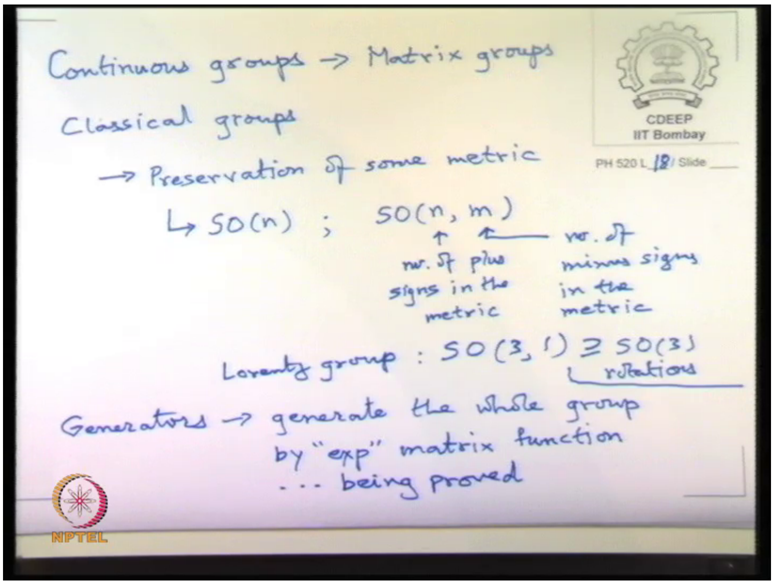
Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 33
Generators, Discussion of Lie's theorems - I

So, we are in the second half of the course and where we are looking at continuous groups.

(Refer Slide Time: 00:23)



And we began to look at the so, called classical groups. So, in continuous groups I can write essentially matrix groups, sets of matrices with well-defined group properties. And, then we are looking at classical groups and then we characterize the classical groups by preservation of some metric.

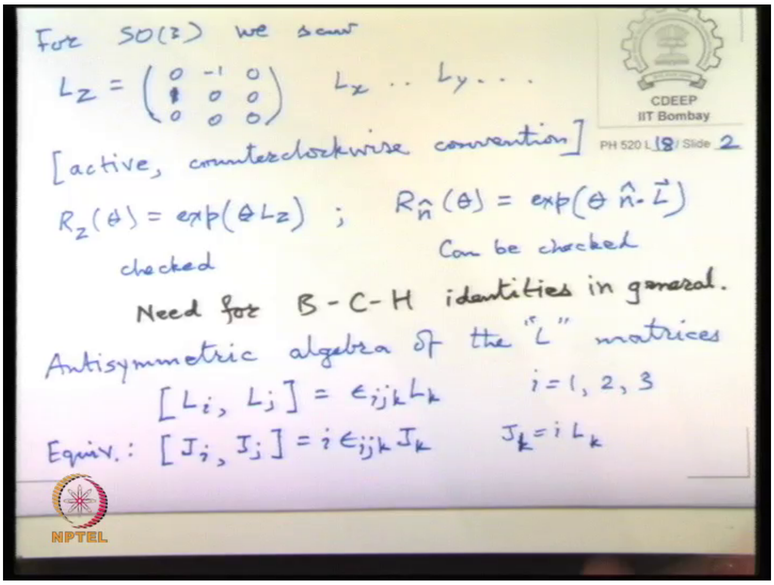
Of course, so right so, if we go down the chain of preservation of some metric, then we get the S O n groups; the real groups that preserve Pythagorean metric of n dimensions or we can also get S O n comma m, where the Pythagorean metric has n number of plus signs and m number of minus signs, in the metric and number of minus signs in the metric.

So, from this point of view Lorentz group becomes S O 3, 1. And the rotation group is it subgroup it is the first part S O 3 without fourth axis. So, it is just it properly contains S

O 3, which are rotations. We will see more about this, what we did next was to look at the generators.

So, these are matrices in terms of which we can generate the group by exponentiation, where the exponential is now a matrix function, but it is defined exactly the same way by using the power series expansion. And we have not seen all of it so, this being this is sort of being proved ok. We are gathering the evidence that this is possible to do. So, we already saw that for S O 3 we had the generators.

