Theory of Group for Physics Applications

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Lecture - 36
Group Algebras; SO(3)-SU(2) Correspondence - II

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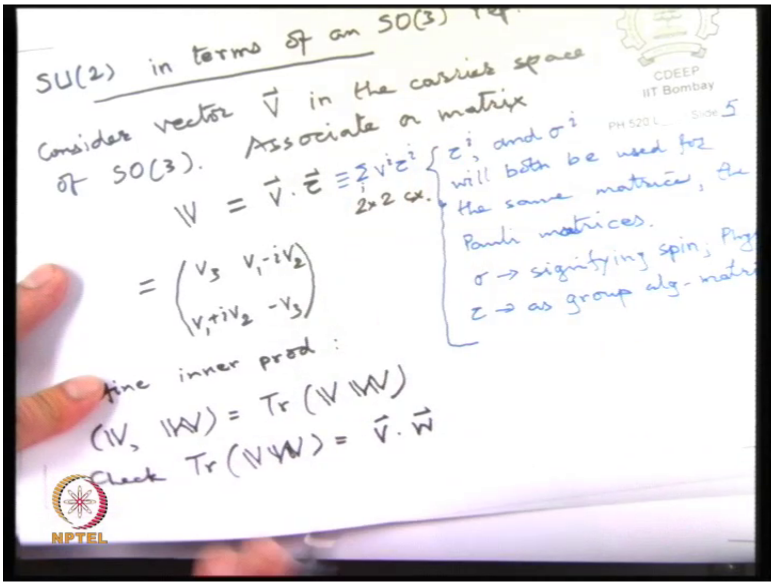

So, consider vector V good old physicist notation, which is in the carrier space of SO 3. Associate a matrix and now I put this what is this called it is called blackboard font in tech. So, you double this line. So, V equal to V dot tau, where tau are the sigma matrix or pauli matrices. And let me write over here a little note about the notation tau i and sigma i will both be used for the same set of matrices the Pauli matrices.

Sigma, if it has to signify spin spin and Physics tau in group theory. At least that is the convention in my mind and you will find that that is roughly true in many books. So, it is just there the same matrices, it is just that we call them tau 1 tau 2 tau 3. So, we identify we define 2 by 2 this is of course, 2 by 2 complex matrices, V made up of this. So, if you want we can write this out in detail what it will become is you know. So, V 1 times tau 1 will basically put the V 1 here then V 2 times tau 2. So, minus i V 2 and plus i V 2 and V 3 times tau 3 so, V 3 minus V 3. So, this is what this matrix looks like.

So, out of our 3 d vector we create this 2 by 2 complex matrix like this. What is interesting about this is that we can define a we can define a inner product V comma W equal to trace of V times W ok. And, we will see that this reproduces V dot W, because. So, how does it work?

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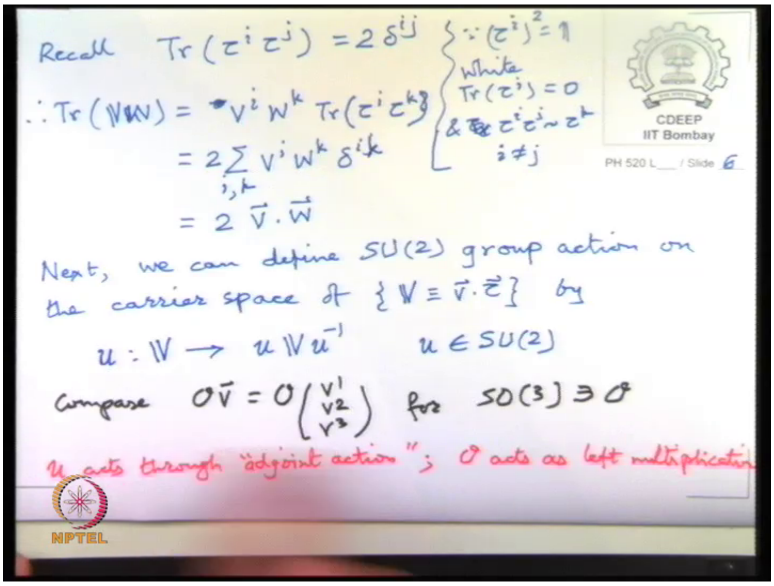

Remember that, trace of tau i tau j was equal to 2 times delta i j right, because the tau matrices themselves are traceless the sigma matrices are the Pauli matrices are just traceless. And, if you take a product then you remember that they actually satisfy tau i tau j equal to epsilon i j k times tau k, they basically give the third Pauli matrix if you take tau and tau 2 it gives tau 3 and so on. If you take trace of that you will again get 0.

