Theory of Group for Physics Applications
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Lecture – 32
SO(3) and Matrix Exponent – II

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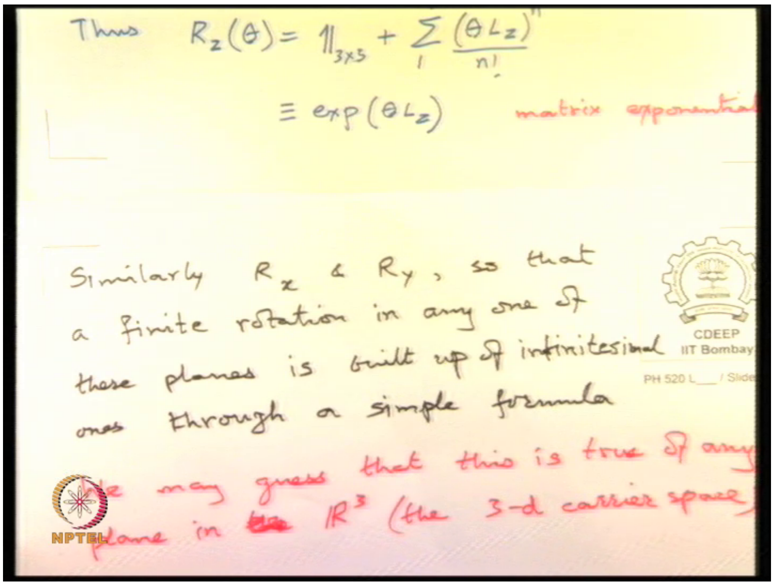

A finite rotation in any one of these planes is built up of infinitesimal once in a very simple way of course any finite one is made up out of the infinitesimal versions, through a very simple formula. So, that makes us wonder what fault did other planes do in the three dimensions, you did not have to have x y y z it was just some choice you made. Suppose you rotated all your axis, then the meaning of x y plane will change meaning of y z plane will change.

What is wrong with those planes? They must all actually obey the same thing there is no real difference ok. So, we guess that in R 3 ok, in the 3 D carrier space. So, it is only a limitation of the coordinate system that this is happening. So, we consider. So, for example, we have a plane defined by we can define a plane by unit vector normal to it.

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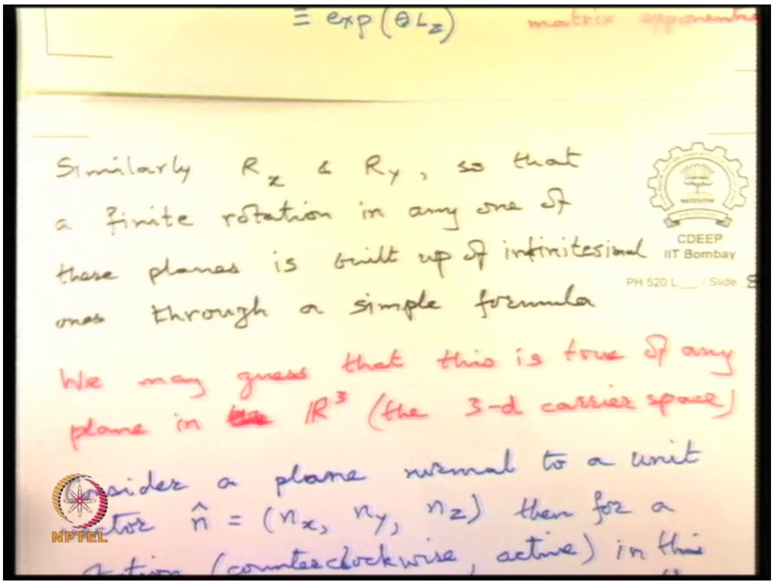

What do we expect for a finite rotation by angle theta in this plane? Of course, counter clockwise active rotation. In this plane what matrix should we write? We have everything in front of us.

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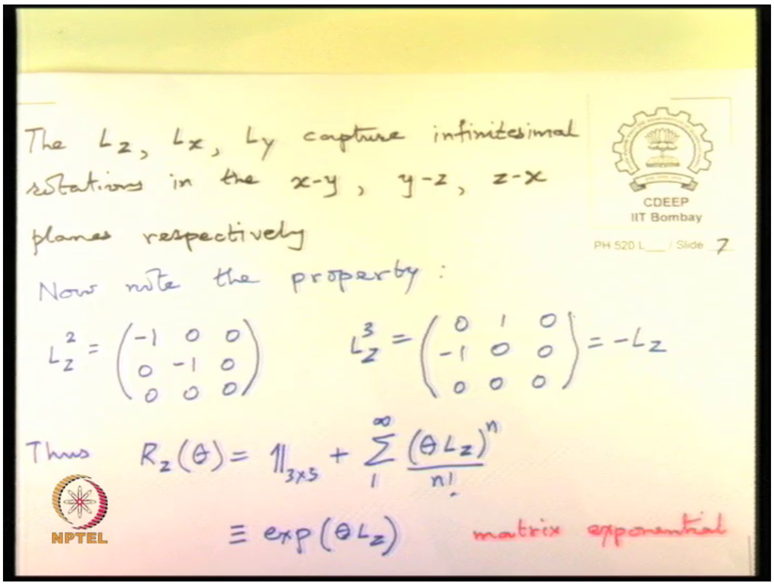

