**Theory of Group for Physics Applications**
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**Lecture - 34**
**Generators, Discussion of Lie's theorems - II**

Now, what we do is we expanding the where a when the parameters are small. Let me ensure you that public work it backwards ok, not this people knew Pauli matrices before then you SU 2 in general. So, they must have worked out what it becomes with a given Pauli matrix.

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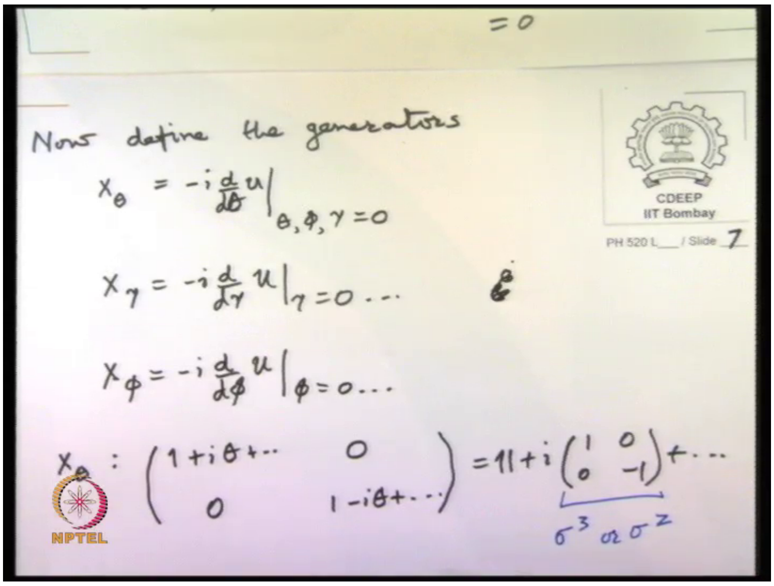


So, we can now see that and I will tell you a very very simple way of guessing Pauli matrices. If you know spin spin up and spin down then it is very easy to see how you get Pauli matrices.

So, now you define the generators at first let me just say j x theta equal to minus i d by d theta of u at theta equal to 0, all the angles equal to 0 theta phi and gamma equal to 0 and similarly. So, x gamma equal to minus d by d gamma of u is just like gamma equal to 0 etcetera and x phi.

In fact, we can do it with other angles being 0 at a time. So, if you look at the e raise to i theta matrix; should I work here or below. So, for x theta instead of doing this differentiation all I can do is expand that it becomes equal to 1 plus i theta plus dot dot dot and the lower component is 1 minus i theta plus dot dot dot and I can set others to 0 for the time being, set other parameters as to 0.

So, this expansion after removing i this is equal to 1 plus i times 1 minus 1 this is called sigma z or sigma 3 matrix.

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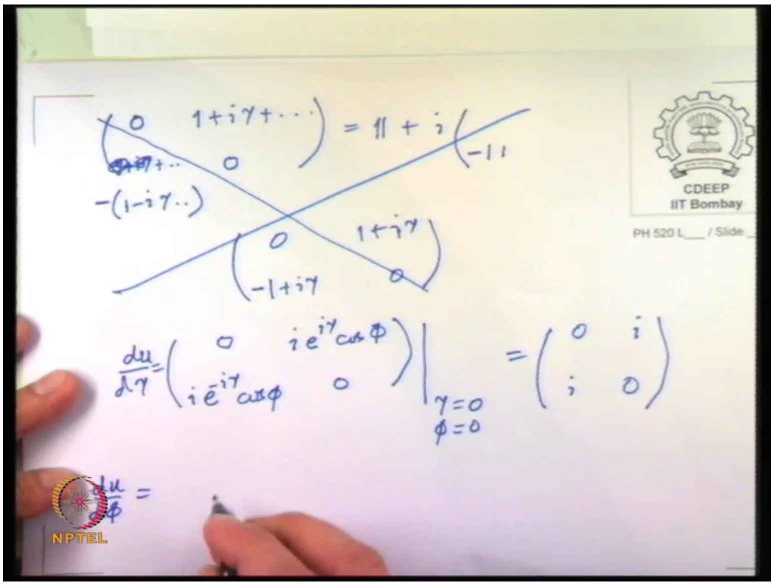


Then let us look at the gamma dependence e raise to. So, is I said at phi to phi to 0 so, that will make these things 0. So, I am looking at this matrix and in it now I am interested in the gamma part that you see. I am interested in determining what happens for small gamma. So, I said phi equal to 0 that does not even require me to worry about theta because that sets this to 0 and these two factors to be 1. So, I am left with 1 plus i gamma plus dot dot and 1 minus i gamma.

