Theory of Group for Physics Applications
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Lecture – 38
SO (3), SU (2) Representations- II

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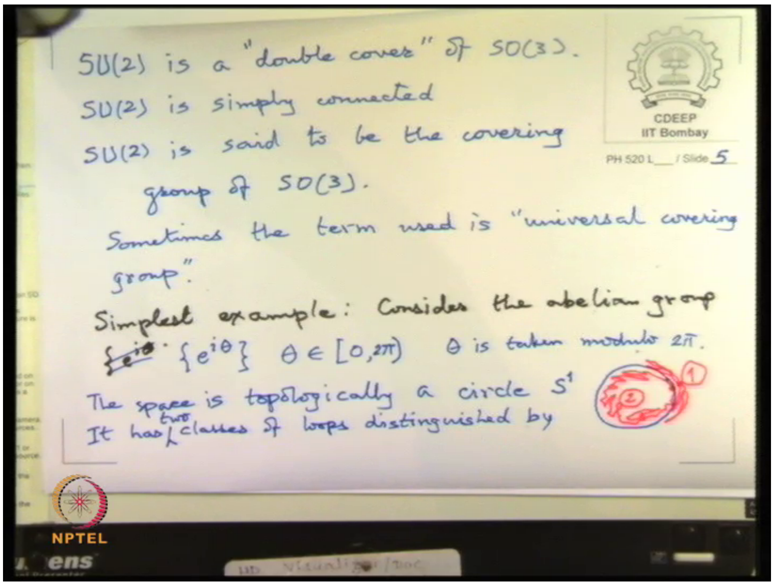

This statement is that, SU 2 is a double cover of SO 3, SU 2 is connect simply connected it is called the covering group. So, this is called the SU 2 said to be the covering group of SO 3, and idea of covering group always it is that it has to be simply connected. So, sometime this is also called universal covering group. Let us see the simplest example of a covering group consider the abelian group e raised to y theta y theta this pen is not good with theta going from 0 to 2 pi.

So, this is like the unit circle and under multiplication. So, theta and theta is mode 2 pi taken modulo 2 pi. So, you can multiply group elements in which case by simple multiplication, you will get theta plus theta prime in the exponent, if that number is become bigger than 2 pi then you subtract 2 pi maps it back into this range. So, it is taken modulo 2 pi. Now this space is not simply connected well it is a circle right you cannot shrinks circle.

So, you can the space is topologically circle call S 1 by topology people, S is reserved for circles for what reason I do not know, but its S and that one is the dimensionality of that space. So, it is one dimensional sphere; circle is not simply connected it has to classes of loops. Actually it has many classes of loops distinguish by how many times? Actually I am sorry I as for as this is concerned it is circle and only the one copy of it, it has 2 classes of loops distinguish by whether they are. So, I can take this circle; class 1 starts from here goes back goes forward, goes back goes forward, goes back goes forward and comes back to this point.

So, this is category 1 I can just keep going zigzag back and do are random walk, without actually going around the circle, then such a walk I can always shrink to point. But the class b is which starts from their do whatever hells you want here, but eventually you go around and then end up here. So, this is category two loop class.

That one you cannot shrink to the first type, because you gone all the way around and once you ended this was the question for the circle also you start from this point and if you come all the way around and reach here, that form circle was loops. Now, that loop if you are allowed to make the beginning an end point is joint then of course, no magic is left, but the point is that their connected in the end makes the close loop and then there is no way of shrinking it.

So, there are two classes of loops and they are distinguish this way; the covering group of the circle or of the u 1 is the real line, you open this up and then you allow it to take all the values and do not restricted by modular 2 pi. Then topologically then they are mappable into each other because their both continue one is both are open sets, you know you once you opened up the circle that is semi finite.