We know L x, L y and L z are like the what unit vectors of rotation. And finite rotation for any one of them at least is given in this way. So, what will happen for an arbitrary plane? Yes you can guess.

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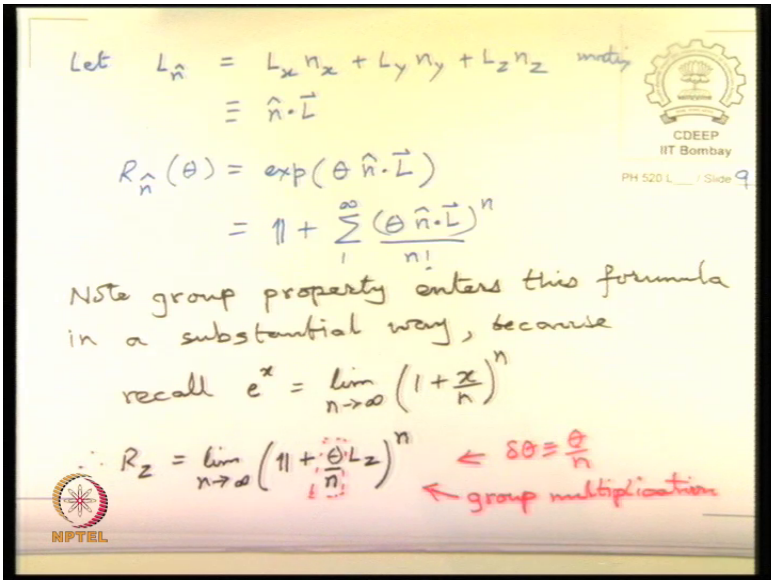

It is that let L subscript n cap be equal to L x and n x plus L y n y plus L z n z matrix right. And we symbolically write this as n cap dot L, mathematicians will faint at this kind of notation, but it does not matter they are not in this room. So, the L's are matrices they are not in any sense vector, but putting that arrow on top saves us lot of writing. And yeah really you have to see mathematics books, to see how much they struggle to retain purity of language, but we can sacrifice language to just get on fasting life. So, then R n cap of theta is simply equal to exponent of theta times n cap dot L, which is equal to 1 plus summation 1 to infinity theta n cap dot L to the power n over n factorial.

There is ok. So, this is very clear right, this is obvious that this is how it will work. There is one other aspect I did not emphasis enough, note also that this is about building up using infinitesimal actions, where group theory enters in a very substantial way. Recall e raised to x is also obtained as a limit; n going to infinity of 1 plus x over n to the power n right, good hold Euler formula. And therefore, say R z just to avoid writing too much detail; R z is going to be equal to limit, n going to infinity of 1 plus; what was it? Just theta times as z yeah we are writing this exponent, now using this Euler version of it, 1 plus theta times L z and let us write the n below the theta to the power n.

So, R z can also be written as this limiting operation, which amounts to infinitesimalizing theta. This is the delta theta, theta sub divided into n pieces and building up the big one as a multiplication of 1 plus this 1 plus this 1 which is all are group multiplication ok. So, it is by group operation that we obtained the finite group element from the small ones. So, that is the important feature of this formula.

And that this is how large number of group elements being multiplied and of course, limit being taken and remember we have said about Cauchy completeness. Any such so, you think of the sequence consisting of any finite n and then letting that n to infinity the sequence R z superscript n, where you multiplied only up to n, but then you let limit of that sequence in the limit n goes to infinity that should be contained within your set. This is the meaning of a continuum of a compact set.

So, these groups are all compact sets and because that Cauchy complete because, they after all involve real numbers theta over n and the reals are complete, so, they are Cauchy complete. So, what we have concluded is quite a deep result. Which is that?