So, this becomes equal to 1 plus i times ok; I am not going to get all the matrices after all those thing oh there is a minus sign good. So, there is a this minus sign. So, that gives, but there is a (Refer Time: 04:18) so minus times 1 minus i gamma. So, I am getting minus 1 plus so, I have this is not expansion we expect because we want it to be identity plus something. So, what happens if we do d by d gamma of this ok. So, these terms are not important because there anyway not dependent. So, I get i times so, d u by d gamma is equal to i times e raise to i gamma cos phi ok; that is the way to do it and then when I differentiate this I get i times e raise to minus i gamma times cos phi; so if we evaluate this at and our definition was removing one factor of i after evaluating at gamma equal to 0.

So, if I evaluate this at gamma equal to 0 and of course, phi also equal to 0 then we get the matrix 0 i i 0. So, I make the mistake of the trick that works on the diagonal of course, does not work in this case because you cannot extract a diagonal identity matrix.

So, you have to do the matrix differentiation as a matrix function which means you differentiate each of the elements individually with respect to the parameter. This did not contain any gamma so, I set them to 0 and there sign phi. So, I set them to 0 leaving cos phi open at first and then we get from differentiation this matrix.

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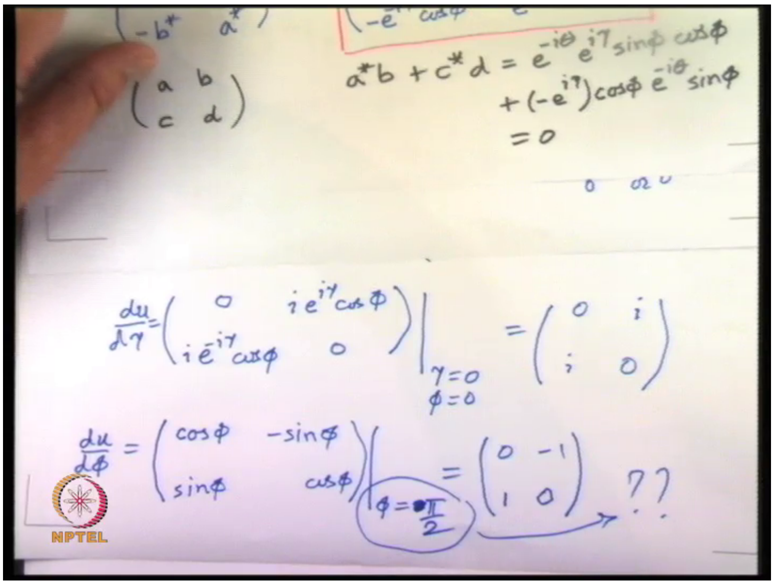

Similarly, doing for the phi there I do not have to worry because now may as well.

So, here I can set theta and gamma 0 from the beginning; it does not matter and then I get d u by d phi equal to cos phi cos phi and minus sin phi and cos phi sorry sin phi evaluated at phi equal to 0 becomes 0 minus 1 ok. So, let me show you the other note that I am looking at.

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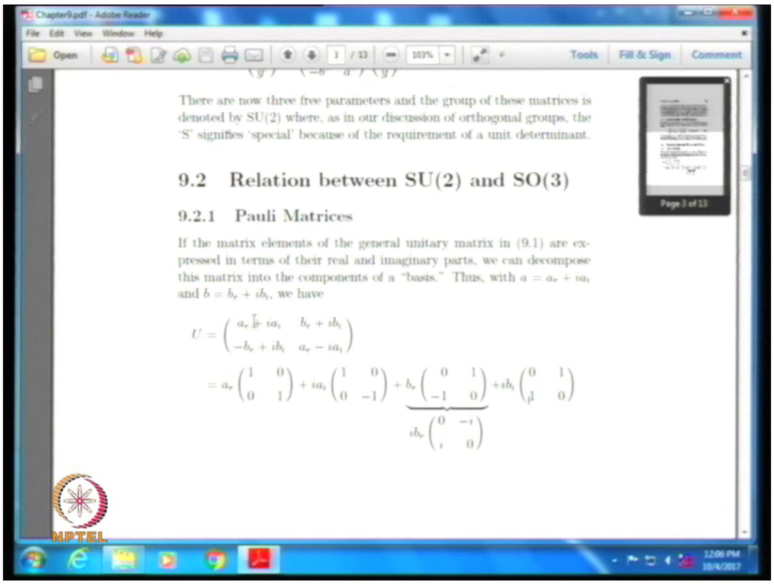


So, this one simply begins by writing out the structure; we wrote a a star a minus b star right, we had a minus b star b a star.