So, it gets mapped on to semi infinite line, but you take the whole line to have universality you take the whole line, but once you opened the circle and make it into a piece of the. So, what when can say is the real line is an infinite cover, in not just double cover. The SU 2 was the double cover its only exactly twice has many point, there are 2 2 1 mapping. For an interval with goes to 0 to 2 pi there are infinite number of such intervals in R m in R. So, R forms a infinite cover.

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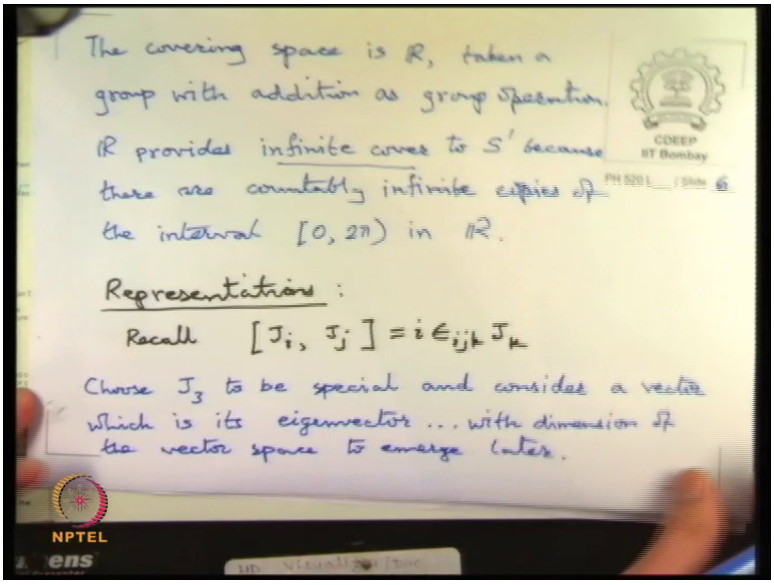

The covering space is R taken as the group, which simply integer which simply addition.

So, R provides any infinite cover. Countably infinite copies of the interval 0 to 2 pi in R. But it is simply connected set R does not contain any unshrinkable loops, the reason you can keep random walking on are as long as you like you never form close loop. So, so much about the story of s. SO 3 and SO 2, and now we go to their representations which actually already know because you have done quantum mechanics right.

Now, to find a representations, there is the standard trick we use the algebra. So, remember we have and I will use the for the time being the J notation the complex notation. Because both tau and l algebra falling the same class one can convert this into. So, now, suppose we choose vectors space such the and we choose the eigen value of J 3 to be the basics. So, now you will ask me what is the dimensionality of this vector and for the time being we will not declared what its dimensionalities ok.

And we will find out what the dimensionalities because the dimensionalities going to the dimensionality of the representation. So, with dimension of the vector space to emerge later therefore, we go to the abstract notation of quantum mechanics.

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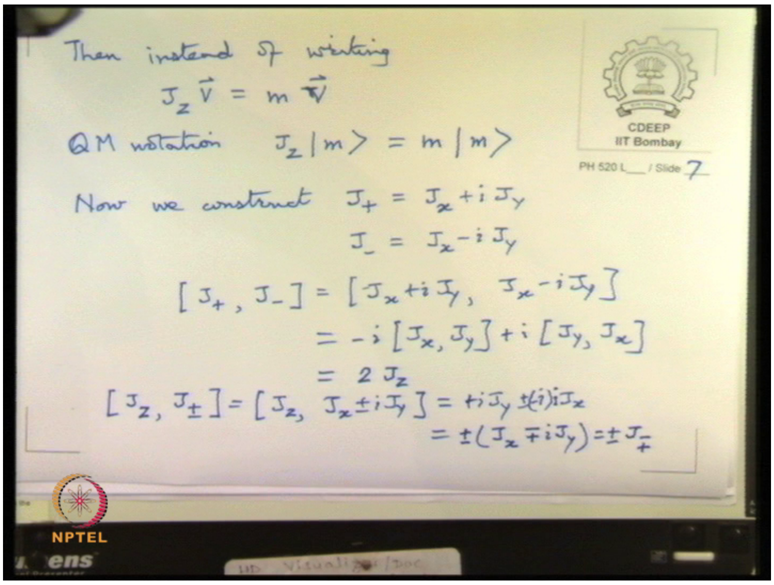

