Theory of Group for Physics Applications
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Lecture – 35
Group Algebras; SO(3)-SU(2) Correspondence – I

So, we have been discussing SO 3 and SU 2.

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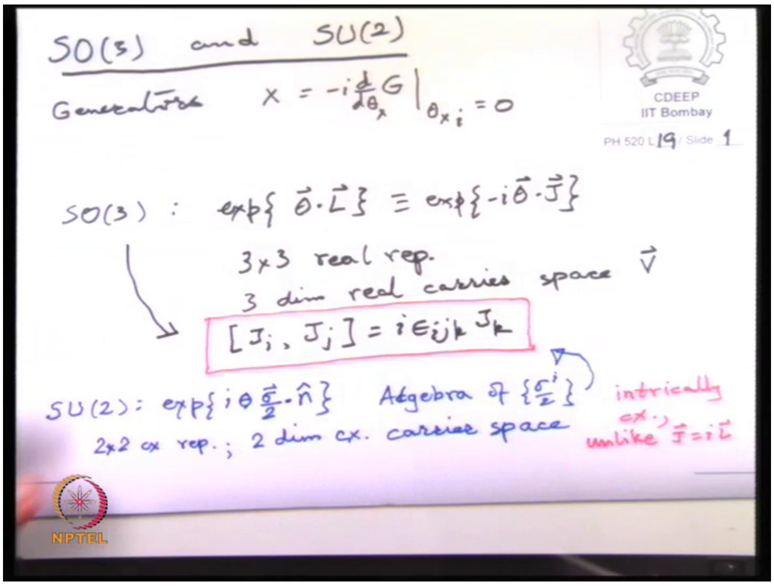

And, we defined generators as infinite decimal version of the group element near the origin of the parameter space. So, I just want to say again that we are dealing with continuous groups, which means that the elements of the group are quote parameterized by values from a continuous set ok.

And, what is meant by continuous set is a set, which at least in some neighborhood is isomorphic to r n. The n dimensional real numbers and therefore, it has all the connectivity and continuousness properties of r n, we have not said anything more than that as far as parameters are concerned and this is what is called topology.

It, just talks about connectedness of points the set, the parameter set is not supposed to inherit the Pythagorean distance of r n; that is not really required. In fact, we will find that for many of these parameter sets the notion of distance is not good in fact, it will hinder then rather than help.

And ultimately the therefore, the whole connection is that if you have understood the continuum nature of r the real line. The r n inherits that continuous property and all the parameters sets we are talking about inherit that continuousness. So, this continuum property is as we discussed Kantor Dedekind all these people who generalize from rationals to continue to the whole real line r and so on.

So, this is the idea of this is the specific sense, which we mean continuous sets or continuous groups parameterized by sub sets. So, we had this SO 3, which exponential theta dot l we also said that by some clever trick we could also write it at e raised to minus i theta dot J, where J are Hermitians. So, this turns out to be a 3 by 3 real representation. So, as we know the minus I insertion is essentially fake it is all real, but it has with it a 3 dimensional real carrier space, the vector space V of physics write in 3 dimensions.

And most importantly these obey the algebra J i J J the generators, which generates the group obey the algebra j i J J equal to i epsilon i j k J k, and I will put this in a red box and I am running slightly out of space below I wanted to display both SO 3 and SU 2 on the same which we will try to do it now.

So, SU 2 basically was also exponent of and we learnt I times theta times sigma dot sigma by 2 dot sum n cap it has this form, the generic SU 2 elements can be written out in this form this we checked by checking I will come I will just remind you how that came, but the point was SU 2 can be written like; this is a 2 by 2 complex representation because it is a 2 by 2 matrices sigma r 2 by 2 matrices you exponentiate them it is a power series in powers of 2 by 2 matrices. So, it is also 2 by 2 matrix.

So, this is 2 by 2 complex representation. In fact, and it has a 2 dimensional complex carrier space right, it has a 2 dimensional complex carrier space. The beauty of this construction is that the sigma by 2 matrices obeys exactly the same Algebra as the j s is exactly the same.