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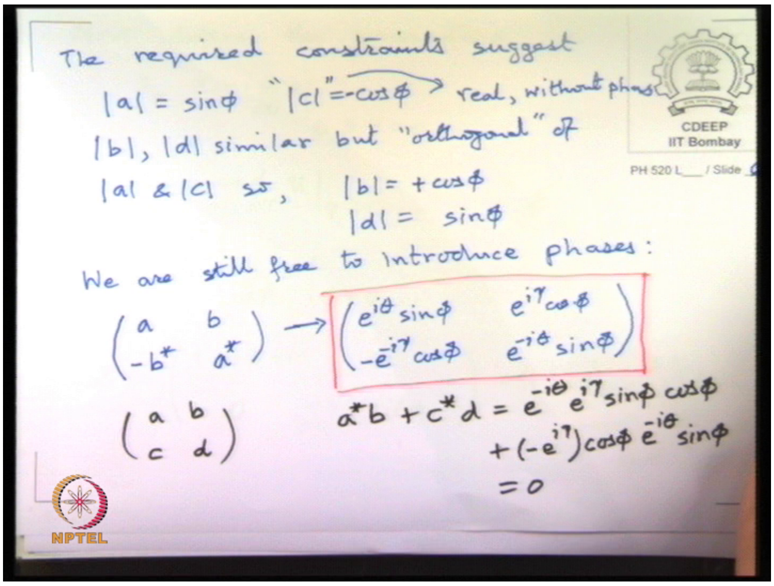


So, if we write out as a minus b star and b a star. So, the right real part imaginary part real part imaginary part; This is a, this is a star, this is b and this is minus b star and now you just expand this in real part of a imaginary part of a making treating them each of them small at a time.

So, it is a r times identity b r times 1 minus 1 i times imaginary part b times 0 1 1 0. So, the unitary matrix not this special unitary because we are not in this we have not put the determinant condition; which is product of this with and this plus a square plus b square equal to 0 as not been imposed. So, the unitary 2 by 2 matrix so, let me over here would the conclusion to this by saying that it.

So, this is giving sigma 1 and this will give me the Pauli matrix sigma 2 matrix provided this is evaluated at pi by 2 not at 0. So, I put the question mark whether this is a good way of deriving it, but certainly this characterization does not go wrong because it is correct. It is not so, good for extracting generators because I want to extract generators by using the point equal to 0.

So, I am sure this is good for some purposes, but not for the purpose we are trying. But we can go back to the that this characterization seems to be good, but it is not so, good for deriving the generator because it relies on evaluating at pi by 2 which is not what we want to do; you can use that convention, but we are not going to use it.

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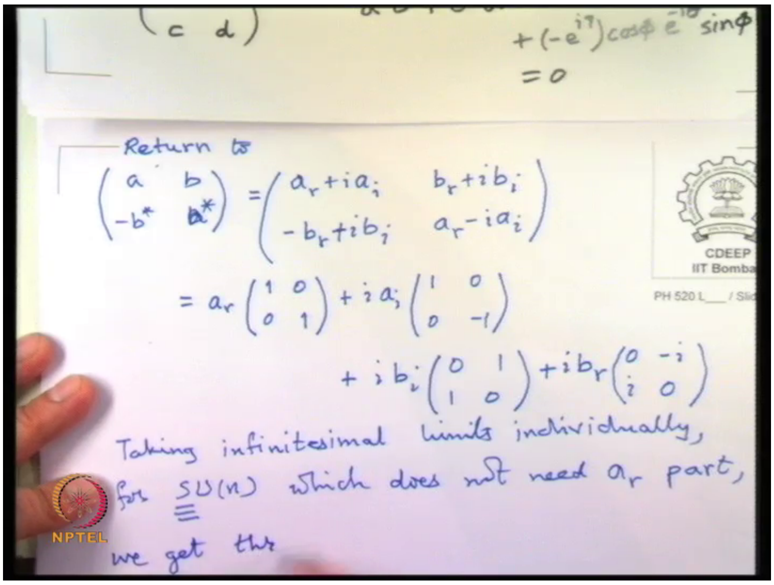


So, let me just go back to the a b convention. So, and write it out as a sorry a a star; write it out as a real part plus i times a imaginary part b as equal to b real part plus i times b imaginary part minus b r minus i times. So, b star will make it minus i, but this minus i will make it plus i times b imaginary part and a star is a r minus i a i.

