us try to write a multiplication table of e a and b; I have 3 elements e a and b. So, what can I do with them well this first row and first column are fixed already, the only thing I can do with a and a is to make it into a b right. If I did a and a also into e I will be first to do a b and b also into e and I will not be able to say a into b equal to b. So, the uniqueness of multiplication forces that the table is precisely of this form.

So, I can only have this particular configuration that I am writing. So, b a = e and $b^2 = a$. The general point to note is that uniqueness of multiplication. So, a b is mapped into another group element, but this group element has to be unique, that ensures that in a horizontal row, no element of the group can be repeated, it can occur only once, you cannot have a times b giving c and a times f also giving c because, then the inverse will not be unique.

So, the row can contain an element only once. Similar the column can also contain an element only once because, a column is nothing, but the reverse I mean you can think is also a multiplication. So, these two facts or this fact that the multiplication gives unique elements and the inverse has to exist that puts a strong restriction on the table you can write, combined with associativity which is additional, but for a small order group associativity does not enter. This itself as fix the group table completely ok, you can think about it. So, if you try to do a a = e. So, suppose I say is a times $a^2 = e$.

Then I have filled in a and I filled in e, I am forced to fill up b over here ok, but; that means, that a b = b, but that would mean that a is identity, but identity is unique. So, you cannot have two identities. So, the uniqueness of multiplication and uniqueness of identity forces that this is the only would mean.

So, thus uniqueness of thus let me just say group properties, you know all the 4 properties enter closure, a multiplication, existence of identity which is unique and existence of the inverse which is unique, those things force the table to be quiet uniquely fixed.

And I will left this space to write, in each row and column an element of the group can occur only once. In each row and column every element can occur only once. This is one of the utilities of the table, you can use the multiplication table to decide what kind of group you can possibly have to that order. And it turns out that to any finite order you will have only finite possibilities that you can have ok. So, the next thing I wanted to do was one more example of a group which is called the 4 group.

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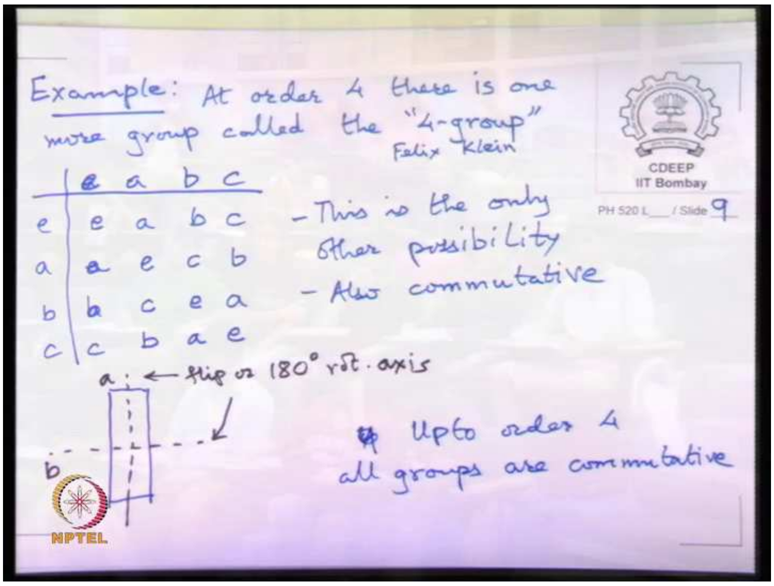

So, at order 4 we do have two possibilities and, we do get a non Abelian group, sorry I think this group is also Abelian. So, at order 4 there is one more group called the 4 group called that because, some great man called it once not some great man Felix Klein.

So, this is in the early days of group theory and so, let us bring up that table for order 4; So, we already draw this table as a possible multiplication table, let us see if there is any other way of drawing multiplication e a b c. So, if we compare with this group, then here we put a a = b. Suppose we try to put a a = c that is another possibility.

But if we try to do this what will eventually happen is that we will basically trade b and c all that, we will do is end up getting another table, where we have traded b and c the other possibility therefore, is that we put e over here a^2 also equal to e, then fixes rest of the multiplication table because a b now must be equal to c it cannot be anything else and then this has to become b ok. So, this table is filled out by making all the elements squared equal to e. And then the rest is filled automatically because, b times a has to be different from b from a and from e. So, it can only be c and b time c has cannot be b cannot be c cannot be e so, it can only be a and we see that it each element occurs once in each row.

