Theory of Group for Physics Applications
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Lecture - 39
Representation on Function Spaces - l

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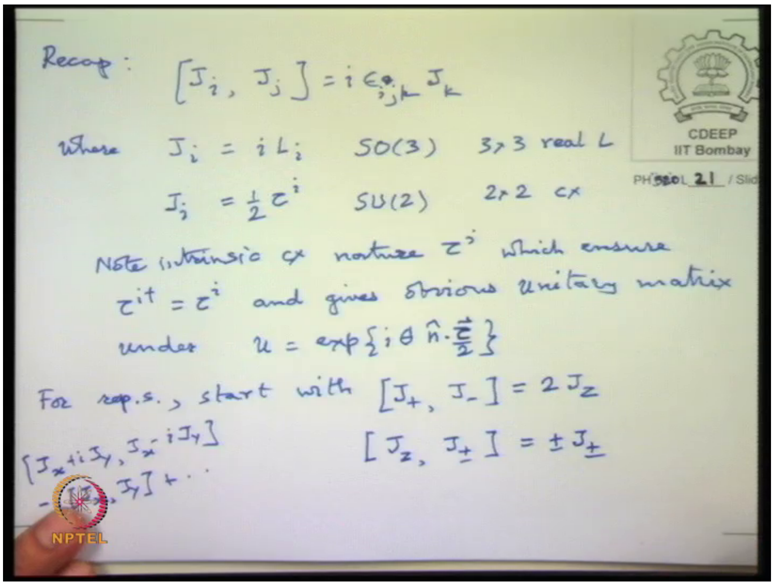

To recapitulate; we showed that there is this algebra of the J i J j equal to i epsilon i epsilon i j k J k; where if J are equal to i times L i then we have SO 3 group and 3 by 3 matrices and if say J i. Now we will pretend as if J i can be either of the two, if J i are equal to one-half times the tau i or Pauli matrices then this serves SU 2 algebra, but these are 2 by 2 complex and these are 3 by 3 real in the form of L, but you can see because there is an overall multiplication of i it is really a real representation; whereas this is intrinsically complex you cannot make all the 3 tau matrices real.

If tau 3 and tau 1 are real 1 minus 1 1 1; then the tau 2 is necessarily 1 minus i. So, that they all remain Hermitian and that is how we get unitary matrices because intrinsic complex nature of tau which ensure that tau i dagger are equal to tau i. If you somehow try to make tau 2 matrix real by pulling out an i from it; you can have all 3 real, but then you will have this i will get messed up and they will not remain all Hermitian, but only if you do the standard choice these are Hermitian and gives obvious unitary matrix; under exponentiation.

So, when you exponentiate a Hermitian matrix with an i in front of it you automatically get a unitary matrix. So, that ensures all that and so that is a good convention to use and this is the standard one. Now next way so we saw that topology that the SU 2 is a double cover and all that; now we will see the representations. So, for representations we started with the J plus J minus construction and who remembers this is equal to plus minus sorry J plus J minus is just 2 times J z I think, but the sign is I hope correct, we can do a little rough work on this side. So, J x plus i J y comma J x minus i J y right; we will have J x minus i times J x J y and plus the other one and that is J x J y is equal to i J z. So, with this it gives plus J z. So, this is just equal to 2 times J z and we had that J plus mine sorry J z acting on J plus minus. I am calling acting on because actually if you write it in this order J z J plus minus then it is easy to see what happens to the state; I will show it again.

So, this I think we found to be equal to plus minus J plus minus and as a result we find that; thus if J z acting on m is equal to m times J z.