Instead of writing J z V equal to m J z n times V that is eigen value equation, where we had the indef I do not know how long how bigger column vector this is ok, but what I will do is instead use the quantum mechanics notation, where we right a cat vector and this causes are little confusion, but the point is the vector V we write as an angular bracket like this and its label is exactly the same as the eigen value it has corresponding to J z that is the notation.

And so, we are not going to write its components this is infect abstract notation of this type, but here the vector is labeled by its eigen value with respect to J z, now we construct J plus and J minus. So, have you seen this, this has be in done in we are classes yeah. So, then become go little bit faster. So, then we learn that. So, who remembers this J plus J minus is equal to what minus plus J z or plus minus J z. So, we have to work this out. So, J x plus this is going to be equal to J x plus i J y comma J x minus i J y. So, J x J x will commune.

So, minus i times J x J y, the good thing about the commutater of the Poisson bracket algebra is the linearity you can bring things out of it and the bracket of sum is sum of brackets. So, that and then plus i times J y J x, but J x, J y is i J z times minus i becomes 1. So, I get 2 times J z, one from this and other from. J y J x will be equal to minus i time J z. So, I get this. So, J J plus, J minus is equal to 2 times J z and J plus or minus or other letters write it here as J z comma J plus or minus is J z comma J x plus or minus i J y that is equal to that is going to J z, J x is minus i times J y and plus minus i times J z

I am sorry. So, Z x is just y plus y plus y and J z J y is going be minus i times J x. So, what is this coming out to be now? So, we take out plus minus sign, there is this i of course. So, it is plus minus i times, you can plus minus i times J z J y which is equal to minus i J x there is another i over here. So, that makes this is the plus 1. So, if I take out plus minus it becomes J x and because I take out plus minus I get minus plus i J y which is equal to plus or minus J minus or plus we agree with this i times J y then plus minus which was just the sign here.

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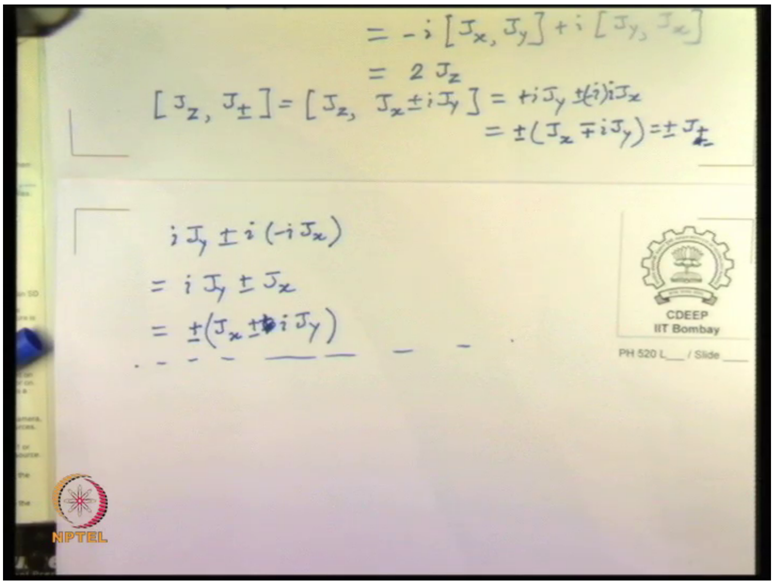

So, let me put plus minus I times J z J y is minus i jx. So, that is equal to i time J y and minus I and this I unsigned I put together becomes plus 1; so, plus or minus J x. So, I take out plus minus out J x and having taken plus minus out of this, I get minus plus i times J y yeah yes, yes, yes, not minus plus good right thank you. So, it is just the same way good.