So, similarly here all I can have is c a that has to be b and then c b has to be equal to a. So, we get a group like this. And we can check that this are all the possibilities we have at order 4 there is nothing else. So, you can of course, rename b equal to c b and c, but it will not change the multiplication table. So, this by Felix Klein was called the 4 group, I guess in the old days it was a interesting discovery. This group is also commutative, as can be seen from the table you can see right.

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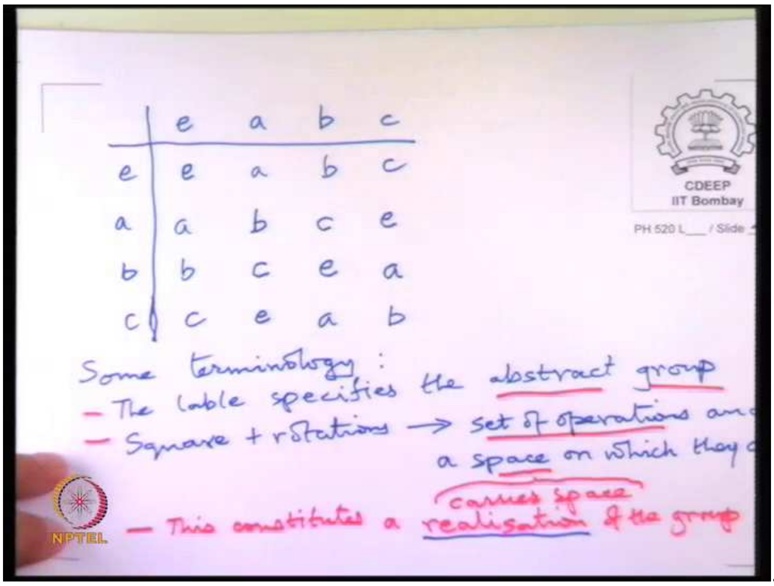

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If you look at that multiplication table, if it is symmetric along the diagonal, then it is clearly a commutative group. So, this is symmetric along the along the main diagonal across the main diagonal flipped across the main diagonal. This group that we are now drawn up we can now take a look at it and, we see that yes it is also symmetric about the diagonal. So, it is also commutative. This is the only other possibility and is also commutative.

Now, we manufactures an abstract group by hand, just by writing up a table by force by the group properties and trying to be different from this other group the cyclic group C_4, but now we may ask the reverse question, where can we realize such a group. What is the what is a physical, or geometric setting where it is realized? And it turns out that it corresponds to flips off a rectangular figure the hint is the fact that square of every element has to be identity. So, if I take if I take a rectangle not a square and, then allow for flips about these two axis.

So, if I flip say 180 degree rotation, then I get back the same rectangle, if I do it twice I get back to identity. So, this flips squared is identity like we see. So, we can call this a and we can call this b or call this yeah. Whatever and the other one we can call b if I flip once I get flip once I get I should get well anyway the flips squared is identity that is what characterizes the diagonal elements here.

If I flip both a and b, then also if I do it I get identity, if I do it twice I get identity. So, the c element c is basically combination of a and b ok. So, this basically is one realization of the Klein 4 group. So, that more or less completes our examples of the lowest order and, we see that the lowest order examples are all commutative up to order 4 all groups are commutative.

So, what we will next so, is start with what are called permutation groups. So, there are two settings in which the finite groups arise, one or all geometric you can have a rhombus you can have various kind of you can have lattices in condensed matter physics are you can molecules. So, there are geometric shapes and they have their symmetries, the other possibilities more abstract which is just permutations, you permute objects among themselves and the permutations form a group. So, we will take that up next and we will see why it is actually fruitful to study permutation groups.