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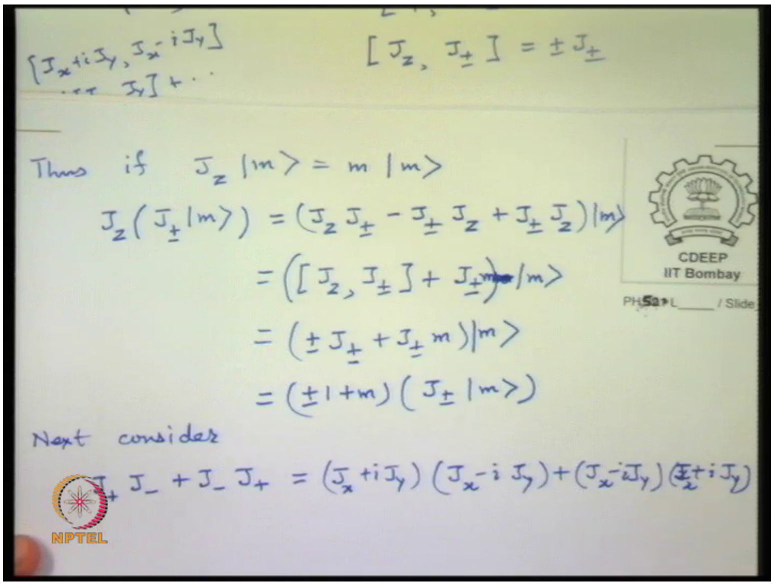

So, we choose the basis to be Eigen values of J z then what happens with application of J plus or minus on m. So, suppose m is an Eigen state of this J z. So, if we apply J plus minus to one of these Eigen values of J 3 and then check again what; it has done to the Eigen value. So we query what is the effect of J z if I take an old Eigen value, but hitted with J plus minus; then we can rewrite this is equal to J z J plus minus J plus minus J z and plus J plus minus J z acting on m, but that becomes equal to; so this is just adding and subtracting and that becomes equal to the commutator J z J plus minus and plus J plus minus acting on J z m, but that is just m times m so sorry; so I should write like this first and then that becomes because J z commutator is this plus minus J z that becomes equal to plus minus J plus minus and m I have to put here thank you and plus J plus minus times m on m. So, i get back m plus minus 1; 1 plus m J plus minus m. So, the Eigen value m is augmented or reduced by one by application of this J plus minus.

Now to prove the next things we need to use some tricks and the trick is that if you take; consider the product J plus J minus plus J minus J plus ok. Now, this is equal to J x plus i J y J x minus i J y plus J x minus i J y J x plus i J y.

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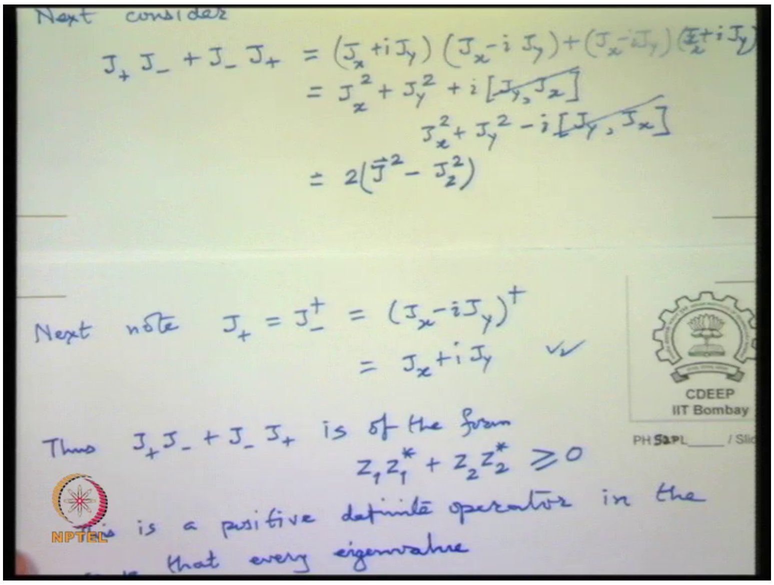

So, it becomes equal to J x squared then here it becomes equal to plus J y squared and then we get a cross term which is equal to i times J y J x minus J x J y. So, we get the J y J x commutator and then from the second term we get again J x squared plus J y squared, but this time we get minus i times the J y J x commutator; sorry there was an sorry J x plus i J y right.

So, yes so we got J x squared plus J y squared which is equal to so these terms cancel and we got J x squared plus J y squared twice. So, we can write it as equal to 2 times the whole of J squared, but minus J z square; because J x square plus J y square plus J z square and minus J z square. So, the this 2 times J x square plus J y square is just 2 times this, but now we know that J square commutes with all the J x, J y and J z. So, J square is a number it is a time something times identity and J z in our basis the J z is also a number.