So, only if the 2 taus are same delta i j, then it becomes identity matrix tau squared is so, this has this depends on while trace tau i equal to 0 and trace and tau i tau j is tau k if i not equal to j right. So, since if product of 2 taus gives you a third tau and if trace of all taus is 0, then this is always going to produce 0 unless i and j coincide and we get identity in which case we get 2. And therefore, if we take this trace of trace V W equal to trace of so, now, we will take out V i W k trace of we get tau i tau k right, because V is just sum also we could have writ10 here sum over i V i tau i that is what it is. So, V i are numbers and tracing has to be done over these matrices ok.

So, we can pull out the numbers and do trace over this and this trace is equal to 2 times delta i j. So, except for a factor 2 we got the inner product back k. So, I have to supply a factor 2 or you can define it as half.

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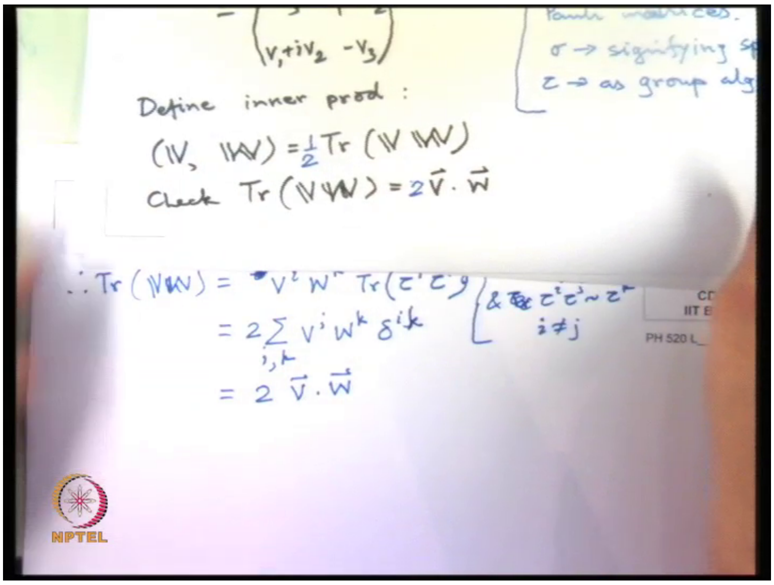

So, we do not have to worry. So, we checked that this is equal to 2, trace this is of course, twice so, you define the inner product to be half of that. So, that it takes exactly the same inner product right. So, what is going on from the 3 D language, 3 dimensional rows and columns we switch to some 2 by 2 complex matrices, but we are recovering the same geometrical or mathematical properties of this V and w. In fact, we can now check that these act as SU 2 representations.

So, if u belongs to SU 2. So, compare O V equal to O acting on V 1 V 2 V 3 for SO 3 now the same where O is belonging O belonging to this sorry O belonging to SO 3. So, the same real numbers V 1 V 2 V 3, which appeared as column vectors and with one helping of O, are now transforming here by 2 helpings of u. If you have an SU 2 element then there is a adjoint action or Simi an action like a similarity transformation.

So, you acts through similarity transformation, what is called "adjoint action" whereas, O acts through left action left multiplication. So, on the same carrier space we can have we can have both kinds of actions either the left action by the real matrices O or the adjoint action which looks like a similarity transformation by the SU 2 matrices. And, which act that that action SU 2 actions preserves this trace operation.

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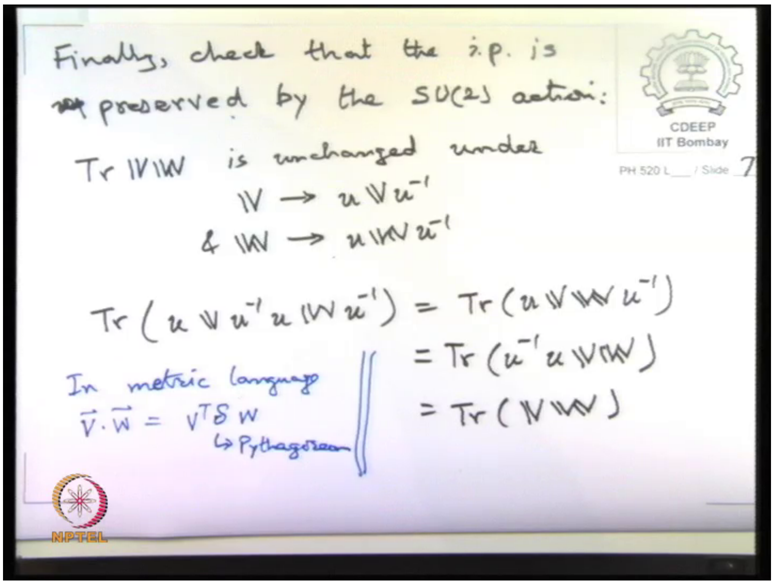

Check that the inner product is preserved by the SU 2 action, because we need to check that trace of V u V W I am sorry trace V W well is going to be invariant, it is obvious you do not have to compute anything right, because you insert these in the trace.