And unlike the j which were sort of fake complex because there was just i times some real matrices the sigmas are essentially complex, because 2 are the matrix is are actually real, sigma 3 and sigma 1 matrix are 1 minus 1 1 1 of the diagonal, but the sigma 2 matrixes i minus i. So, it not matter of just multi of multiplying all of them by some complex number. So, this is happen to be intrinsically complex unlike J equal to i times l. So, main conclusion we now make is that both SO 3 and SU 2 share the same algebra.

So, this leads to some interesting questions what does SO 3 do verses what does SU 2? And, we what we are next doing to try to prove is the Homomorphism between SU 2 and SO 3.

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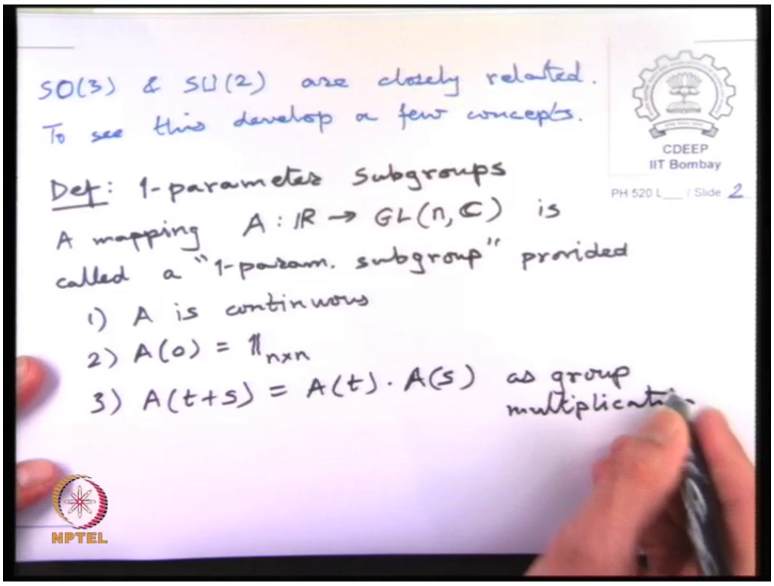

So, to see this we develop some specific concepts 1 of them is the idea of a 1 parameters of group. So, 1 parameters of group is. So, we say that a map or a yeah a mapping a from reals into G L and C as we all agreed most of the thing we are going to do are essentially matrixes group.

So, we see specifically in context of this the is called a 1 parameters group, we have already used this term before, but now we have, now define it specifically here provided A is continues, A at 0 is equal to the identity element right because a is the map from reals to G L n C. We ensure that when you are at 0 and the real axis you map into 1. And finally, A of some t plus S is equal to A t times A S by group multiplication.

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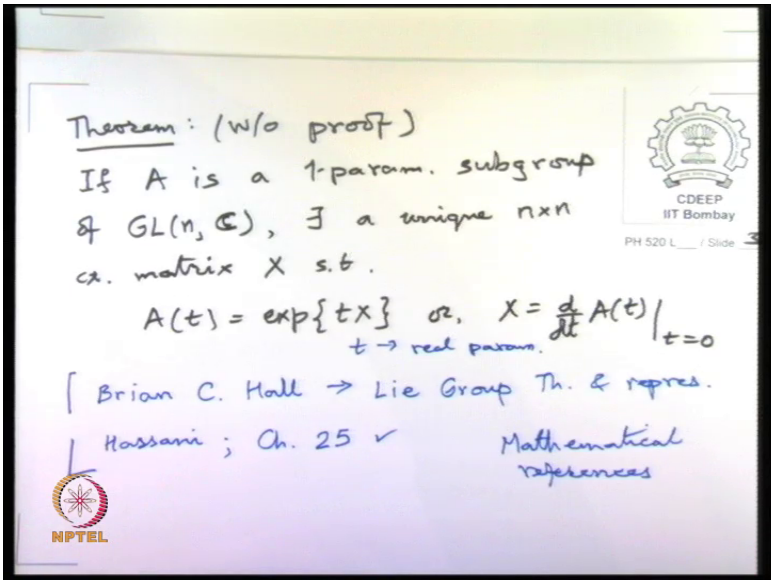