So, we summarize this by these two things and then now trying to wonder if I was supposed to put a square root 2, but square root 2 sir to remove these 2, but that 2 is not going to if I put square root tools here then I remove this 2 from here and the rest of it hide end up getting nothing else would change right. So, we have concluded that thus J plus comma J minus equal to minus times J minus J plus is equal to 2 times J z and J z, J plus minus is equal to plus minus J plus minus; we are sure of all this right yeah when I take out plus minus sign that has to also be plus minus sign good.

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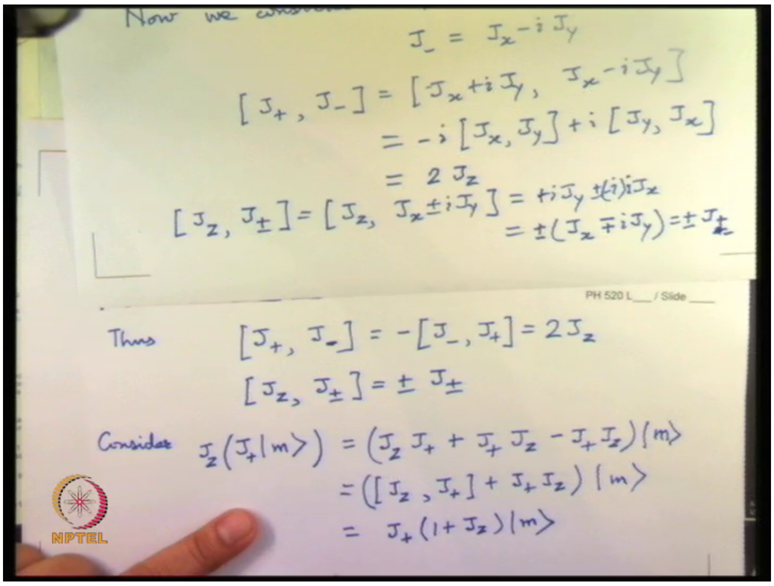

So, now, that statement is that consider the eigenvalue m of J z and act on it with J plus what is this object. To find out what this object is I hit it with J z to find out what it is, but this is same as J z, J plus plus, J plus J z and then add also minus J plus J z acting on m.

So, that is same as commutator J z, J plus, plus J plus J z acting on m, but the commutator J z J plus is J plus into bracket 1 plus J z on m right on this bracket has become J plus, there was J plus on the left side of this which I can take out, but then one plus J z heating m will return m plus 1 right.

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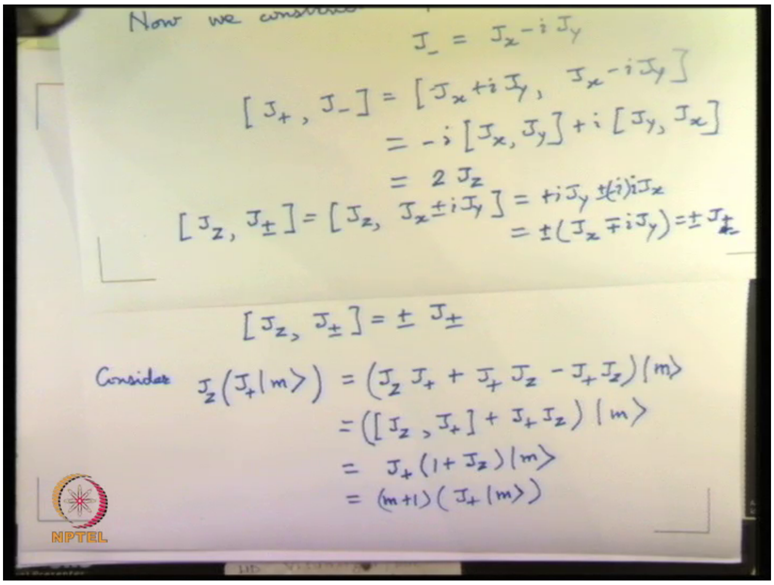

So, if I have any vector with eigen value m, then action of J plus increases the eigen value by 1. Now, if you do this minus 1, then I will get J z on J J minus m will be equal to the commutator of J z comma J minus J minus plus.