Firstly, this u inverse u is going to multiply in the middle and then under trace you can always cyclically change the order of the matrices. So, it will become. So, it preserves the inner product. In other words it preserves the metric meant for the 3 dimensional vectors, you know in the in metric language metric language this V dot W was actually V transpose the delta matrix of delta metric of Pythagorean metric.

So, it basically preserved this delta and we see that here SU 2 is preserving the same delta. So, actually the 2 groups are have a very close correspondence, but it is not isomorphism.

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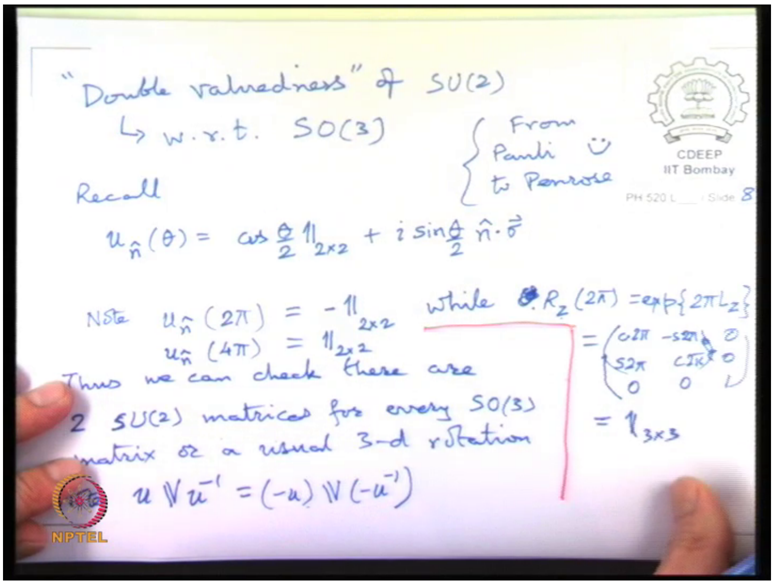

"Double valuedness" of SU 2. So, this double is of course, with respect to SO 3. And this has been a source of great mystery and mysticism even for very important physicists in quantum mechanics, because the tau matrices or sigma matrices were first invented for spin. And this double valuedness remained a big mystery for everyone from Pauli to later wheeler and roger Penrose ok. So, from Pauli to Penrose they have all been amused by this and Penrose of course, has a whole programme of quantising gravity using the representation similar to what I said, but for the Laurence group not rotation group ok.

So, what is this double valuedness? The point is that SU 2 u n cap theta was equal to cos theta by 2 times identity note lot of books do not write this identity, but it is important to remember it is there ok. This is how the group works out to be.

Now, let us look at the range over which this theta goes ok? So, we can see that when theta is equal to pi by 2 sorry 2 pi what am I saying. So, you when you go to theta equal to 2 pi you get cos pi. So, you get a minus 1 and sin of pi is 0. So, at 2 pi you do not return to identity, while the O elements let us say O generated by theta along z axis. So, what was our notation that R R z of 2 pi, which would be exponential of 2 pi times l Z right, but what is this exponential was just cos theta sin theta etcetera. So, it was just cos 2 pi cos 2 pi minus sin 2 pi and sin 2 pi. So, that is just equal to identity. 3 there is a third 0 0 1 ok.

So, it was just equal to 3 by 3 identity. So, R z 2 pi or R x R y anything you want, it just basically became cosine on the diagonal and with the whichever axis that was not being turned this one just migrates here if you change to x or y migrates up the diagonal, but the other elements are just cos and sin and exactly at 2 pi the you get back to cosines becoming 1 and the sins becoming 0 with no overall sin appearing, but here at 2 pi these 2 by 2 complex matrices of SU 2 do not return to identity, but only to minus 1 ok. And so, what we find is that there is a 2 valuedness, there are 2 SU 2 matrices and of course, we know that u and cap of 4 pi would be equal to 1, if you put 4 pi then you will get back identity.