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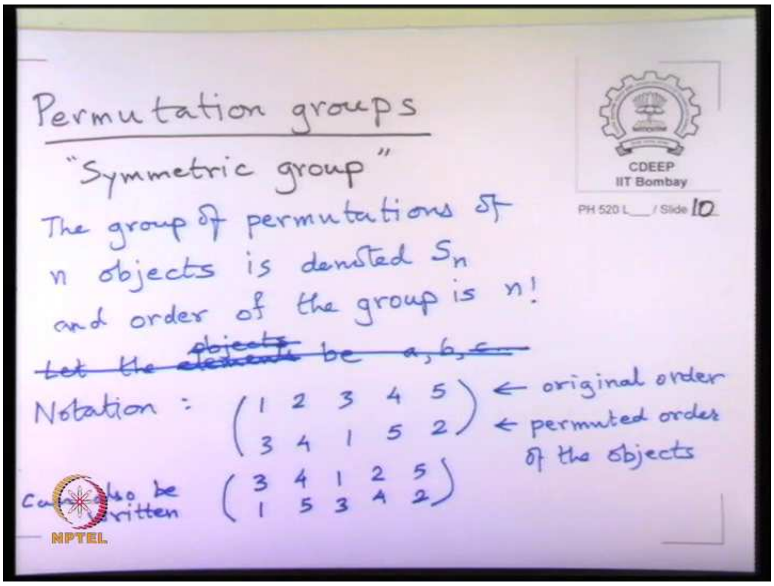

We will next do permutation groups, sometimes these are also called symmetric groups, we call it S_n. So, a symmetric group the group of permutations of n objects is called S_n and will have order n!. The group of permutations of n objects is denoted S_n symmetric group S_n, but this is only a notation because the number of elements in it is n!. So, the notation we have instead of a, b, c,…. we will just write 1, 2, 3, 4,…. that is the difference, then I tell you the reason because, I want to reserve the small letters a b c for group elements.

So, we say we write something like (1 2 3 4) and we are not gone to 5 so, far. And then some configuration that it goes into. So, this permutation this is what this represents is that my original order, after the permutation goes to this of the objects ok.

So, this is the notation we will use as a result well so, one thing to keep in mind is that there is nothing very sacrosanct about the top row be in (1 2 3 4 5), I can now actually exchange the columns, I can also write it as trial. We can just permute the columns I am sorry to use the word permutation too much, but for example, we could do this.

It means the same thing it says order number object number 3 went to look place 1, object number 4 went to the place of object number 5 and so, on ok. This we may have to do occasionally we will not in general do it gets confusing, we will start with this and say what it went to, but these two are equivalent.

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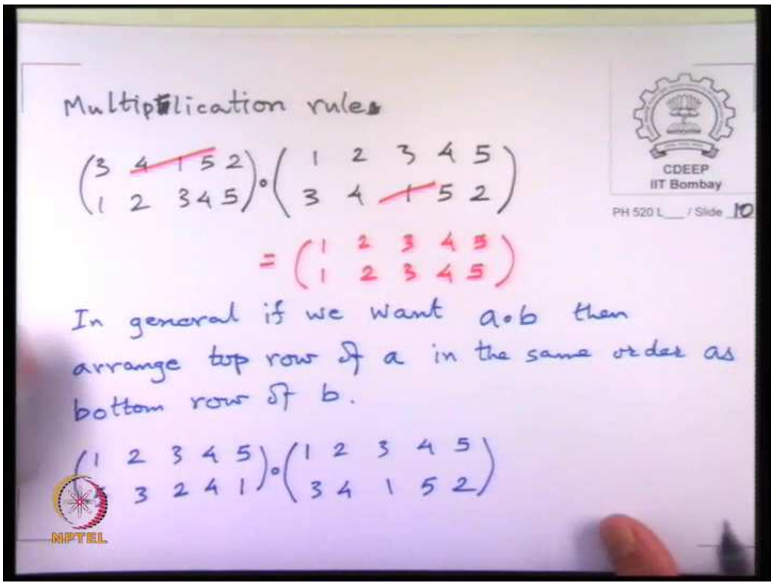

The advantage of that is that you can actually make a multiplication rule, if I am given some particular thing $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{bmatrix}$, I can write another element and just to keep things simple let me say I wrote this. Now, according to the comment we made here because that this. So, long as it gives correctly 3 → 1, 3 → 1, then it does not matter in which order I specify the permutation indication, then this is clearly also a particular permutation of 5 objects write it, just says 3 →1, 4 → 2 etc.

But now I can use this as a way to multiply, in which I simply cancel this and this ok. So, if this row is the lower row of this element is same as the top row of this element, I can cancel the 2. And get equal to what identity then, I just compactify and write this down. So, to multiply two elements what you do is and you know the order in which to do it.

