Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 37
SO(3), SU(2) Representations- I

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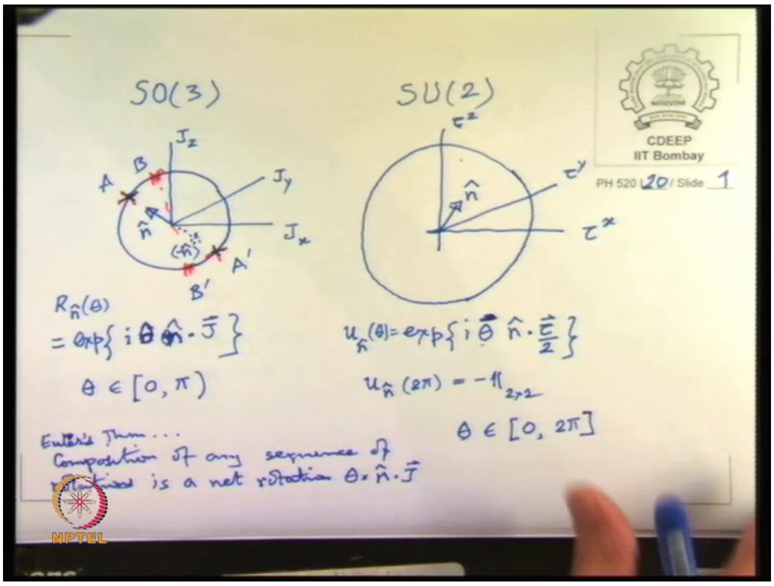

So last time we saw that we had we drew this pictures for SO 3 and SU 2 and the pictures were that SO 3 is a ball of radius pi and psychologically we can draw SU 2 of double the size. Although remember there is really no size in this space and these are the direction tau x, tau y, tau z or tau 1, tau 2, tau 3 I think, but whatever.

So, this n you can erect your n cap here somewhere here wherever and the expressions here were exponent of if you put j then you put i also i n cap theta n cap dot j times theta ok. So, e raised to i theta n cap dot j. So, this always reminds me of Euler’s theorem that every rotation is ultimately a rotation about some 1 axis.

So, there is an n cap. So, you can do all kinds of rotations, but the beginning position and the final one are always connected by a single rotation which is the rotation about one axis ok; composition of any rotations is a net rotation theta times n cap dot j. So, n cap is the access of rotation theta is the angle of rotation.

So, totally there are exactly 3 parameters connecting any one orientation of body to any other even when it has gone through all kind of complication in complicated paths. So, this is what I would like to call it is part of the Euler's theorem and we saw that this theta goes from theta belongs to and let us write it as 0 to pi and there is the whole. So, it is 0 to pi like this but we will see more about that in a little bit.

On this side we have exponent i times theta times oh god theta times n cap times tau n cap dot tau by 2. So, either you put 2 under the theta which is often done, but to remind you that in group theory since it is tau by 2 which is the generator you should put half there, but the net connivances the same. The so this is u of u n cap of theta and this is what we called R n cap of theta rotation that is u n cap of theta and the interesting point is that u any n cap of 2 pi becomes equal to minus 1 because this has expansion cos theta by 2 plus i n cap dot theta sigma time sign theta by 2.

So, when theta becomes equal to 2 pi you get cosine of pi. So, you get a minus sign in front of the identity matrix and whereas, here at you get either get plus pi or if you flip the n cap to minus n cap you are basically doing the opposite rotation. So, you cover the distance from minus pi into pi by choosing the n cap appropriately, but it never reaches minus 1 it does so, ok.

So, the outer most point of this are all independent, but the opposite points are identified. So now we are coming to the topology part of it which is that.