So, there are 2 SU 2 matrices for every O matrix. And, that is because of the adjoint action u V u inverse, which is how you transform any V is same as minus u V minus u inverse. So, if u carries out the required rotation. So, does minus u and minus u is actually a distinct matrix it is not just u itself. So, u and minus u are both doing to vectors V the same thing that the same thing and for which there will be only 1 corresponding rotation matrix ordinary rotation matrix.

So, there is a 2 valuedness and that was what was observed here both the 2 pi value and 4 pi value will look like identity operation as far as the V matrices are concerned, whether you put u n cap of 2 pi which would give minus 1 and a minus 1. So, it will look it will look like ordinary rotation of a 3 d vector and so, will 4 pi.

However, the representation which is the complex 2 D vector.

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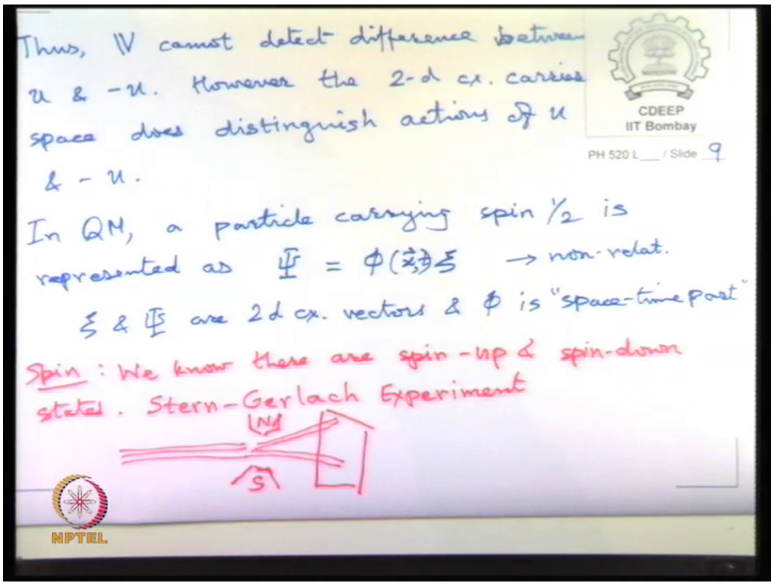

The 2 dimensional complex carrier space does know the difference between u and minus u, in quantum mechanics we simply call this spinners we call these spin matrices or spin wave functions times psi, where psi and psi are 2 dimensional complex vectors and phi is space part well you want to put t you put t space time part.

So, we split the wave function of the electron. So, this is of course, non-relative all non-relativistic, but you can split the wave function into a spin part and a space part and you will find this language in lot of the nuclear physics and condense matter physics literature. And, now we come to explaining how young Pauli was maybe 23 years older something like that when he invented the Pauli Matrices. It is very simple. If you we know that there are spin up and spin down, this was what is what has come to be called stern gerlach I should not say what has come to be called, but I think historically it was the experiment that was performed by stern gerlach stern and gerlach, but that was not origin of Pauli's considerations he had other thing.

But, I am just saying what I meant by saying? What is come to be called is what we see what can see most clearly through stern gerlach experiment? Is that, if you send a mixed beam of electrons through an arrangement of Magnets North and South Pole, then this beam will split exactly into 2 parts. You have to repair an arbitrary, you just have you are heating some silver or something like that.

So, you are just some vapour of silver coming out it is all mixed up, but now if you apply this magnetic field classically you would have expected that, it will range over you will see on the screen you should see a continuous set of possibilities; this spin is here, spin is here, spin is here, but quantum mechanics just mix out 1 and the other. So, this magnetic field is inhomogeneous.

So, that it actually can separate out the dipole moments as you know right homogeneous field would not do much, but in homogeneous field we will separate out the dipole moments, but when that happens classically you expect the spring to be anything. So, it will pick out any projection that is perpendicular to these magnetic fields. So, it will get attached to it, but quantum mechanically you either find this or you find this and nothing else, it is impossible to get anything in between.

So, this is the classic experiment that tells you that regardless of what kind of initial state you start with, if you make a measurement you only find the Eigen values. The measurements do not return continuous values of the observable in any 1 observation. The way you recover the sort of port average is if you do a lot of experiments, then of course, the weighted average of the these 2 together will equal the weighted average here the average spin here, but as the answer will come only as an average, but any 1 process of 1 observation if event will only return either spin up or spin down.

So, we know that there are spin up and spin down states so, what would we do logically and if we believed now the next point.