So, I have an element a and I have any other so, in general. If we want a b, then arrange top row of a in the same order as bottom row of b. And here the order is important b is what is happening first you know we will because, multiplication is not commutative we have to remember that writing a • b means that b will act first on whatever it acts on, it is an operation.

So, b will act first and a will act later, that is the meaning of this multiplication. And so, we bring the top row of a in the same format as the bottom row of b. And let the bottom row of a follow whatever fate it has because, you have to move them as blocks vertical blocks, then you can cancel of the this row and this top and this bottom rows. And compact if I and whatever you get is the answer ok. So, we can do another example.

So, let us see so to do that we have to check the thing. Let us do it in a bit of a detail. So, suppose I say that I will take that already as it is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}$$

Suppose I want to do this, then what I will do is first bring this top row in the same format is this ok.

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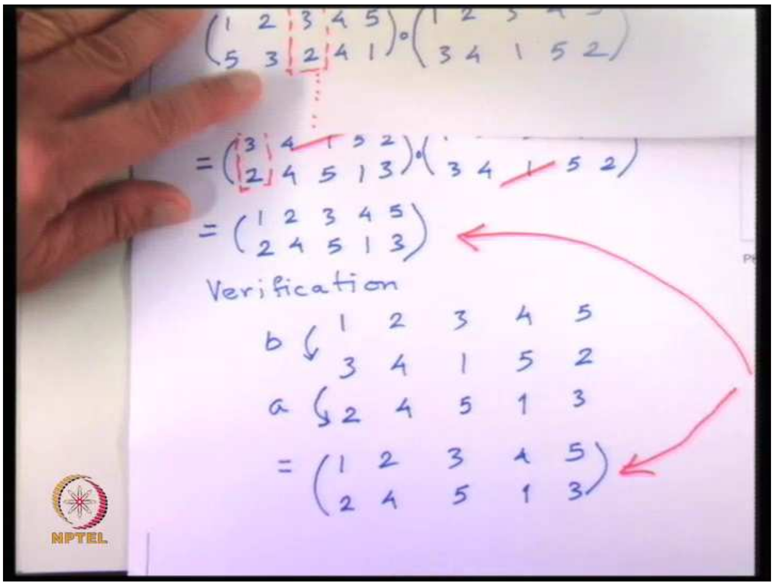

So, we write this as equal to leave this as it is. Now, I make this also (3 4 1 5 2) and, then carefully look here where there is 3 there has to be 2 below, where there is 4 there is, 4 where there is 1 I have to have 5, where there is 5 I have to have 1 and where I have 2 I have to have 3. Now, that I did this I am free to cancel this and this and write it as

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}$$

Now, how do we verify that this is correct? So, one way to think is that you could have drawn it in a drawn it like this. Instead of committing yourself to any notation just go back to basic enumeration a representation. So, all I do is write (1 2 3 4 5) and then I look at what my element a was well it sent 1 2 3. So, that I will write as it is (3 4 1 5 2). So, this operation is my entry b. Now, if I want to do a now if we want to do a well let us look at what is a, I bring back here. What is a? a says that 1 $\rightarrow$ 5; that means, that I have to replace this by 5. 2 → 3. So, this is 3. 3 → 2. So, this becomes 2 and 4 remains 4 and 5 → 1.

So, I had done operation b and then on the result I have applied a. So, what did I get I got let us check if it gave. So, it is equivalent to doing it is equivalent to deleting the middle row now, because that was intermediate step and I have to if we want to return to writing in our notation. Then the notation says that I write original configuration first (1 2 3 4 5). And the resulting configuration which is the bottom row ok.

And then we compare with what we got, I mean it is not it is not a major magic trick, but I am just saying that what you if you would do so, this two tally ok. That is so, that is why this method which is rather simple, but you have to all you have to do is enumerate carefully, when you are reordering the columns remember to keep each block together, each vertical block has to be kept together, some we can say something like this (3 2). So, from here to hear going from here to here the (3 2) → (3 2) ok.

So, you have to move block wise each column block wise, then you will be doing what you would have done logically by writing out everything in detail. So, you will reproduce that answer ok.

So, I think we will stop here, we have just introduce permutation groups and we will continue with that next time.