(Refer Slide Time: 04:10)



We saw that we have L Z equal to 0 0 minus 1 0 0 0 0 1 0 0 0 0 0 and similarly L into L y there you can reme work them out and there was some philosophy about active rotations and so on.

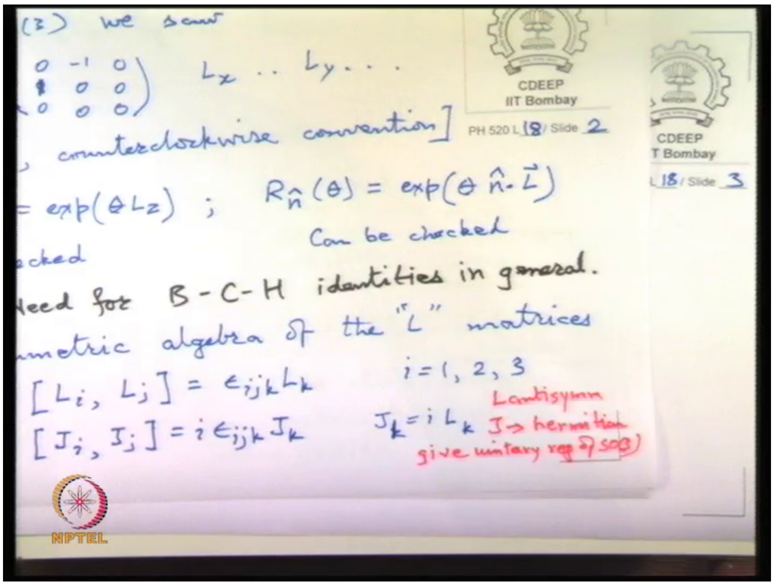
So, this just defines the convention in which we did it, because you can get the signs opposite, but then you have to change signs everywhere and they may corresponding to what you may call for passive counterclockwise or you may call them active clockwise etcetera.

Know that other thing which act is that so, we check that R Z theta was exponent of theta times L Z and in general R n cap with an angle theta can be written as exponential of theta times n cap dot L, where L is short form notation for the 3 matrices L x L y z we write it as if it is a vector, but physicists will understand this notation right away.

Now, what we are doing next is that we are looking at SU 2 ok. So, so, this all we worked out explicitly we took a z axis rotation and then we made an expansion and then check the these are infinite decimal generator we checked this. And we can see the reasonableness of this we have not actually proved, but it can be proved and we saw that need for baker Campbell. So, this can be checked this is checked and need for baker Campbell Housdorff identities in general.

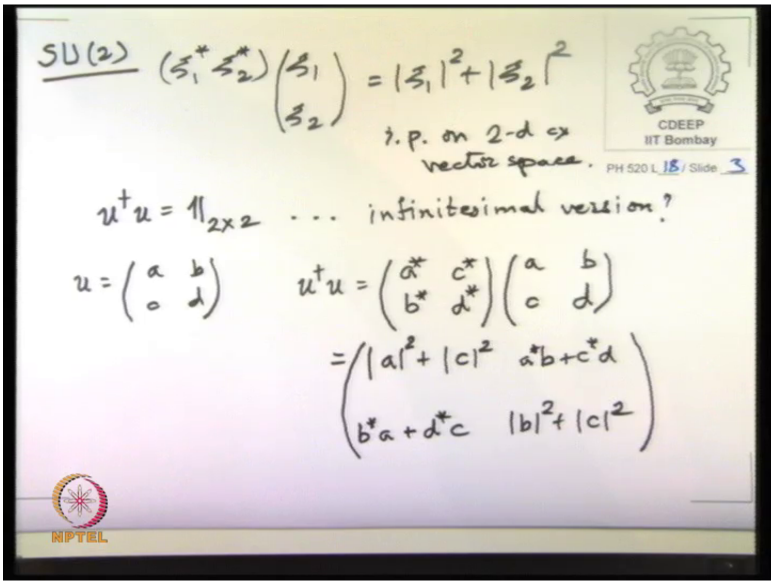
So, we will see more about it a little bit later we what we do next is to look at the group SU O and certainly most importantly this satisfy the algebra. So, the Antisymmetric algebra of the L matrices, which is L i, L j equal to epsilon i j k L k, where i j k equal to 1 2 3 instead of writing x y z. Equivalently we can introduce J i J j equal to i epsilon i j k J k where J equal to i times or J k equal to i times the corresponding L k. So, we complexity the algebra, but then the j become Hermitian generators. So, L are Antisymmetric J are Hermitian

(Refer Slide Time: 08:43)



So, the fact that J is hermitian will make it is exponential unitary matrix. So, it will give a unitary representation of S O 3.