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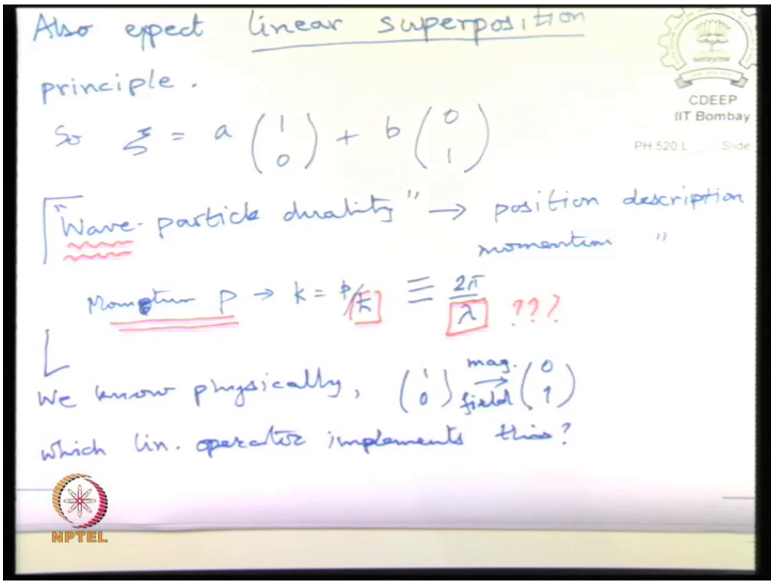

The very very important point of quantum mechanics linear superposition principle so, generically ok. So, the very crucial thing is because there is linear superposition you write vectors vectors are linear spaces. So, if this state was supposed to be obeying linear superposition, then it should be expressible as linear superposition of a basis and that basis if it is 2 dimensional would be just 1 0 and 0 1 ok.

I just I do not want to launch here into a big lecture, but I just want to tell you very briefly, that linear superposition principle is the main positive content of quantum mechanics. Unfortunately the uncertainty principle is emphasised there is nothing uncertain about quantum mechanics, it is pretty certain, it is commutation relations will give you the (Refer Time: 30:45) Heisenberg uncertainty relations and it is it is linear superposition principle, which is very wrongly sold as uncertainty principle.

It is true that your classical expectations are not born out you will not be able to measure position very accurately or momentum very accurately, but to couch this as a principle is wrong the principle is linear superposition, which allows you to have Fourier transform Fourier series representations.

And this delta p delta x relations can be derived in any Fourier transform theory, the width of the function in x space will be inverse of the width of the function in the complementary space. So, that is just a result of Fourier transform and Fourier transform works, whenever there is linear superposition, whenever you can ex obtain functions as linear superposition's of basis functions like sin and cosine like done in Fourier analysis.

So, it has all to do will linear superposition and nothing else ok. So, likewise the so, called wave particle duality is just an oversold point they are just particles you can the wave people are confused, because if you measure a momentum Eigen state then it it is characterized by it is wave number.

So, essentially when people say wave what they actually mean is a momentum Eigen state and a momentum Eigen state I may as well write this down. So, these are the 2 big misnomers of quantum mechanics and sold to public and public as (Refer Time: 32:44) you will find innumerable philosophical discussions going on about this it is all garbage.

So, the "Wave particle duality" basically is position description or momentum description and we do know that the there is this complimentarity, you cannot have both. When you have momentum P what am I writing momentum P, you can associate with it. So, you can associate with it a wave number k, which is equal to p over h cross this is because you had the fundamental constant h cross available ok. So, to momentum p you associate a number of dimension 1 over length and which you define in your wisdom to be equal to 2 phi by lambda ok

So, this lambda is a product of your fertile imagination, there is no such lambda. The main point is that there is a fundamental number h cross which allows you to think of momentum as a length scale ok. And once you introduce a length scale you say oh my god, but particularly either here or there you know it is a wavelength like this there is no wavelength it is a momentum Eigen state either you have a momentum description or. In fact, it neither of them completely precise, because of the earlier part the linear superposition, but the main principle of quantum mechanics is that there is linear superposition and in any measurement you can observe only 1 Eigen value, you cannot observe all POS mixture of all Eigen value it is just that x is a continuous space. S

o, the set of Eigen value is continuous. So, you can come out with any 1 of them with some weightage. If and so, that settles this issue of what is the wave there is no wave the wave is essentially a brain wave ok. And that is because it is possible to associate a wavelength lambda with a momentum Eigen state. So, that the real truths of quantum mechanics is momentum Eigen states.

Now, coming back to this because of linear superposition, you can write spin as a linear superposition of up or down vectors and in fact, those are the observed Eigen states. And so, you what you observe of course, is the magnetic moment physically, but that it is up to a mu multiplying this spin vector it is the same thing. So, now, what does 1 see, we can always go from up to down states right, physically up can go to down by application of a magnetic field, if you apply a magnetic field you can flip the spin.