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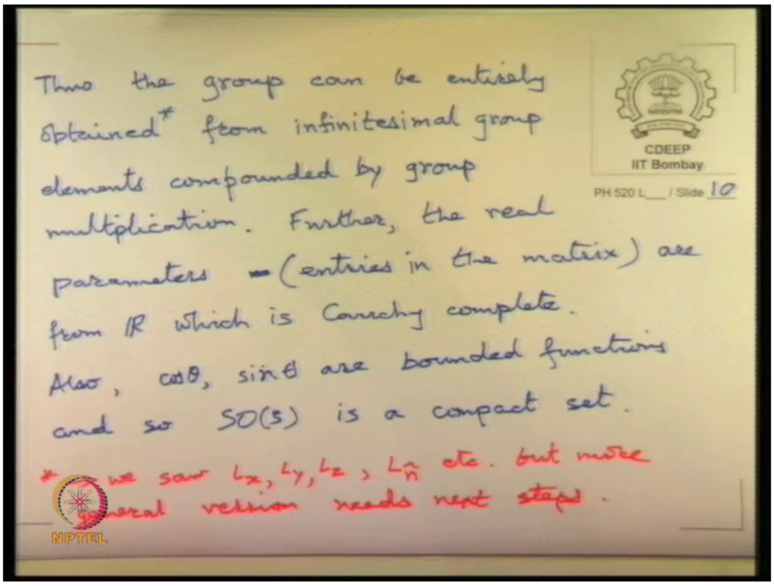

Well I did not quite prove this yet, but so, I will put a star on it, can be entirely obtained from infinitesimal group elements compounded by group multiplication, it has that is the entries of the matrices are from reals of course, because they are real.

Finally we note the most important thing that the entries are all cosine thetas and sine thetas. So, they are all bounded is a compact. Now, the star part; we have yet to prove, we saw L x y z and L n cap etcetera, but we need to see it more generally. So, what is this? Next step; so, we go back to; so, this part is clear right that this group is essentially built up, I mean we conjecture because, this guess that what happened to any one plane should be true of any other plane. So, that is a geometric intuition not an algebraic one.

But in any case starting with that we saw that at least for any given plane, we have a compact set of matrices representing those rotations and now we want to see what really happens if you exponentiate n dot L ok. For L x or L y or L z we could just easily take square cube and see what happens, what happens in general. This is not obvious not obvious also, we get to more general Lie groups where this is a standard feature and this was my theorem stated in quote marks.

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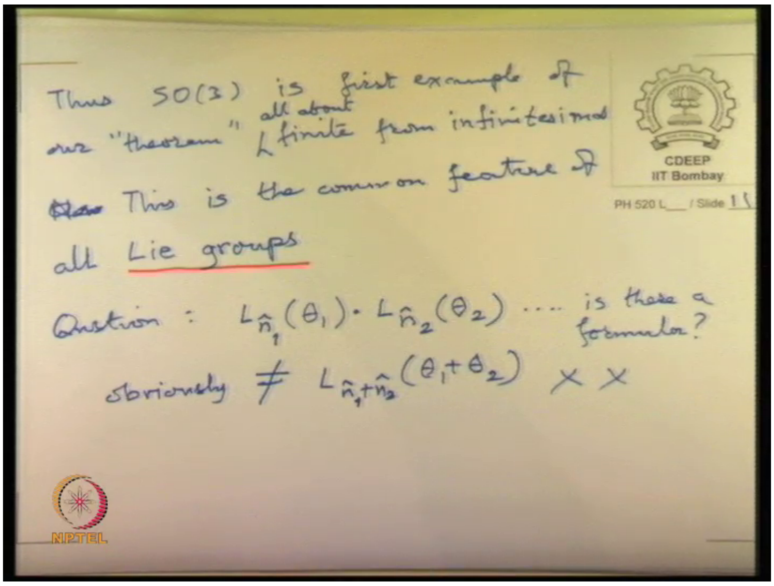

All about finite from infinitesimal which are way you want to stated, but we now need to check and this is a feature of common feature of all Lie groups. So, now, we go to the next steps which is to check what actually happens if you take higher powers of something like n cap dot L. Generically what happens is that so, question: If I take L n 1 theta 1 times L n cap 2 theta 2 do, we have a easy formula? We know that this is not equal to L n 1 cap plus whatever you want to call it, it is not some simplistic answer like that ok. There is this question of what happens, if you multiply like this and the answer has to do which have particular property of the L matrices.