(Refer Slide Time: 09:23)

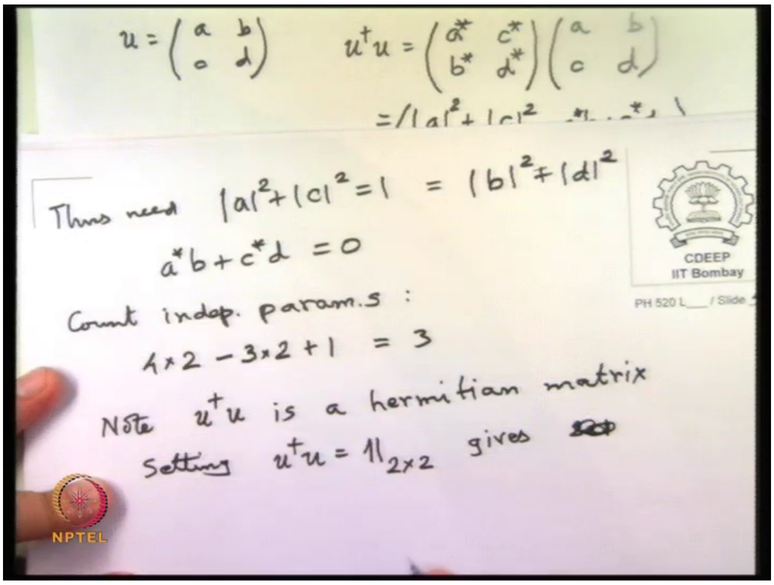


We come to SU 2. So, as you remember these are the matrices that preserves the metric on the complex vector psi 1 star psi 2 psi 1 psi 2 star, this is the inner product on 2 dimensional complex complex vector space.

So, in any case the condition on the unitary matrices was that u dagger u has to equal 2 by 2 identity matrix, we want to characterize this matrix the same way we characterize the rotation matrix. So, let. So, we want infinitesimal version. So, let u be equal to a b c d this is the most unimaginative unimaginative, but effective way of working it out.

So, you can imagine what is u dagger u going to be when you start writing down, now this has to be said equal to 1 identity matrix.

(Refer Slide Time: 12:05)

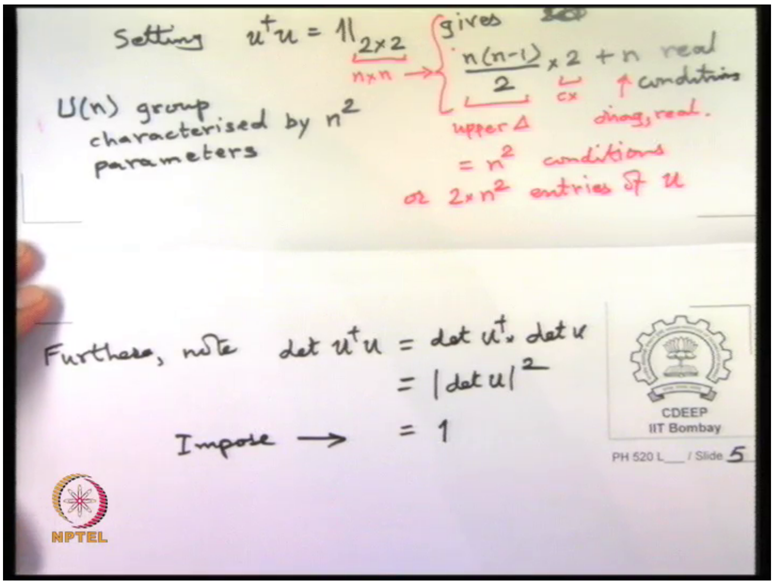


So, it requires that so, we need and if we examine the off diagonal elements, then we see that a star b plus c star d equal to 0, the other element is just the complex conjugate of that. So, there is if this number is 0 then it is complex conjugate will also be 0. So, essentially we got that many conditions. In fact, for SU n we can do this calculation.