How will you represent this in quantum mechanics? How will you represent it mathematically well it is a linear space? So, there should be a linear operator ok. So, which linear operator will do these implements this? So, I am very sorry to say, but this business of wave particle duality all the books, that claim themselves to be modern quantum mechanics they miss it completely.

They write fat books and they are very popular, because they reinforce what people like to hear you know miracles are liked by people. If you tell them the truth they do not like it because it is a little too simple to understand, but if you tell them something is very difficult to understand and in fact, tell them that it is something never understandable then people are drawn to it like honey you know honey bees to honey.

So, most people reinforce this idea that quantum mechanics is not understandable. So, then people rush to them. If you tell them that by the way what I am telling is nothing new it is what is written by Dirac in his 1929 book, you have to read Dirac's book please do not read any other. So, there are only 2 books to read in quantum mechanics, Dirac and Schiff of course, (Refer Time: 37:57) and Schiff is always there volume 3.

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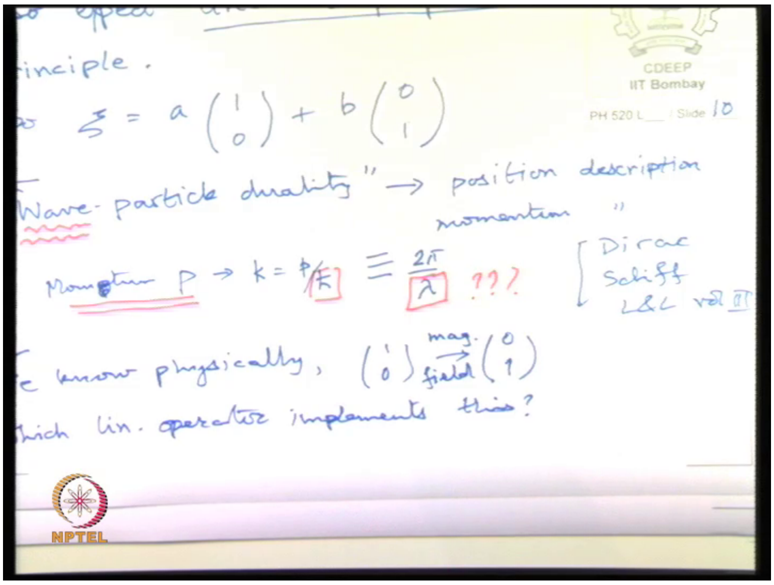

So, these are the books to read in quantum mechanics and they will tell you on and on truths, which are Dirac's is the best and written in 19 29 when people were still puzzling over the interpretations of quantum mechanics. So, essentially that is what, but nobody learns from Dirac. And the somewhere I tried to tell someone and they said do you really believe Dirac, I felt like telling really believe all the else anyway that is how it is?.

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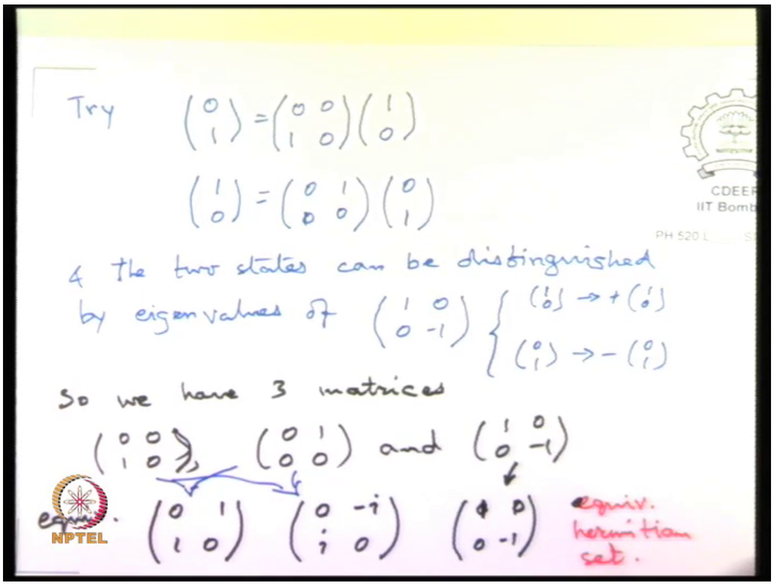

So, how do we get from down to up or up to down we apply a lowering operator equal to something acting on 1 0 and; obviously, this can be done by this row into this column. So, I have to have 1 right. So, this into this 0 and this into this will give 1. So, this produces this similarly of course, you can do the reverse by putting a 0 1 1 0 0 on the down. So, you can convert down to up and up to down. And the 2 states are Eigen states of can be distinguished this matrix 1 0 0 minus 1, which on 1 0 will produce plus 1 0 and will produce minus 0 1.