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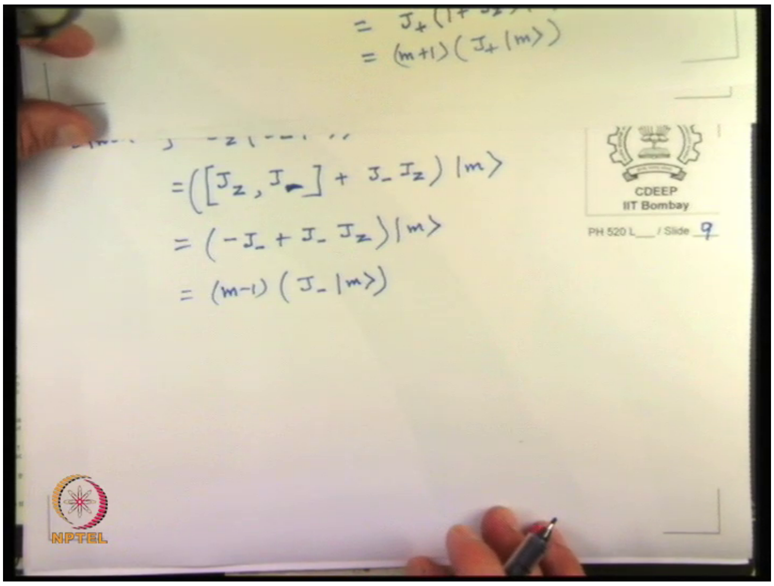

So, the same logic repeats except that now, I have J minus J z acting on m, and J z J minus is equal to minus J minus. So, that will return me m minus 1 times J minus 1 m.

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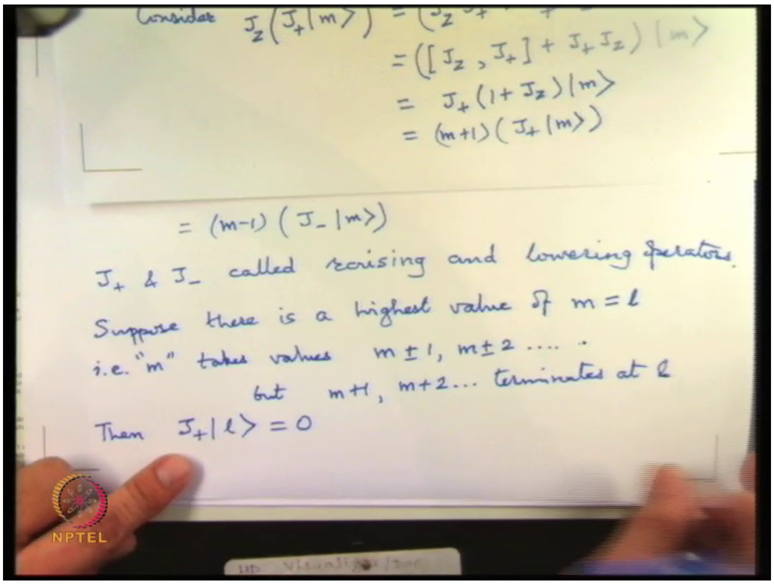

So, J plus and J minus are called raising and lowering operators. Now, the point is that there existed highest value for this m ok. So, suppose there is. So, suppose there is a highest value m by which we just mean.

So, we call it L i e m take takes values m plus minus 1, m plus minus 2 we saw that this is going to happen right when apply to this operators, but it is its biggest value is sum value l. Then we find that to termit to have this happen, we should require that J plus on this l should be equal to 0 you cannot rise on you further.