This can be split into a r times the 2 by 2 identity matrix plus i times a i times 1 minus 1 plus i times e i times 0 1 1 0 and plus b r, but I pull out a i for good luck. So, if we do that then I have to put minus i here i here 0 and 0.

So, I think this is the simpler and cleaner derivation because now in fact, it is not a it is a linear function of these parameters. So, taking infinitesimal limit individually for SU n which does not need a r; we get three generators called sigma x or sigma 1 equal to 0 1 1 0 sigma 2 equal to 0 minus i i 0 and sigma 3 equal to 1 minus 1 0 0.

We need to argue why the identity is not part of the special unitary 1, but subject to that this is these are the generators we get.

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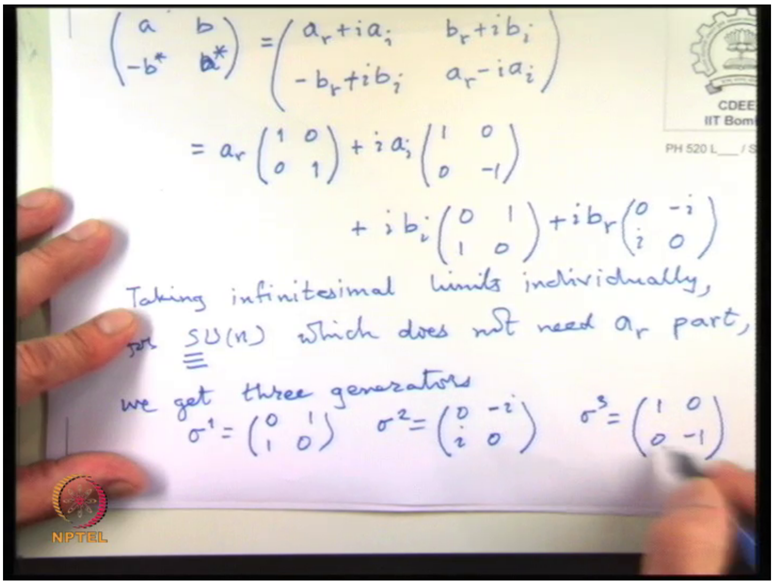

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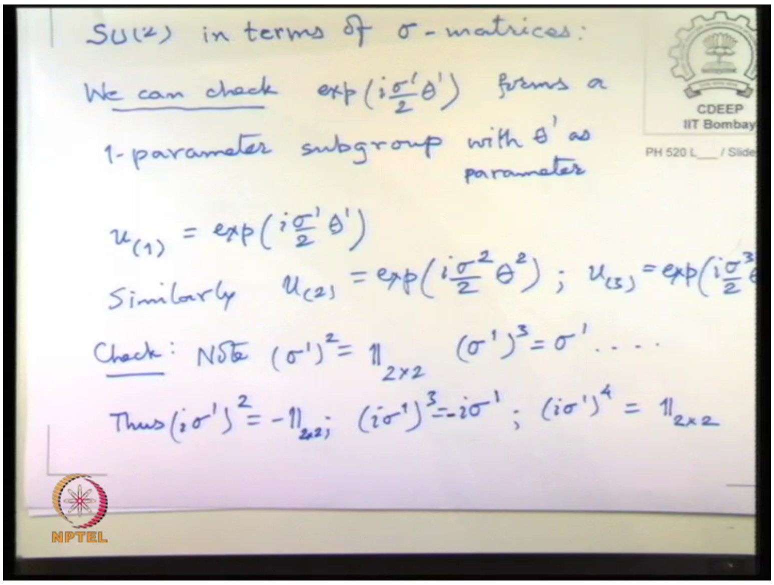


Now, the interesting part of this is that we can check exponent i times sigma 1 by 2 times theta 1 forms a subgroup. We can see that it forms a subgroup because as i for with theta 1 as parameter. So, this we may call u x or u 1.