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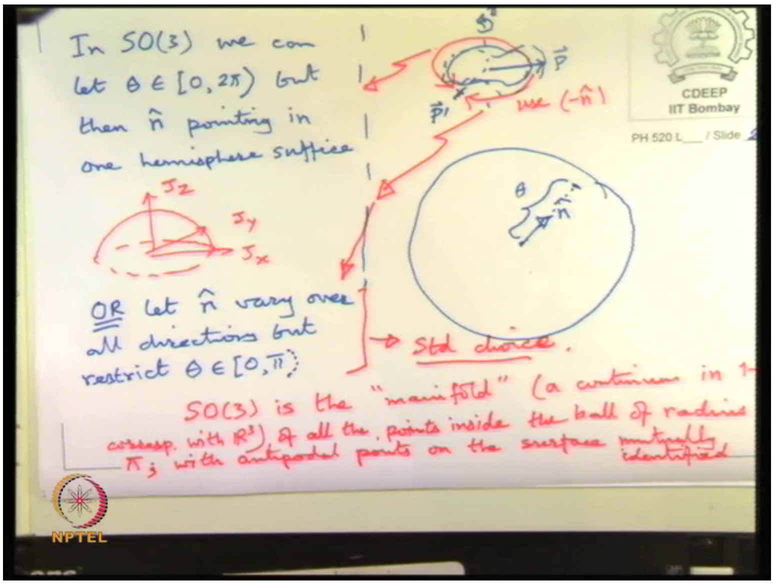

So, in SO 3 we at theta equal to pi we can take I will just leave the this side for SU 2 statements. We can let theta go from 0 to 2 pi but then n cap pointing in only half of the hemisphere are required, in one hemisphere suffice, right this is an important statement this is the important statement. So, suppose I take the hemisphere corresponding to the upper ball which is right. So, the one defined by j x j y plane and suppose we take only all the n caps pointing in this and not go below, but we go 0 to 2 pi, then we have covered all the rotations or we can go 0 to pi and then take minus n cap as well opposite the all of the n caps.

But then only restrict ourselves to go from 0 to so, sorry, either I take one hemisphere and 0 to 2 pi all of the possible rotation so I come back to the original, but then there is no point taking the other side because the same orientation I can obtain by doing the larger rotation here or I restrict myself to or let n cap vary over all angles all directions but restrict theta to belong to 0 to pi and now I will draw a dotted line here.

The point is that if I do a pi rotation so, if I do a pi rotation about this direction; so let us look at this diagram. If I do a pi rotation I reach this point. So, as you now in this diagram, theta is the magnitude that by which you scale this n cap vector because theta multiples the n cap vector. So, theta is the magnitude of the vector. So, the amount of rotation is given by the magnitude. So when I reach here I have rotated by exactly pi. So, I will reach the, I have done this rotation.

But to go to further you now greater than pi and suppose to now start the other way and reach that point by using minus n cap. So, I can reach this point, but if I now start off with the opposite direction n cap minus n cap then I will be double counting if I reach the opposite this point ok.

So, n cap are unit vectors and the amount of rotation is how for you going this so, this is theta. Now let us draw a physical space picture let us take a cube let us say cube is I do not know whether what to take actually not the cube, cube is not a good idea. So, suppose you take a slightly uneven shape and you rotate by pi then this point comes here and you get the; you know the corresponding opposite shape. If you want to rotate further, so this is pi rotation. If you want to rotate further either you can keep rotating further up to 2 pi which is this choice or you can say if you want to reach so, suppose this is the you now some p vector p. Suppose I want p to be here, p to reach here, I can either rotate by this way or I can now start rotating in the opposite direction which; so I should have this is what I should have emphasize. The theta rotation is always entirely counter clockwise looking down on n cap that is our usual convention. So, with that convention if you now want to reach this you can also reach it, so either by rotating counter clockwise this way or by taking minus n cap and rotating counter clockwise this way.

So, you can reach this point in SO 3 into different ways either like this or like this. So, this is the case 1 and this corresponds to this choice the lower choice sorry where you let n cap go only go over all directions. So, here you choose this happens using minus n cap minus n cap and counter clockwise from looking from below. So, either I use minus n cap, but then I have to restrict to 0 to pi only because this reaches up to here and this minus n cap can reach up to that. So, I have to restrict my rotation angles 2 theta to pi and the point is that the actual point pi is double counted in this method because either you can reach it by pi rotation by n cap or by pi rotation were minus n cap.