That is the highest possible value you can have and if this is true then by applying J minus to l enough times, we can also check that we get the spectrum of eigen values l minus 1, l minus 2 separated by integers. Now, the point is that in this spectrum if any number m is an eigen value then so, is minus m ok. To see this one other important point is that.

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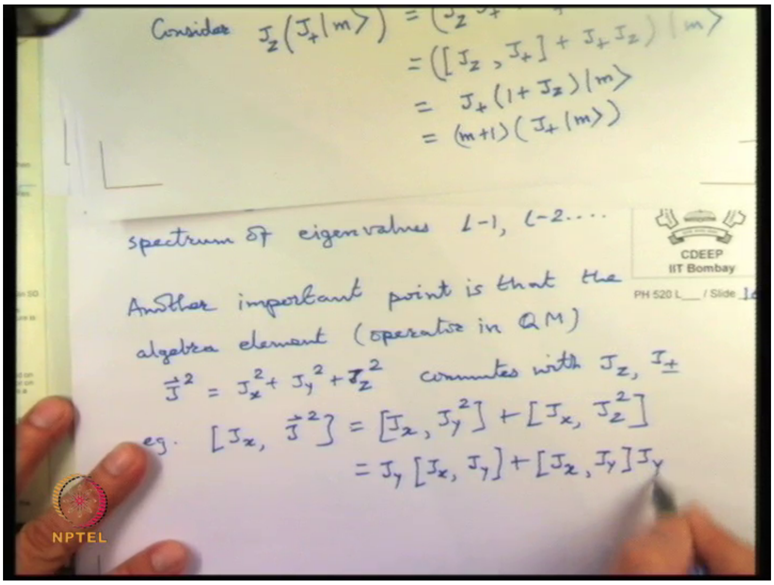

So, leave this further time being the other important point is that the operator or the what is it called the algebra element J squared. This commutes with all the with J z and J plus minus or with J x J y J z everything.

This is easy to see because you start commuting for example, if you take J x J x, J squered the first time would give 0 anyway and would become equal to J x times J x comma J y squared plus J x comma J z square, but remember the rules of this commutator it means that I can take J y out, 1 J y out on this side and next time I can take out a factor on the other side.

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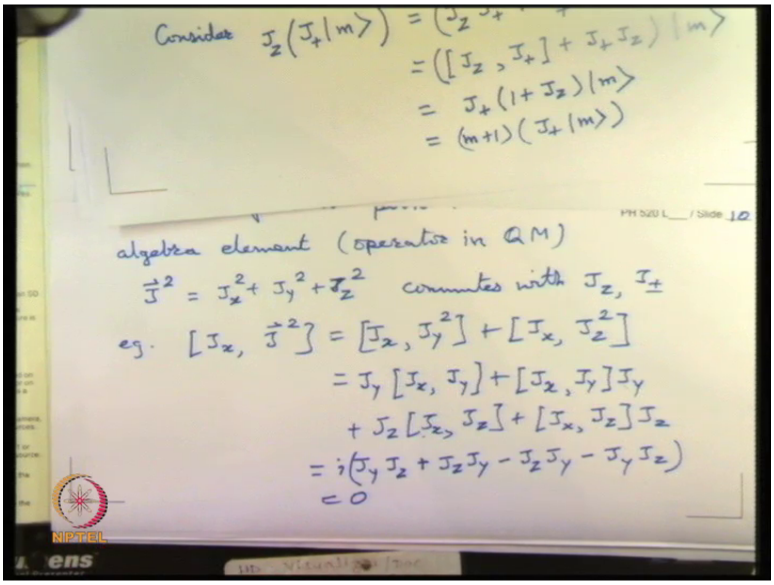

You understand this right because this is this is two factors a and b, I can take out a on this side and do this in the next step I take b out on that side, but here a and b are the same. But have to remember which side I take out a or b, but this is a very very important property of the bracket. So, and this should all add up 2 0 because J x, J y will give i J z this will. So, it will give i times J y times J z and here plus J z times J y and here we should get x z is equal to minus i.