So, if you apply this matrix on this or on this, you can distinguish the up spin and down spin as Eigen states of this matrix 1 minus 1. So, we have a fundamental set of 3 matrices 0 1 sorry 0 0 1 0 0 1 0 0 and 1 minus 1 0 0. Now with good mathematical sense you will say why do not I make some symmetric matrices out of this. So, equivalently what happens if I symmetrize I get 0 1 1 0 which is sum of these 2 divided by 2, but the minus 1 if I make I will get 0 say minus 1 1 0, but I also wanted to be a hermitian matrix. So, I make it minus i i 0 it is simply i times the difference and then there is 0 1 0 minus 1 ok. So, these are the equivalent hermitian set and what is nice about this is that these 3 hermitian matrices exhaust the basis for unitary matrices in 2 dimensions.

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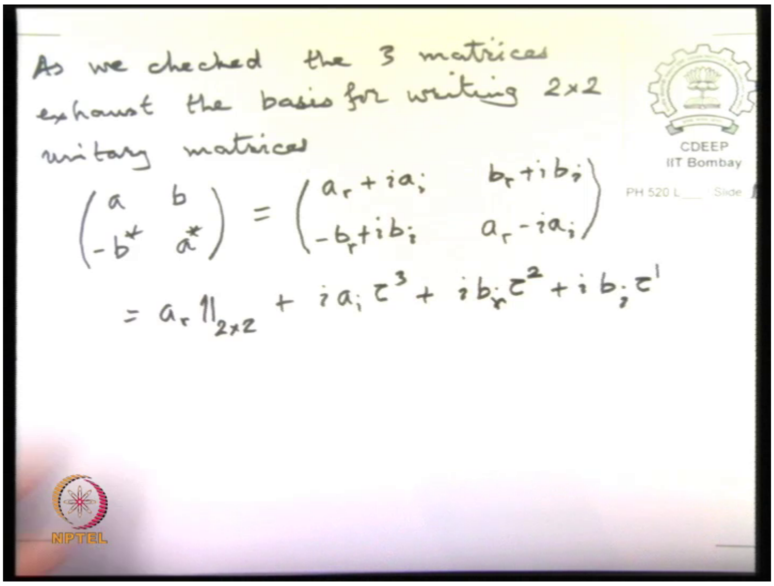

It, is quite a remarkable thing, because we had this a a star and b and minus b star this is how we could characterise a unitary matrix. And, then that we wrote out as equal to a r plus i a imaginary part b r plus i b imaginary part minus b minus i b imaginary part sorry with plus sign now. And a r minus i a i. So, it just became equal to a r times identity and plus this a i times i a i times tau 3 plus i b i times tau 2 b r times tau 2 and plus i b i times tau 1.

So, we have 3 hermitian matrices, which help us to understand the quantum mechanics of spins which obeys linear superposition principle and has this property that only Eigen values are returned in as a result of measurements. And amusingly the 3 hermitian matrices are the basis for the corresponding unitary matrices this is an accident of 2 dimensions ok.

But, a very significant 1 because helps us to do lot of things. So, we have seen the origins of both the types of representations in last 5 minutes maybe I tried to therefore, explain to the topology of SU 2 and SU 3 which I find the most most imaginative exercise usually not done in physics books, but done in Schiff's book.

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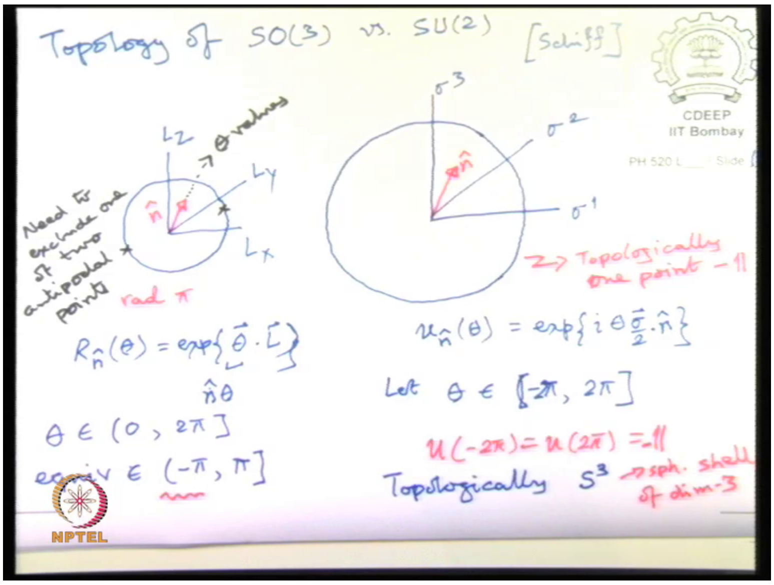