So, we get these numbers for J squared and J z. So, next note that J plus is equal to J minus dagger because this is J x minus i J y J minus is J x minus i J y; if I dagger it I get J x dagger, but which is J x and then plus i times J y dagger, but which is J y ok. So, J y J plus is just J minus dagger therefore, this left hand side J plus J minus J minus J plus is a positive definite operator you are taking operator time it is own Hermitian conjugate operator time it is own Hermitian conjugate; is of the form z 1 z 1 star plus z I have called plus z 2 z 2 star i.e. greater than or equal to 0.

So, although they are operators; you know they are they are if you represent them as matrices then they will follow this same rule that if you take matrix and it is own dagger then all it is Eigen values will be positive. So, this is positive operator; in the sense that all it is Eigen values are positive your positive definite.

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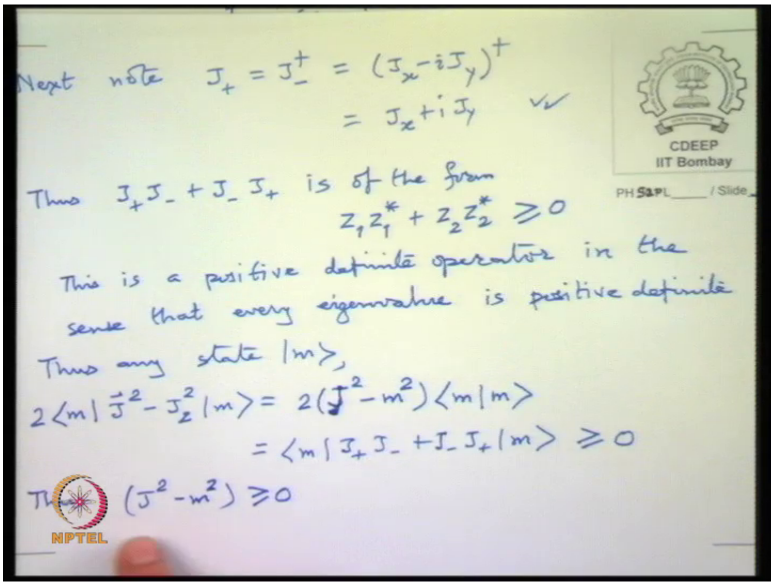

So, this is getting to be a slightly boring proof, but I will just go through it because actually you will probably do it in quantum mechanics class. I was about to skip it last time, but then I taught I may as well finish the argument. So, we are now observing that I have a positive definite operator and this side are things that become numbers thus on any state m; we would find that 2 times m J square minus J z square m which should be equal to 2 times and let us call this for the time being J square i think we will just call it J squared you know where it is some number.

J square minus m square and then there is basically m. So, it is the norm of a state m and this is equal to m J plus J minus plus J minus J plus m, but which much be greater than or equal to 0; because this is just calculating the expectation value of this positive definite operator. So, it means that J square without any arrow I have not put any arrow because that is the now it has become a number; the vectorial operate any way it was vector in the sense of having J x square plus J y square plus J z square all the squared operators become numbers.

So, the whole number we are generically calling J square minus m square is greater than or equal to 0. Now we had seen so this is as sort of long winded proof and you have to spend some time reading it again. Now we are already proved that this m are raised and lowered by plus or minus 1. Our first statement here, but we also learn that the J square which is a fix number it does not change with m because it commutes with all the operators. So, it is it is a something times identity.

So, J square is a fix number for all the states related by J plus J minus. So, m cannot increase indefinitely and this was our proof that you cannot there is an upper bound on how much m can go.

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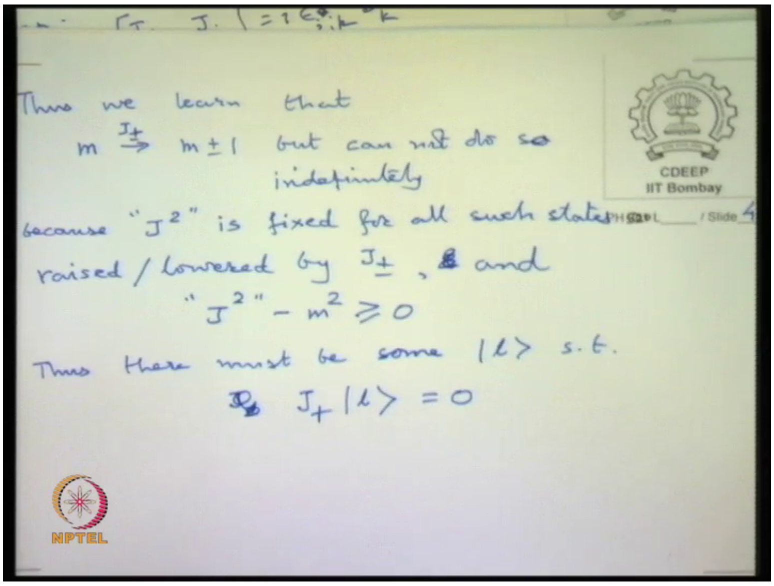