So, and I will tell you why I am putting a half ok; the theta is some parameter I will I will use the parameter theta by 2 for a reason that we will quickly explain. Similarly, u 2 equal to exponent of i sigma 2 by 2 times theta 2 and u 3 equal to exponent i sigma 3 by 2 times theta 3.

So, we can check business can be done easily because all we have to do is take square of sigma 1 squared is equal to identity matrix. And so, sigma 1 cube is sigma 1 and so, on; and so, sigma 1 to the 4 is back to being 1 and so, on. So, if I take exponent of i times.

So, when I have substituted an i in it sigma 1 squared with i i with i squared will give minus 1 i time sigma 1 cube will give, ok. So, thus sigma i sigma 1 whole squared is equal to minus 1 and i time sigma 1 cube is equal to i time sigma 1 minus i times and i sigma 1 to the fourth power returns to being i returns to being 1.

So, the power 2 gives minus identity and power 4 gives plus identity, third power gives minus times the original matrix. So, this is cosine and sine series because alternating in the even powers and also alternating in the odd powers.

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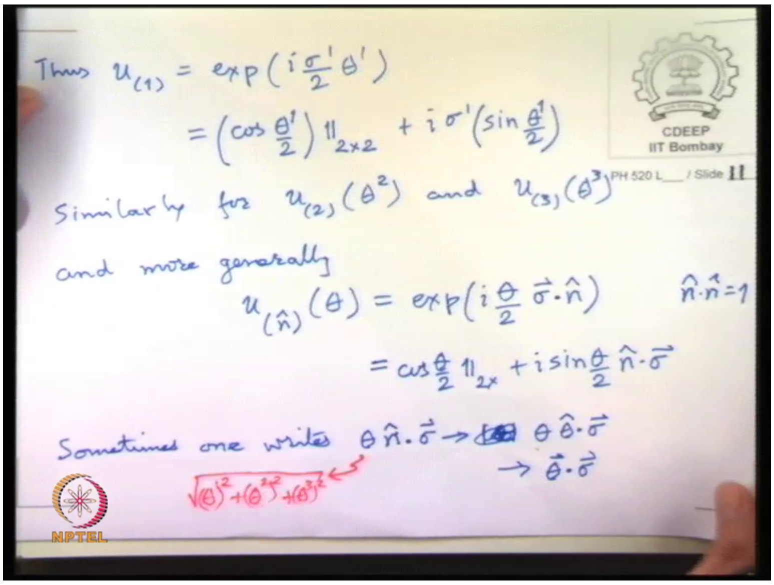


So, we get that now, writing very very important formula and I chose for some strange reason parameter theta by 2. So, we will stick to it for the time being and we will later see why we pick put the half factor. So, this times the 2 by 2 identity matrix and plus i times sigma 1 by 2 say i time sigma 1 times sin theta by 2.

So, the half factors we will go with the theta parameter because sigma is themselves are just the squares are ones or back sigmas. Say same thing holds for u 2 with theta 2 and u 3 with theta 3 and more generally u and I will put now subscript n cap of some theta with the convention we put theta that we put theta. So, let us put it down exponent i times theta by 2 times sigma dot n cap ok; where n cap is a unit vector.

So, if you exponentiate this combination what you have to do is check the sigma dot n cap matrix for its square and for its cube and so, on. It will have exactly the same properties that we saw so far and will give cos theta by 2 times identity plus i sin theta by 2 times n cap dot sigma ok.

Sometimes people also write the theta times n cap dot sigma is converted to be as if it is modulus I should not say modulus because that is confusing things, but theta times a theta cap call the n cap to be as if it is the direction vector of some vector called theta and so, people write theta dot sigma as well; where theta is now directly a 3 entries.

So, this is complete misuse of vector notation because theta 1 theta 2 theta 3 do not form a vector in any sense by any stretch of imagination, but there are 3 parameters theta; this is all it means. There are three parameters theta 1 theta 2 theta 3 and you could have just taken theta 1 time sigma 1 theta 2 time sigma 3 sigma 2 theta 3 time sigma 3; then extracted the modulus of them.

So, in this language theta becomes equal to square root of theta 1 square plus theta 2 square plus theta 3 square. So, that is another way of thinking about it, but it is a bad one because there is not any kind of vector is a set of three parameters.