Therefore, in this picture these two points, the antipodal points on the surface are actually one and the same but they are not they are different but this point is distinct from say this point. So, this point has an antipodal point here. So, this point is same as this point, this point is same as this point.

So, now the space of SO 3 rotations is the set of points which are inside this ball; some n cap, some size theta and when you reach the boundary of the ball you can include only half of the surface of the ball because the antipodal points are already are have to be excluded on the outer surface so, SO 3 so with this is the standard choice but of course, the topology will remains same even if you that we can argue separately.

So, if you make this standard choice then SO 3 is the manifold which is essentially a continuum set which can be in 1 to 1 correspondence with R 3, is the manifold of all points all the points inside the ball. So, we avoid the words sphere because sphere sometimes means only the shell. So, to be specific mathematicians call the full filled sphere the ball. So, all the points inside the ball of radius pi with antipodal points on the surface identified; mutually identified ok.

This is the key fact about SO 3. Now you can also as a exercising thinking think of the other convention where you take the upper convention. I will let n caps be only in the upper hemisphere, but I will go I will let my sphere go to size 2 pi instead of pi. If I do this then again I have going up to 2 pi, but then the outer most points are all included now; except that on the hemis the circle the great circle on the hemisphere, I can take only half of the points because again the antipodal points are there.

So, remember that when I take this and when we said we identified antipodal points, it means that I have to leave out the outer most peel or outer most shell of this lower half out; not only that on the equator I can take only half of the equator not only that I can only take the semi closed equator; include the point 0, but leave out point pi because pi and 0 are also identified. So, it is a slightly strange manifold in which the outer surface is only half of it is there. But the rotations that are about along this direction remain independent of rotations around this direction because when I reach pi by using this axis it is different from having reached pi by using some other axis. As the, so the configuration may looks similar but you can never actually continuously deform this pi rotation to that pi rotation.

So, they are actually all the antipodal points are identified this point remains distinct from that point. So, suppose we call A and B and A prime and B prime. So, A prime and B prime I will write over here. Here I would not being so, careful. So, let us put is again. So, let us say there is A and B and I have A prime and whatever I am drawing here (Refer Time: 18:20) B prime. So, A prime is same as A; B prime same as B, but A is not same as in any group theory sense A and B are distinct.

Now we come to SU 2, SU 2 story is a bit simpler and nicer because SU 2 simply says take n cap and let theta go from 0 to 2 pi and actually there is no harm including the 2 pi because when it reaches 2 pi it becomes minus 1 in the group in the group. So, regardless of which direction you reach that minus 1 it is minus 1 and in SU 2 case, the entire outer surface of the ball is one point topologically one point, one and the same point.

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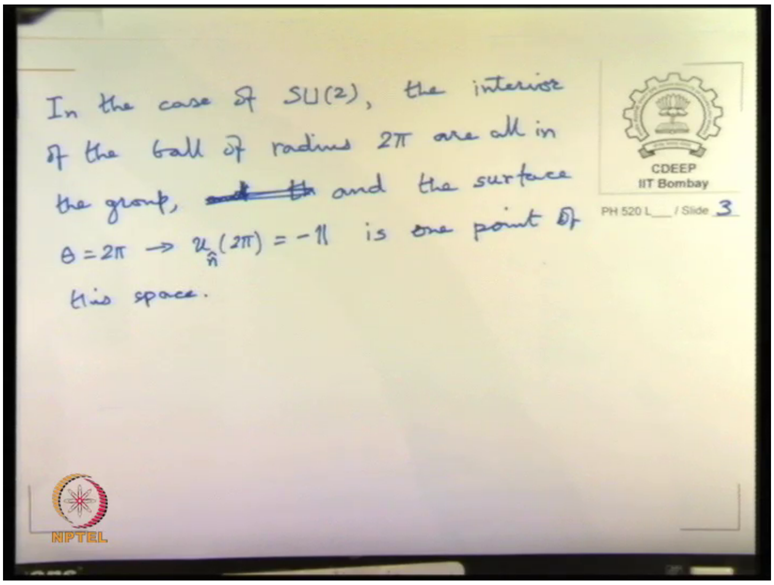