So, we can do accounting here we started with 4 complex numbers count independent parameters, there are 4 into 2 to begin with, but minus 3 real conditions. The so, there are also because the generic matrix is complex. So, they appeared like there are 6 independent conditions because there are 3 conditions on complex numbers, but we have to restore it to the fact that the fact that it is unitary already reduce, it by 1 so, plus 1 ok.

So, it becomes a 3 independent matrices. So, the reasoning is that u dagger u is a Hermitian matrix right. So, setting this equal to 1 in this case 2 by 2 basically gives 3 independent conditions, because 2 into 2 minus 1 sorry and any in 2 and minus 1 by 2 gives 3 independent conditions, that 2 elements on the diagonal and 1 off diagonal is 2 and this, but the conditions on the diagonal are automatically real ok.

(Refer Slide Time: 15:22)



So, this gives n into n minus 1 by 2 into 2 plus n real conditions ok. So, suppose this is if you take the case and this is general n by n then, because it is a Hermitian matrix the you need to consider only the upper diagonal upper triangle then the diagonal. The upper triangle has n into n minus 1 by 2 entries because it starts with n minus 1 goes up to 1 right. So, it has n into n so, this is upper diagonal entries upper triangle I am sorry and because of complex numbers there is vector 2, but the diagonal does not give complex conditions because the matrix itself is Hermitian the diagonal gives only real conditions.

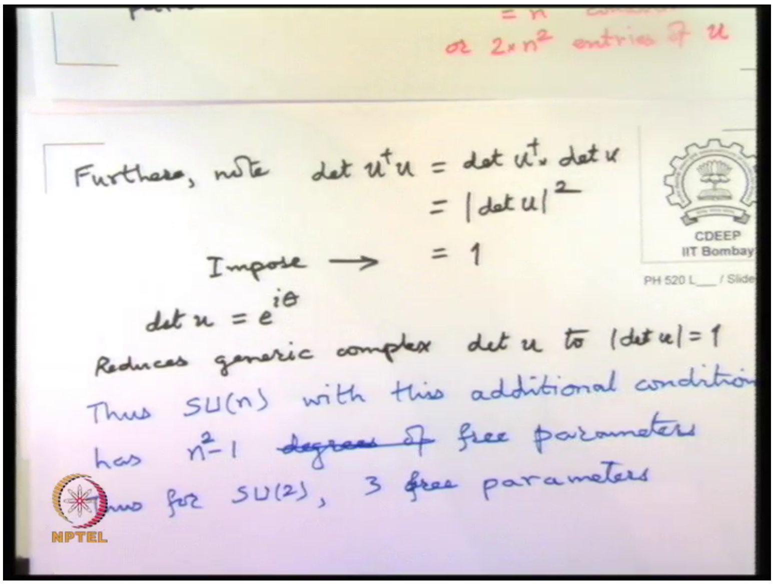
So, this is the diagonal ok. So, it gives this many conditions which is equal to n in 2 n minus. So, n square 2 cancels so, n squared minus 1 plus n sorry n square minus n plus. So, it gives n square conditions on in principle 2 into n square entries of u there is still something I have to count we are starting with SU 2 right. In fact, let us just think temporarily of SU n what does SU n mean I have some matrix u such that u dagger u has to be equal to n by n size identity forget all the other things that we did just look at this condition.

How many conditions are there in this? And the conditions are that this is a Hermitian matrix and that you can see on the right hand side also it is identity matrix which is a Hermitian matrix. If it is Hermitian matrix it is contents are this many of diagonal conditions n into n minus 2 in upper triangle by triangle times 2 because of complex numbers. So, there is n minus and in 2 and minus 1 from the upper triangle. On the diagonal we have only n real conditions because automatically, because of hermiticity they do not have any imaginary parts. And so, totally we have n square conditions here this gives the this so, for gives conditions on U n. So, U n group characterized by n square parameters.