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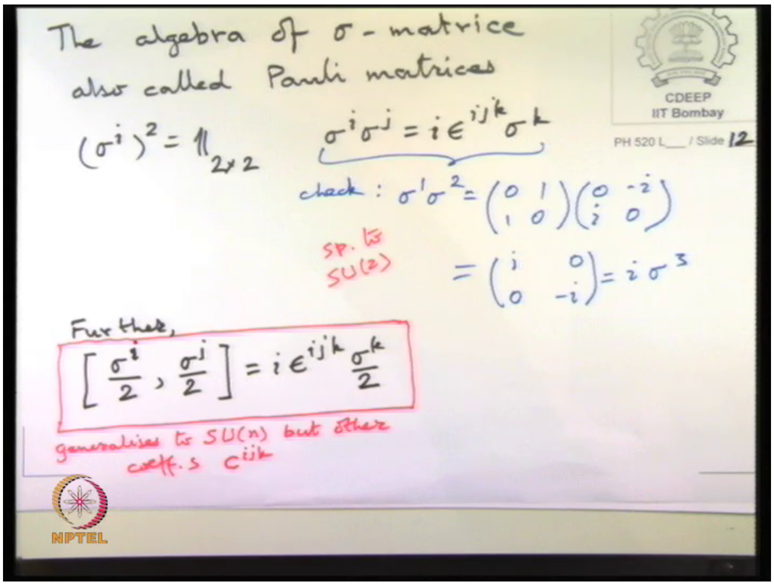


So, now, we come to the interesting punch line of all this is that the algebra of sigma matrices which are also called Pauli matrices and in many books sigma matrices are written tau matrices u symbol tau. I will tell you when to use tau and when to use sigma.

So, also called Pauli matrices and the algebra is clear any of the sigma i square is equal to identity matrix, but more interestingly sigma i sigma j is equal to i times epsilon i j k sigma k. This is the special property so, we check this. If I take sigma 1 sigma 2 that is equal to 0 1 1 0 time 0 minus i i 0.

So, this becomes this row into this column i, this row into this column is 0 this into this is 0, this into this is minus i. So, it is equal to i times sigma 3 matrix. So, you can cyclically check this for sigma 1 sigma 2 and sigma 3 and you will essentially get this epsilon so, cyclic property. But this also, but this is in this is a special property of sigma matrices which does not generalize to SU n ok, this is special to SU 2.

But what generalizes to SU n actually follows from this or is related to this which is the commutator algebra. So, if we take sigma 1 or sigma i and now I insert my favorite factor 2 of the other. So, we can check this; if you like. So, can be generalized to SU n, but other coefficients C ijk ok; not epsilon ijk, but some other coefficients, they are called structure constants of the group.

And this you want to check it, you can check it the actually. It just requires multiplying 2 by 2 matrices. So,, but the half factor is important without which you will not get the, if you do not put half then you have to modify this coefficient by a factor 2. You can multiply both side by 4, if you multiply by 4 because it will cancel these 2s cancel and 2 will be sticking here.

So, the constants will in involved this arbitrary 2. So, people prefer putting it here. So that, then they are like normalize the epsilon ijk matrix is just either 1's minus 1's or 0 so, it is unit kind of size. So, this is that convention and that is what we put in the that is why we put theta by 2 here as well. So, actually the half factor belongs to the sigma and not so, much to theta.

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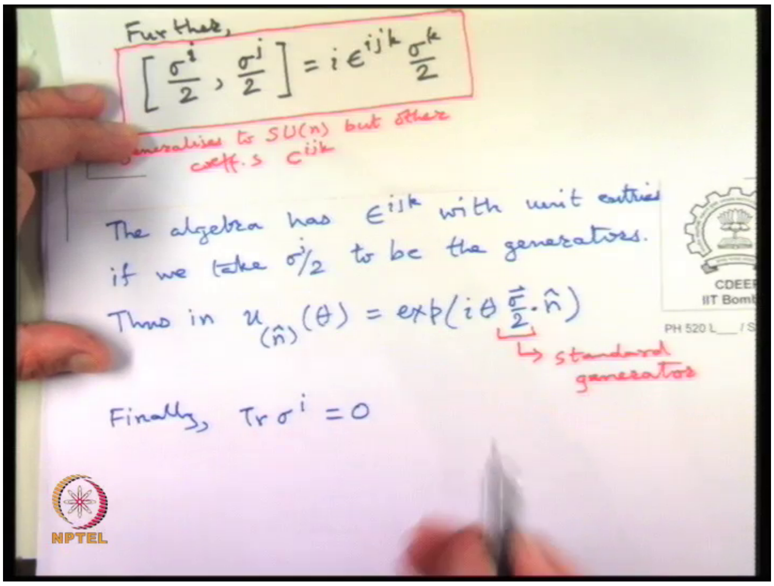