Topology of SO 3 verses SU 2 the point is that so, we will try to draw these 2 spears and this will be SO 3, and although I promised you that topology has no sense of distance in it we will draw double the size here. So, we want to represent something like R n cap theta is exponent theta times theta dot l ok. So, for simplicity just take this so, suppose you drew axis corresponding to L x. So, these are not real space axis L y and L z then the n cap vector.

So, this theta is often written as n cap times theta, n cap is a vector in this space it has composed. So, it has dotted with l right. So, you can dot it with the L x L y L z. So, this is n cap and the rotation amount theta you can show by drawing along it how much you rotate. So, theta values, but they stop at theta equal to pi because if you do a if you do A n cap rota pi rotation and minus n cap pi rotation you have actually covered 2 pi rotation about that n cap axis.

So, by convention so, you can take theta equal to so, theta belongs to let us say 0 to 2 pi equivalently belongs to minus pi to pi ok. And so, that is what we do we make a ray a ball of radius pi. Now, to make SU 2 we draw the sigma x sigma y sigma z axis for the generators of that. So, Schiff has the discussion in his very elegant language, it is so, compact that you will mess it if you have flipping through the pages and has no diagram.

So, put n cap here we have u n cap theta as the exponent of i times, theta times, sigma by 2 matrices dotted with n cap. Except that now theta goes from 0 to 4 pi. So, we let theta belong to minus 2 pi to 2 pi and actually it does not matter. So, we just put minus 2 pi to plus 2 pi. So, including the outer surface of this spear ok, the big difference is that in SO 3, we have to exclude the minus pi, because it will reproduce the same thing as plus pi. I have rotated by pi around this way if I rotate it minus pi I would have come to the same point. So, to not to repeat topologically have a unique set of points I have to leave out this, but as far as SU 2 is concerned both minus 2 pi and plus 2 pi give 1 just gives the same element.

So, there is no ambiguity. In fact, the SU 2 is a space such that the whole of I do not want to clutter up this image, but the whole of the outer surface is 1 point. So, you have to think of so, if you think in terms of forget about n cap because it takes various values, but you let theta increase you get a sequence of 2 spears, until you reach the outer most 2 spear, which is corresponding to 2 pi, but that whole spear is actually just 1 point. Topologically is the same thing the connectivity is such that all of these points are actually just 1 point, they are not distinct point. It is just limitation of our visualization in 3 D. In fact, drawing it on 2 D, that we think of this thing as distinct points, but mathematically they become just 1 point in the group sense they are 1 and the same object.

They are same element of the group it is identity element minus 1. Thank you, identities at the origin is minus the identity. Is topologically the same point minus the identity on the SO 3 space the antipodal points will be corresponding points and you have to leave out 1 of them. So, the punch line topologically this object is S 3, it has the spear of 3 dimensions, which we cannot normally visualize within 3 dimensional space.

And the analogy is the following if you had a disk you draw on it circles of growing size until you reach the outer most circumference, but you identify circumference to be 1 point, then what will you get it is like travelling from south pole of a 2 spear ball to north pole you have made the outer you folded it up and it is like some of the sweets made you take this and also make it into a same point. So, topologically you have created a 2 dimensional shell; out of a 2 dimensional disk. This thing that we did is a sequence of 2 spears such that the outer most 2 spear is joint to be 1 point. So, it is a 3 dimensional shell in 4 dimensional real space ok.

So, it is topologically what is called S 3. So, S 1 is circle S 2 is spear spherical shell in 2 dimension and S 3 is spherical shell of dimension 3, it is intrinsic dimensionality is 3 just as the intrinsic dimensionality of a shell is 2. So, 2 spear although it occupies 3 dimensions it is internsic ant walking on it only detects 2 dimensions. So, an ant walking on this will detect 3 dimensions, but it will be a continuous and homogeneous space, you will not find any joint anywhere. So, topologically the S 2 is actually in S 3 whereas, topologically SO 3 is not and it is a slightly more complicated space, but what happens because of the double covering is that SU 2 double covers this. So, there are 2 copies of SO 3 inside SU 2 and after that covering it becomes a simply connected space SO 3 is not simply connected, but SU 2 is.

So, we will see that next time.