Now, we knew that the determinant of this such a u matrix can be a phase. Further note that u dagger u equal to det u dagger u det u this is general theorem of matrix algebra determinant of product is product of determinants. So, that makes it into det u mod square because determine u dagger is same as the determinant of u star is the same as star of the det u. So, this becomes det u square.

So, if you said this equal to 1 so, impose that this is equal to 1, then; that means, that det u is some phase.

(Refer Slide Time: 20:27)



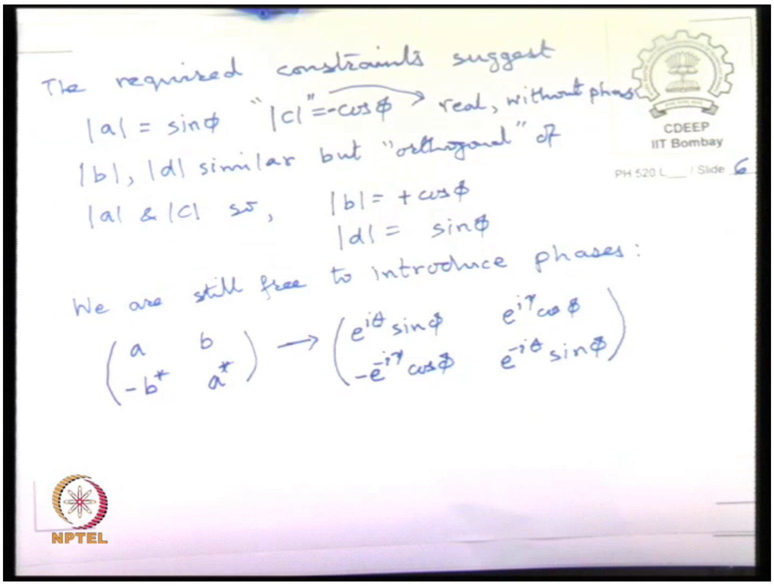
So, it reduces the degrees of freedom in determinant u from 2 to 1 to det u equal to mod 1 mod det u equal to 1 right it reduces to a single complex number. So, that 2 degrees of freedom in the determinant has reduced to 1. So, there is a further reduction by 1 in the degrees of freedom in a SU n free parameter not degrees of freedom, but free parameters.

So, in the case of SU 2 we expect 2 square minus 1 equal to 3 independent parameters 3 free parameters. So, I hope this logic is clear the logic is only arguing the algebraically independent conditions that are being placed eventually the conditions wind up on the 2 n square entries of the u itself all that the arguments all are meant for the quadratic expression involving u dagger u, but that many algebraic conditions reduce the number of free parameters in the original matrix u, leaving n square elements in SU n and n square minus 1 because the determinant condition does not put two independent condition, but only 1 on the complex determinants ok.

So, we have this many parameters now we can go back to SU 2. So, we can look at it like this there is the matrix a b c d, but we got the condition that a square plus c square modulus squares have to be 1 and similarly b square plus b square modulus squares have to be 1. So, we are immediately tempted to set a squared and a mod a and mod c to be cos theta and sin theta some cosine and sine that is 1 way to think about it, this would be some other angle say this is cosine phi 1 square this is cosine phi 2 square phi 1 square phi 1 square and cosine phi 2 square sin phi 2 square. So, let me write it.

The required constraints suggest.

(Refer Slide Time: 24:27)



That we can set mod a to be equal to and now I am using somebody's parameterization for it looks useful cos sin phi and c mod c equal to cosine phi we will see why this comes out useful we will end up with Pauli matrices. So, this is why we are writing like this now in principle I could have set b equal to some other angle psi, but the point is I also have to have this orthogonality condition between the a and the b.

So, that requires that this b has to be, but "orthogonal" to a and c. So, expect that b. So, it is a star b. So, b we can set equal to minus cosine phi and d equal to equal to right sine phi, c equal to minus cosine phi and b equal to plus cosine phi that much is ok. And then there is an overall phase which so, we can now introduce phases; so, let me first rewrite this.