It is better to think of it as exponent i times theta times sigma by 2 dot n cap; so, this x as the unit, this x as the generator. Sigma matrix is also have the additional property that about the traces, trace of all of the any of the sigmas is 0 is on the diagonal in this famous property of Pauli matrices. And the trace of sigma i sigma j is going to become equal to trace of the epsilon ijk is symbol.

So, I can take it out; it will become trace of sigma k. So, this is equal to 0 unless i equal to j.

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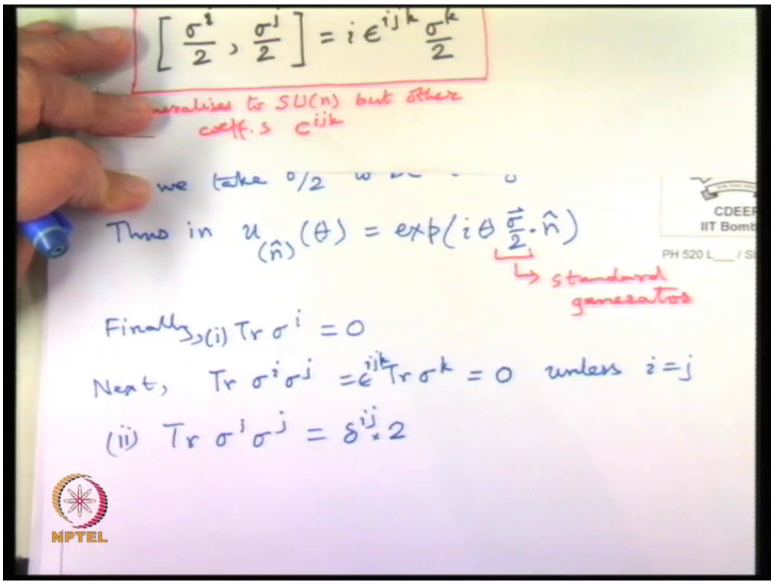


So, trace of sigma i sigma j is equal to delta i j times 2 because when i equal to j. So, the finally, applies to these two properties of traces 1 and 2. Trace of individual sigma is 0 and if you take trace of a product then i is not equal to j; if i is not equal to j we get trace equal to 0 and if i is equal to j then it becomes identity matrix in 2 by 2 matrix. So, it will become 2.

So, now, we see the analogy of this with the rotation algebra it is exactly the same. The algebra of the sigma by 2 matrices is exactly the same algebras the j matrices.

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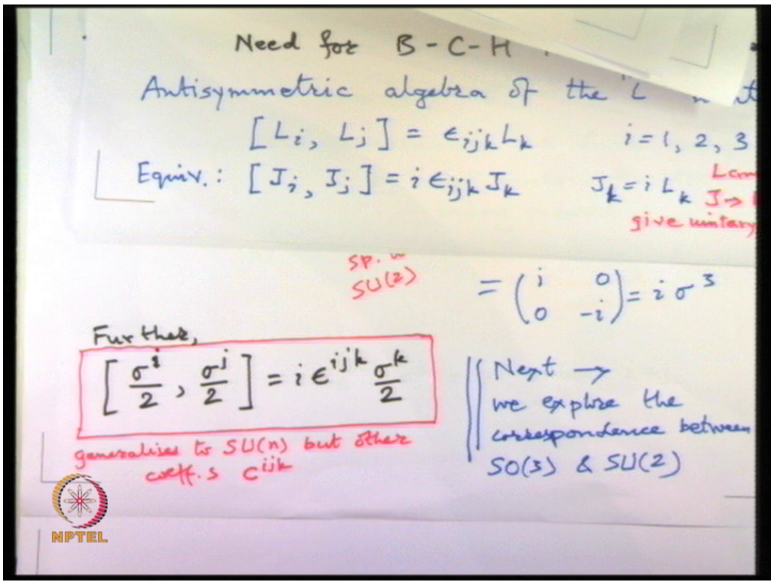


So, we will explore the correspondence between SO 3 and SU 2.

So, that is for next time.