So, our claim now is that m goes to m plus minus 1, but cannot do so indefinitely under J plus minus goes to m plus minus, but cannot do so indefinitely; because there is an upper bound. So, because J square is fixed; for all such states raised and lowered by J plus minus and we learn that J square minus m square is greater than or equal to 0.

So, this is the reasoning for why m as to be kept so thus there must be some l such that J z can sorry J plus very sorry J plus cannot raise it any further. It has to make it 0 otherwise it will keep going indefinitely; so there must exist a highest weight state. Now next we need a sub identity of this; here we directly calculated all this, but there is so yeah actually we can see from here. Now note so actually I do not know how to punctuate it; it is like step 1, step 2 if it was a mathematics book they would have called it lemma 1, lemma 2, lemma 3.

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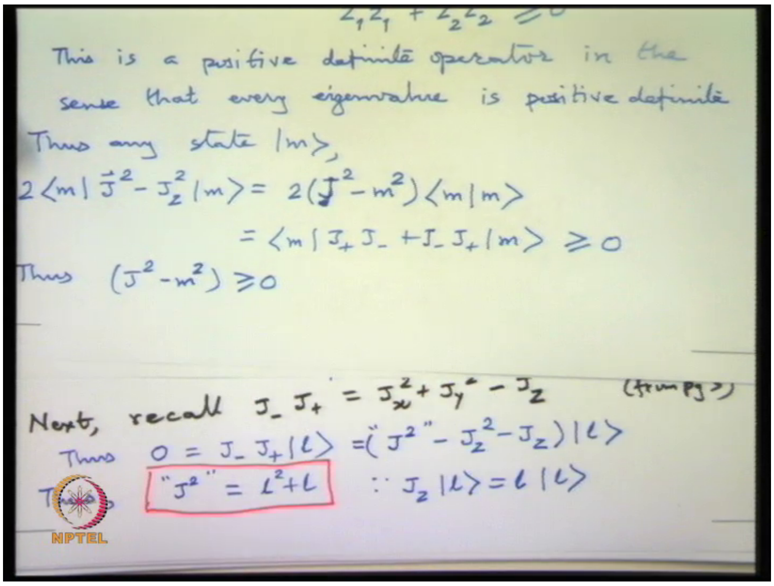

But so next recall what this J plus J minus multiply multiplied out was giving this and we added to J minus J plus which we separately calculated, but suppose we you will drop the second line and second term then recall that actually which one do I need? I need J minus J plus. So, look at actually drop the first term and first line and look at the second one; recall J minus J plus is equal to the second line and then minus i times J y J x. J y J x is minus of J x J y; so that removes this minus sign, but then that becomes equal to i J z. So, i times that i becomes a minus. So, it become minus times J z.

So, the J minus J plus is equal to J x squared plus J y squared; so this is from page 3. Now we go back to this statement and see that, thus 0 equal to you know J plus 1 l is equal to 0, but I can now hit it with a J minus in front and nothing will change right because J plus l was 0. So, I multiply by J minus on the left it is still 0, but then that is equal to this and this is equal to what? My J squared and minus J z square minus J z.

So, if the highest value is equal to l; thus I find that J squared value is equal to l squared plus l because l is the Eigen value of z. So, we may as well retain it on this side acting on l. So, J plus J minus J plus acting on l is same as the combined operator J squared minus the known operator acting on l, but these have a Eigen value l on the state l whatever l is we do not know it right now it is whatever the highest value of m is, but the J squared value gets fixed to be equal to that value. Highest value squared l into l square plus l this is of course, a famous result which everybody knows l into l plus 1.