So, the next thing we say is that if A is the 1 parameters of group then there will always a exist A unique matrix X, then basically there is a unique X. So, that you can recover all of a has a exponentiation f that X, such that A of t is equal to exponent of t X or X is d by d t of A of t, mathematicians do not like to insert i s at this point and this is they come from mathematics book.

So, I should tell you that to develop this course I have gone between trying to be formal and mathematically regresses, but also not lose track of the physics. Many physics books will just tell you all the geo (Refer Time: 12:20) physics things, without worrying to much about the Math's. Whereas people who actually begin to worry about mathematics of lie groups gets some into doing that because it quite involved.

So, if you want to strike a middle course an interesting mathematics book is by Brian C Hall this is lie group theory and representation and I will be living in this two mind the whether to do more just computational and physics oriented things or try to bring out the basic theory of lie groups. The actual theory of Lie Groups does get quite complicated mainly just becomes it does become complicated, the other book we follow of course, continues to be Hassanis that I think a chapter 25 if I not mistaken.

So, in Hassanis book chapter 23 is descried groups then chapter 24 he jumps on to manifolds, which an differential geometry, because as I said the little speech I gave earlier about the continuum and what is meant by continuum. So, to get it all out of the way he launches full fazed in to differential geometry and in the next chapter he comes to this. So, it becomes slightly difficult to adopt his chapter completely, because he is reline on some of the things he has developed in the previous chapter, but if you can skip by or believe some of the theorem that are done later then that is also a good reference.

So, these 2 have the mathematical references and interestingly Hassani has exercise number 20 you, now in the last exercises where he gives the whole recipe for the what I call the complicated theory of lie groups. It given as an exercise because it is too difficult to really do it in the chapter and he probably feels unfair to leave it out. So, he as just the a sophisticated way of introducing it and which realize on chapter 24 terminology. Brian C Hall tries to strict a balance in trying to not go as deep into all of differential geometry, but develop just some definitions and theorems, which will take care of the essentials of lie group theory.

What, I will say is that because I am doing some all this is to say I am gong some selective presentation. And you may have to reread back and forth to see what is the whole picture, because I am not able to give the full picture if we gave that then it will become a complete mathematics course ok.

So, I just want to so, what is good about mathematics books is that the bring out what are the assumptions? What are definitions, which is means we have to propose those ideas and then, what are theorem that can be proved?

So, these things are separated out instead of it all being 1 package has done in physical reasoning. So, Brian Hall does in fact, I read some discursion in various for that is say Brian Hall book is the good is in fact, is a good balance between very regresses books and lie group theory and something that physicists can use?.

So, Brian hall develops this idea of 1 parameter subgroups and basically as a definition. And note the most important thing here is this, which is rather ambitious to say that you have an arbitrarily parameterized space even 1 dimensional, but such that the value of a at some of parameters is exactly group multiplication of the corresponding this is 1 of the most important part of the.

So, we define 1 parameter sub group like this, which has I said the most mischievous property number 3, which takes care of lot of the complications by insisting on some very big connection.

This is actually at the heart of lie group theory, that the structure of the whole groups get define by what happens near identity element. So, now the claim is that if you have a 1 parameter sub group, which satisfies these things then you can always get it has an exponent of some code generator times a real parameter t so, with t a real parameter.

The next thing is that the next thing is the definition, which is again a loaded definition you know anticipating what we want to prove.