So, that you feel a little reassured that we have here essentially a and a star and b and minus b star and we will further parameterize them by writing e raise to I theta times sin phi, then e raised to I gamma times cos phi, then minus e raised to minus I gamma times cos phi and e raised minus I theta minus sin phi.

So, one way to think about it is that this much could be reduced directly from the mod square equals sum of mod square 1 conditions and the mutual orthogonality, that leads to this condition. Because, if you look at this matrix now than it has automatically if you mod a square plus mod b square and those relation are satisfied; Now, that this is still complex, because all we did was fix the module i so, we got sine phi sine phi cos phi and minus cos phi, but we are allow to we have some relative phases.

So, there are two independent phase you are allowed again because of that complex arthagonality condition you cannot choose the this completely arbitrarily. So, we are allowed these two phases theta and gamma ok. So, totally we got 3 angles phi which characterized the mod square condition and then the 2 angles theta and gamma that follow from trying to ensure this ok.

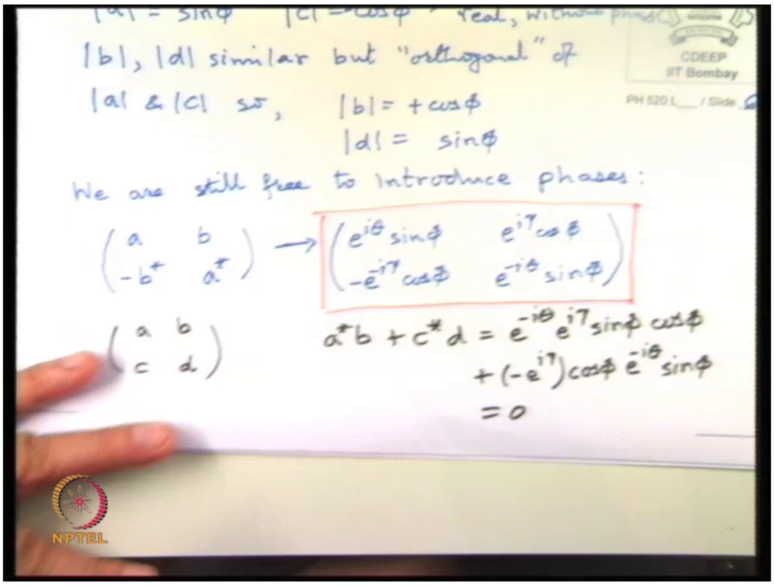
So, this is the way in which one can write out the SU 3 SU 2 matrix real we are doing some rough work on the side, we are just trying to was guess is simply to put the module of the size is of them to be sine and cosine and then we try to when the try to when the try to do the orthogonality thing actually we come out with these conditions ok.

So, you can fool around with a bit and you will see that this is what it is should all I am saying is if you look at this expressions, then you know that you have to put a relative minus sine, then you get to you can put some phases you can now more systematically you can start now with this. After having made a guess that we can put sines you can put sign and cosine right because the mod square as to be 1.

Now, you also know that when you take cross product of the two you should get 0. So, they should be flipped as minus sine and plus cosine or vice versa is just guess work and then you say that I am still left with the phase, which is independent as far as a and c is concerned a and c can have a independent phase, but because of the product b and a are not allowed to have independent phase.

So, that fixes the phases to be opposite this and this and this and this plus (Refer Time: 31:21). So, it is guess work as to what the form should be get this we can see that it satisfies the required conditioned, because mod a square plus mod c square, we can quickly see now compare that.

(Refer Slide Time: 31:44)



So, this originally was a b c d and with mod a square plus mod c square equal to 1, which we can see is true right a and c mod a square plus mod c square mod b square plus mod b d square is equal to 1 and then a star times b is e raise to minus i theta e raise to plus gamma sine phi cos phi and then plus c star b, but c star still equal to minus sine plus e raise plus gamma and the times cos phi and times d is e raised to minus i theta times sine phi.

So, the phases came out now equal e raise to minus i theta plus i gamma, but there is a minus i sine. So, this will cancel and will give 0. So, this parameterization parameterizes the SU 2 matrices.