So, I will put minus sign J z J y right J x J z is minus i jy. So, minus the is outside minus sign and this J z times J y and from here again minus J y J z. So, it is all cancelling I think yes. So, it is equal to 0. So, you can check that each of these gives 0. So, this commutes with each of them by Schur's lemma as and when this get represented by matrices on any finite dimensional space, by Schur's lemma this J squared must be just proportional to a number times identity matrix because it commutes with all the elements of this thus J squared is a number times identity of dimension.

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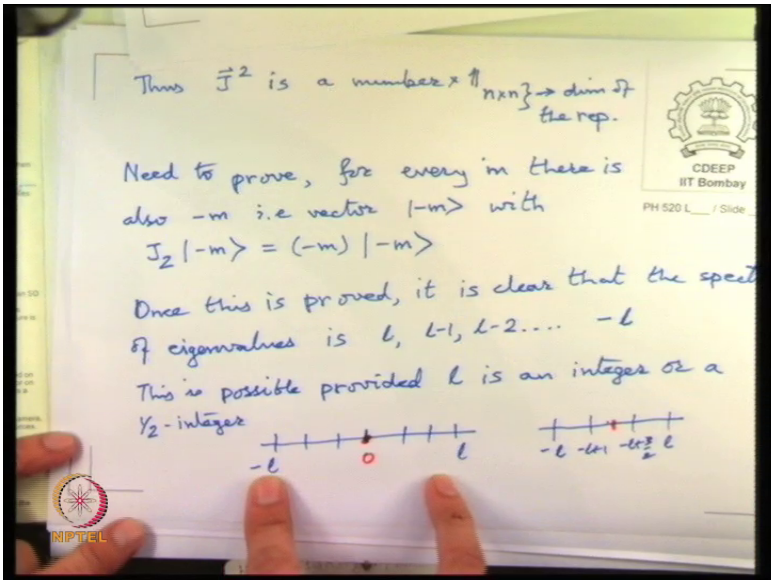

But now, we are trying to find out and also the value of J squered can be found by writing it out as writing this as J x the J plus J minus plus J minus J plus this writing these out as product of J plus J minus and using the algebra again ok. Now the main thing I wanted to prove now was that, for every eigen value m there is also eigenvalue minus m in the spectrum and if that is true. So, if this is if we once we prove this, it is clear that in. So, spectrum of eigenvalues is L, L minus 1, L minus 2 etcetera and reaches up to minus L.

But this also means automatically that L itself has to be integer, because you cannot have integers space spectrum, which exactly becomes minus L in the end. Unless L itself is an integer or a half integer, you can see this by say drawing the line and then say I have l here and I have minus l here I have I can only go in unit steps. So, either I have to have 0 here or I have to as 0 exactly between these.

So, either I have to have 0 here I have a 1, 2, 3, 4, 5, 6, 7. So, I have 0 or I have the 0 in the middle, but these are the only two possibilities there is no other possibility right. So, l plus 3 by 2 and becomes plus l even minus l plus 3 by 2, but becomes l. So, these are the only 2 possibilities for the spectrum. So, we need to just check the m 2 minus m mapping, moving on for the time being this means that the space is 2 L plus 1 dimensional the dimension that we are not specified so far.