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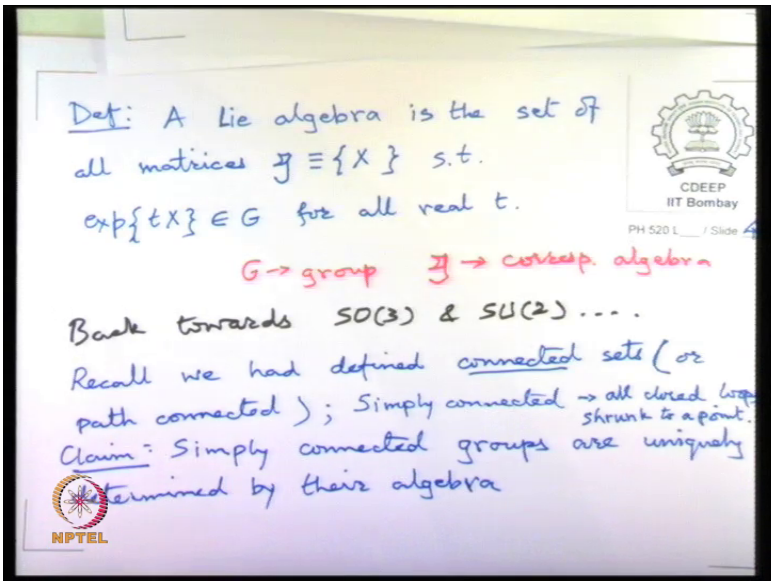

A Lie algebra lie is a set of matrix is X is the set of all matrices is and I am sorry to use these gothic g equal to a list X such that, e equal exponent of t X belongs to G, for all real t. So, this gothic small g is the as a strong correspondence. So, G is group and g is the algebra.

So, algebra is something you exponentiate to get the whole group. This definition looks rather inaquest, because it simply insist that lie algebra is the set of all possible g all possible matrices is X. So, that exponentiating them you get elements of G, it does not talk about any kind of exhaustivity it does not say all of G is exhausted by X in this way, but the properties that we define for 1 parameter groups and so on will than lead to that kind of exhaustiveness.

So, some of the other theorem I will postpone for the later, but right now we just work with this definition of A 1 parameter group the idea that A 1 parameter group can always be got from an element X and the exponentiation. And if you took all possible such X in the group, than you will be able to recover then they I mean we define to be the algebra all the matrices X, such that exponentiating them we get elements of the group G.

Now, with this we come back to our SO 3 SU 2 discursion well not immediately, but some are back towards SO 3 and SU 2. And, now we say some statement definition from topology, which is that a simply connected group.

So, we had defined simply connected a sorry we had define what is connected those are the once that are or path connected, where you can connect any 1 point to any other by having the parametric map of the piece of the real line into that set. So, if you can parameterized find the sequence series of points continuously going from 1 point to the other point then we call this the connected set. Now, we claim that simply connected groups are uniquely determine by their algebra.

So, we had define what is connected and then we are all also define simply connected as the once where the pass can be shrank close all close loops can be shrank to a point. So, you can roughly see you what is the relationships between this ideas, because if you if you have a map like this if you have some a of t this t is the variable, which varies continuously.

So, this idea of connected and simply connected goes over to setting ups such sub such subgroups to be able to setups or identifies such subgroups. And so, when a group is simply connected, it would mean that all kinds of 1 parameters subgroups that you can find in it or 1 parameter path you can find in it can be shrank to a point, but shrank to a point will mean that the identities always if you can shrank to any 1 point and if the group is connected than you can always connected to identity.

So, you will be always able to go from any arbitrary region of the group to the identity and therefore, to the structure of the group near the identity which is essentially the algebra. It is d by d group at the parameter value equal to 0, which gives the identity has you remember we define in fact, 1 parameter subgroups to be once. So, that once you set t equal to 0 you get the identity of the G L and C.

So, this is how the various ideas are going to be connected, but as I said you will have to read them later in context. Today I want to go on move on to doing the SO 3 and SU 2 connection. So, much we have said of about what is 1 parameters subgroup, what is an algebra is the set of matrices is that can generate all of the group through exponentiation. And we have said something about all simply connected groups can be uniquely determine by their algebra. Now, we see some interesting connection between SO 3 and SU 2.