Group and the surface theta equal to 2 pi with u of 2 pi regardless of whatever n cap is equal to minus 1 is entire is one point. So, now, what happens is that the difference between SO 3 and SU 2 is that in SU 2 the range of parameter is double ok. If you take the standard choice n cap varying over the whole sphere then you have to restrict theta to go from 0 to pi, aside from all the fine print about what happens at the surface, but it goes 0 to pi, whereas in SU 2 it goes from 0 to 2 pi and only when you reach 4 pi do you get. So, if you if you do this convention for SU 2 take only upper hemisphere then you can go all the way to 4 pi and then you will return to 2 plus 1. That is express simply by saying that you can go to minus 1 from any of the directions.

But the size of this or you can say there is a 2 to 1 mapping from S SO 3 to SU; 2 2-to-1 homomorphism from SU 2 into SO 3.

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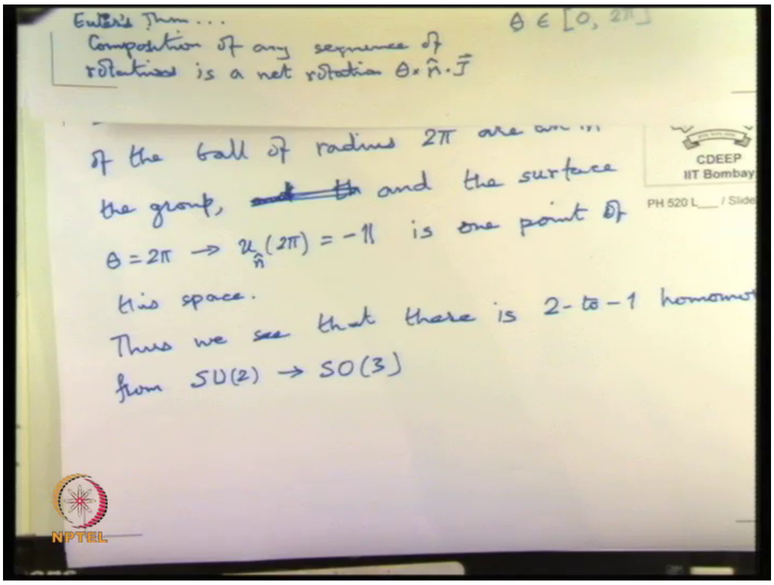

Then we finally, come to the most important topological property. Therefore, SU 2 is a simply connected space. Every single curve in SU 2 can be shrunk to a point but in SO 3 you cannot.

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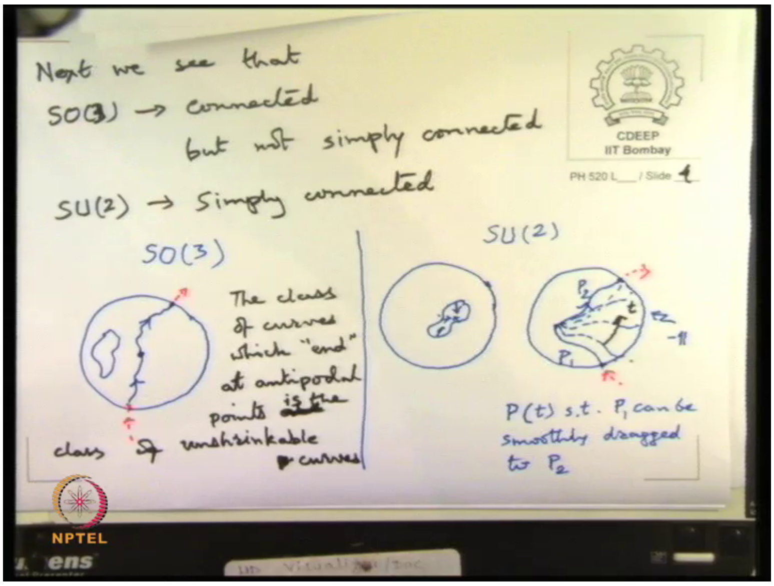

Whereas, SU 2 is simply connected, once it is simply connected it is automatically connected; connected means any two points can be joined by a arc within the set, but it is not simply connected which we will just check which I prove in a minute. Whereas, SU 2 we will find a simply connected. Now remember the definition of simply connected is that every single close curve can be shrunk to a point and not simply connected means that the exits loops that cannot be shrunk to a point, but there will be an equivalence class of them there will be a whole.