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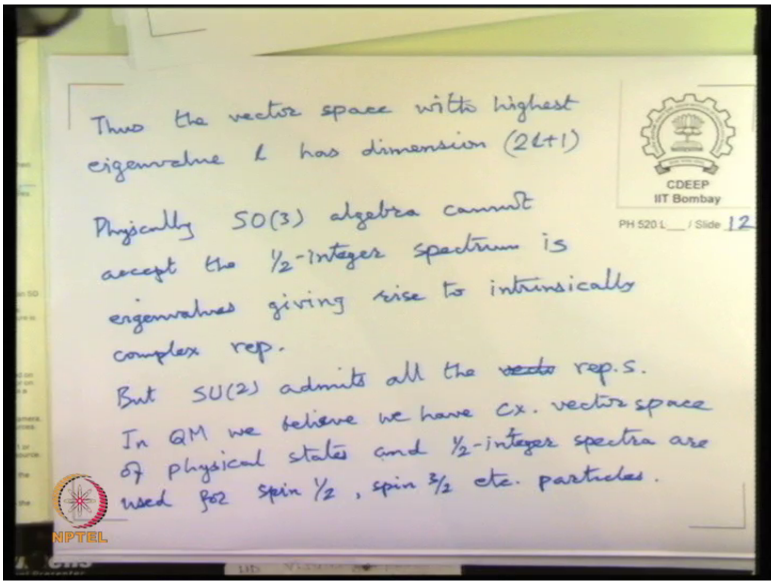

Now, we will see that if the algebra was really of the l of angular momentum, physically SO 3 algebra cannot accept the half integer spectrum as eigen values.

Because the corresponding representation matrices will be intrinsically complex and you cannot have that in the in the real orthogonal group; Whereas, SU 2 admits both admits all the vector spaces or all the representations each is of course, an irreducible representation, and in quantum mechanics because we believing complex vectors space, spectra use for half integers spin particles.

So, the representation space of this can be entirely derived from the algebra alone. So, we will we will see that the next thing I; two things I want to do is one is the Lawrence group SO 3 1 and the other is representation on space of functions. So, in quantum mechanics we know that; so, two topics for the next time.

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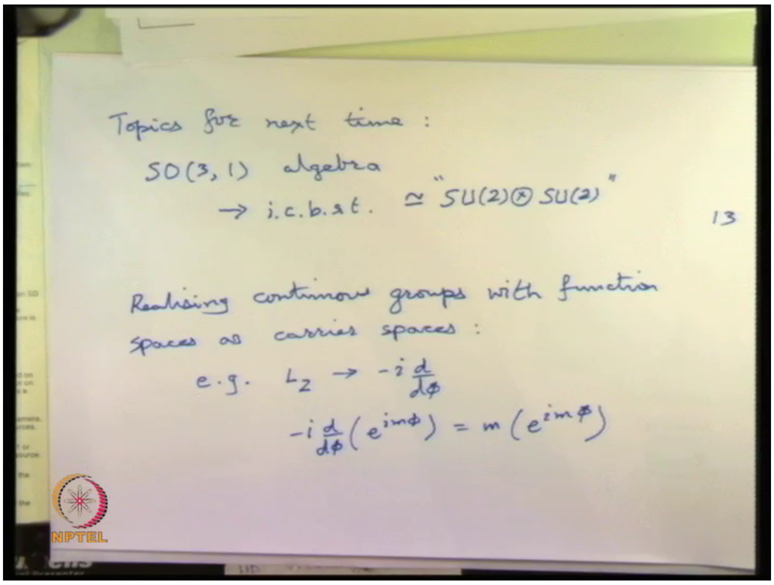

And to give you a preview it can be shown that, it is isomorphic to SU 2 process u 2 with some inverted commas ok. Because this is a real orthogonal group these are both intrinsically complex groups. So, if you complex if I this algebra then you can actually make it into this. So, if the Lawrence group can be made to look like it is to SU twos, then its representations are also all done and the trick is very simple actually well so, will come to it next time.

So, that is one thing and the other thing is realizing continuous groups on function spaces. So, we have for example, L z gets represented by minus i d by d pi and minus i d by d pi on e raised to i m pi returns m as the eigen value. So, e raised to i m pi is a function space realization of the abstract m we were writing, where we represent now the L z operator as a differential operator. So, these are the two things will be continuing with next time.