So, since we already know that this is greater than or equal to 0; we also know that there is a lower bound. So, m cannot decrease indefinitely because this is an m squared. So, m cannot be going negative indefinitely. So, there is some lowest lower bound also.

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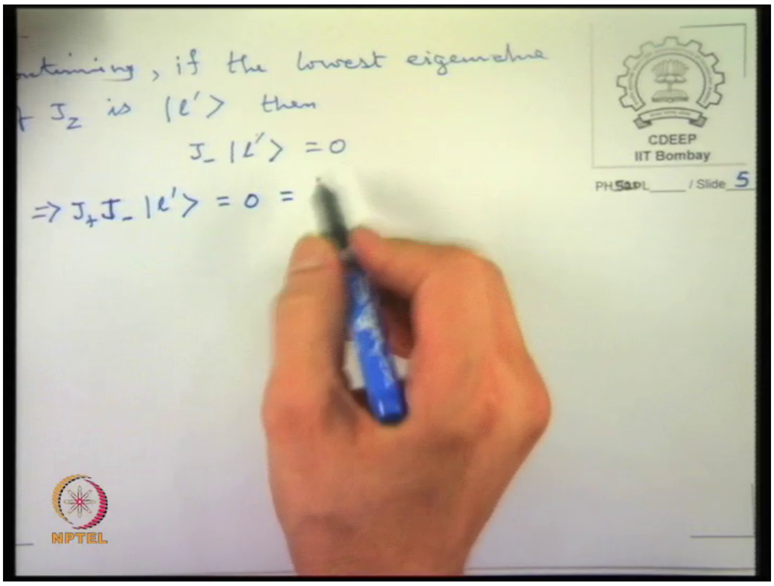

So, continuing; if the lowest Eigen value of J z is l prime something else. Then we know that J minus l prime must be equal to 0 but that implies that J plus J minus J plus J minus on l prime equal to 0 equal to and now we play the same trick instead of writing here on the highest weight we have to apply J plus to get 0 and the lowest weight or lowest Eigen value we have to apply the J minus and then J plus; so we look back and find out what was J plus J minus only the sign was sign changes of this sign.

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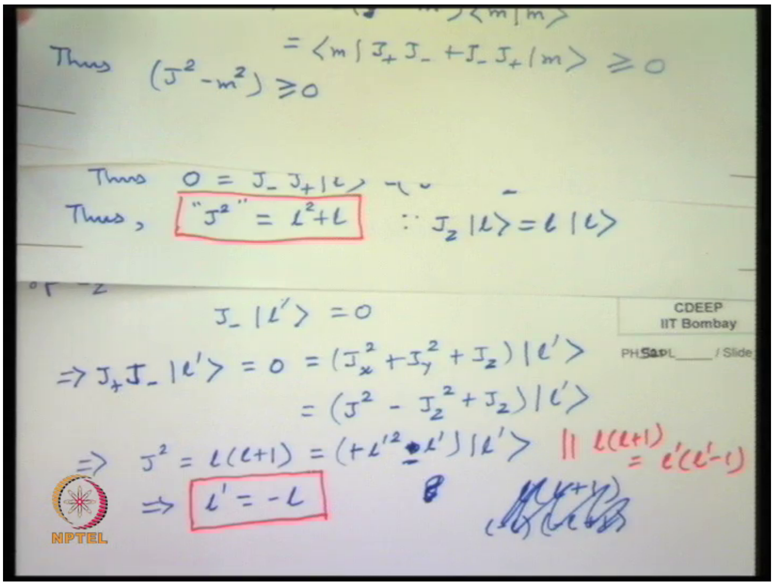

So, it becomes equal to J x square plus; so one step J x square plus J y squared, but plus J z on l prime. So, that is equal to J squared minus J z square, but plus J z on l prime and this is equal to 0. So, this implies that J squared which we already determine was equal to l into l plus 1 is equal to minus l prime squared plus l prime. So, replacing these operators by the Eigenvalue on l prime; that means, that l prime must be equal to minus l.

So, the only way this will be satisfied if this is true alright. So, we have transferred it to this side thank you, so, this 0 equal to this. Therefore, J square on this equal to l prime square minus l prime.