So, there basically there will fall into two equivalent classes in the case of SO 3; either they can be one class which can always shrunk to the origin, the other class which cannot be shrunk to the origin and the picturization goes as follows. If you take SU 2 which is easier to explain think of one class of loops which are entirely inside ok, you can always shrink to this to a point no big deal right. So, this can be shrunk to a point. Take another class of loops which start somewhere in the interior then reach the boundary but now boundaries all one point.

So, I can reenter through here. I go like this. Remember actually this is a little bit confusing because this is not a rotation this is the sequence of many different rotations. So, we are in the space of all possible rotations, but there is a path in this space we goes here. I can re enter through this point. So, I can magically come out of that and re enter through here because they are actually one and the same point and then I come and close the loop because the whole thing is minus 1, it just minus 1. Therefore, I can continue by coming in from any other point, but these points are after all one and the same.

So, there is no harm in slowly sequentially converting it into nothing stops me from doing this and finally, I make this; so the P 1 and path P 2. So, we can define a path P as a function of t such that well that is actually not the argument. The argument is that I can shift this point indefinitely there then it becomes a close. So, P of t such that P 1 can be smoothly drag to P 2 and once it reaches the once the outer most point reaches here that has now become a close loop. So, I can shrink it becomes of this class. So, I can shrink it to 0. So, every single point that every single close curve you can imagine in this space is shrinkable to a point but SO 3 is not like that.

Now in this case if I start like this I reach let us say this point. Now, according to rule if I want to continue I can continue only from the antipodal point, I cannot re enter from any other arbitrary point. So, I can reenter through an antipodal point and then I continue come to the beginning point. So, of course here also there are the trivial class of loops which are just in close like that, but there is this class of loop which is a close loop.

Now you can see that this cannot be shrunk to a point; the reason is that here when I shifted this point along the surface I was just at sitting at minus 1 minus 1 minus 1 regardless where it appears in the 3 D in this 2 D picture, but here the movement I try to shift this point this point will also shift because antipodal point tied to each other ok. So, I cannot start shifting these two point close to each other.

So, this class of curves remains unshrinkable are unshrinkable; actually the if it refers to class then is class of unshrinkable points is the class of unshrinkable points curves. So, there is it is like being in a on a torus or in an annular region. You take the x y plane but you have cut out some central disk or even just remove the origin, then you cannot close the loops to one point. There is one class of loops which do not go around the origin and the class of loops that goes around the origin.

So, the same thing happens here and therefore, SO 3 and SU 2 are fundamentally different first we said that this is the double cover of these, but now what we know is that this set is simply connected such a. So, if I embed this SO 3 inside this SU 2 then I have a double cover, but which is then simply connected. So, this is called a covering group. So, breaking the curve is not an option that is not the then of course, you cannot do much topology.

So, topology deals with the fact that you do continues transformations and they deal with essentially with open sets. Eventually in some way or some sense you do always keep falling back on your intuition of the real line where you have this dense set of points which have the continuum which form the continuum and even abstracts basis that do not necessarily have the sense of distance or magnitude that the real line has. The connectivity properties can be similar which you can study by making out 1 to 1 map and that map is precisely what this map is our page 1 expressions right this is the map.

So, this map maps the numbers n x, n y and n z, n theta and this is because it is a unit vector there are 2 degrees of freedom here and 1 degree of freedom here, 3 via independent real numbers mapped into this matrix which psychologically is representing a point in this abstracts space.