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Right? Yeah. So, this means that l prime is equal to minus. So, we get l into l plus 1 is same as equal to l prime into l prime minus 1 and this quadratic will be solved only by l prime equal to minus l. So, we get the famous result that the lowest value of m z will simply be equal to minus of the highest value.

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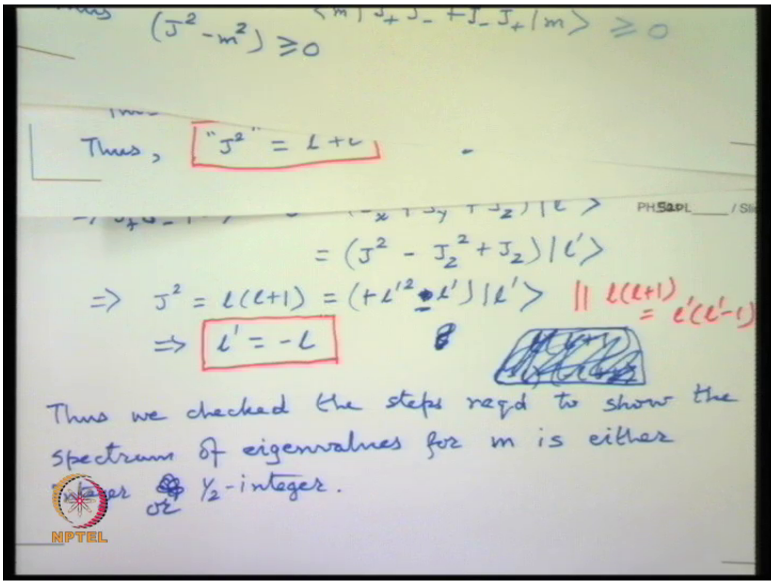

So, these are all the famous results and we can so now we discuss this spectrum for so thus we have proved thus we checked the steps required to show that the spectrum of Eigenvalues. So, i am using deliberately this word spectrum of Eigenvalues because that is what they are called. In all the advanced Hilbert space theory also the Eigenvalue is called Eigenvalue of spectrum. So, if you have ever seen a spectrograph the way it split out the spectral lines it is like spectrum that goes on forever.

So, these we have check the; to show that either you have half integers Eigenvalue or integers Eigenvalues for l; I should say m no for m is either integer or half integer or half integer. Some comment I wanted to make here so correspondingly we get spin the half will not be admissible for the real algebra and integer will be used for the angular momentum algebra, but some one more comment I wanted to make oh yes and right and the point is that each one of these; is one irreducible representation.

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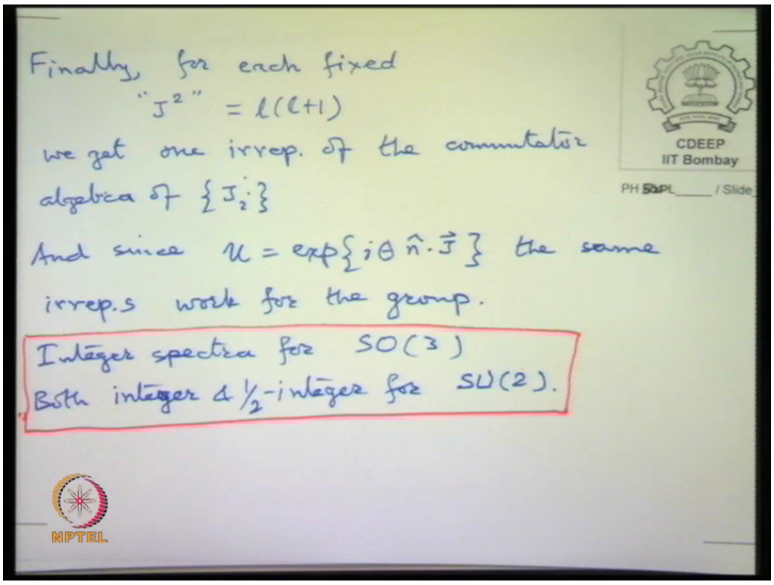

For each fixed J squared equal to l into l plus 1 we get one irreducible representation of the let us for the time being say the commutator algebra of J's J i, but because the group itself is obtained as exponentiation of J i this multiplet will also work as the represent irreducible representation of the group; integer spectra for SO 3 and both integer and half integer for SU 2. So, that is the final summary.