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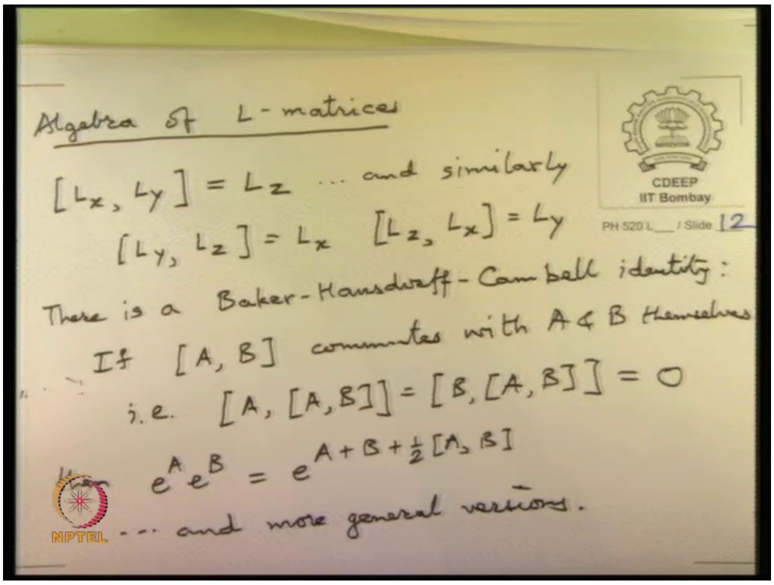

So, we check what happens if we take this commutators. So, what will happen if you do L x L y can you calculate? You are good. So, L x L y is equal to L z I think, you can check this we will not spend time writing matrices here and we can also see also I am I suspect you have seen it somewhere right? You have all done quantum 1?

Student: Yes.

So, that is so we will quickly see the analogy to the quantum mechanical L which is R cross p. So, similarly we have L y, L z, y z equal to x and z x commutator equal to y; turns out that this commutator algebra is what is going to save our day. This is because there is a Baker Campbell Hausdorff identity which goes like thi; if I yeah if A and A B commutator i.e. A comma A comma B equal to B comma A comma B equal to 0, then e raise to A, e raise to B, the product or the exponentials becomes e raise to A plus B plus half of the commutator A comma B. So, it turn turns out this is the first step. If this is not satisfied if the if they do not commute, then you accumulate more terms in the exponent of A commutator with A, commutator B, B commutator with A commutator etcetera, but the point is that the because of the matrix exponent, the exponent a exponents are not going to add that is the main thing.

So, if we know the algebra of L x, L y, L z and more general I will just write and more general version. So, if we know the algebra of L x, L y and L z and it is a closed algebra, and then you can always exponentiated and expected to tally with your n cap notation.

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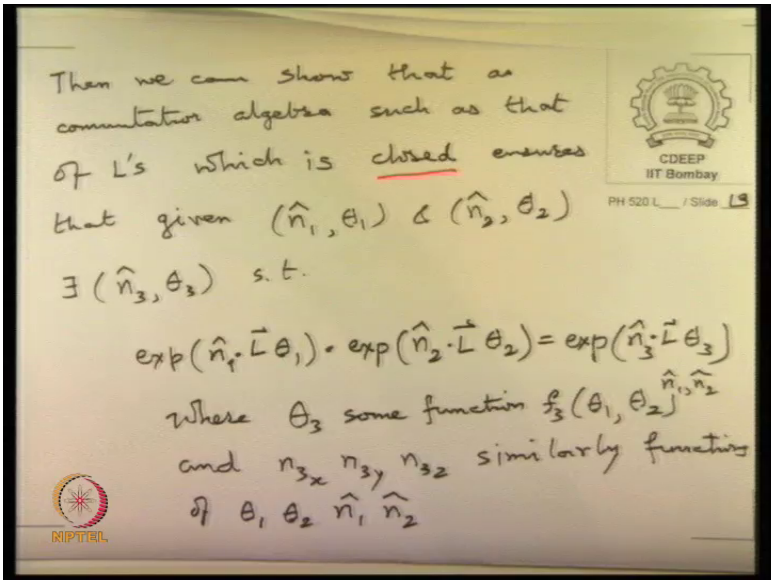

That a commutator algebra, such as that of L's which is closed in the same algebraic sense, that given n 1 cap theta 1 and n 2 cap theta 2, there exists n 3 cap theta 3 such that exponent what did we say yeah sorry. So, yeah exponent n 1 cap dot L theta 1 times so, group multiplication, where theta 3 will be some function of and of course, similarly n 3 x, n 3 y, n 3 z.

Similarly, functions of everything of theta 1 theta 2 right. So, functions of theta 1 theta 2 and of n 1 cap n 2 cap so, the parameter 3 there are effectively 3 parameter rather unit vector in 3 dimensions so, it has 2 parameters and there is an angle so, there are 3 parameters this is 3 parameters. So, these 3 parameters are functions of the 6 that is the statement.

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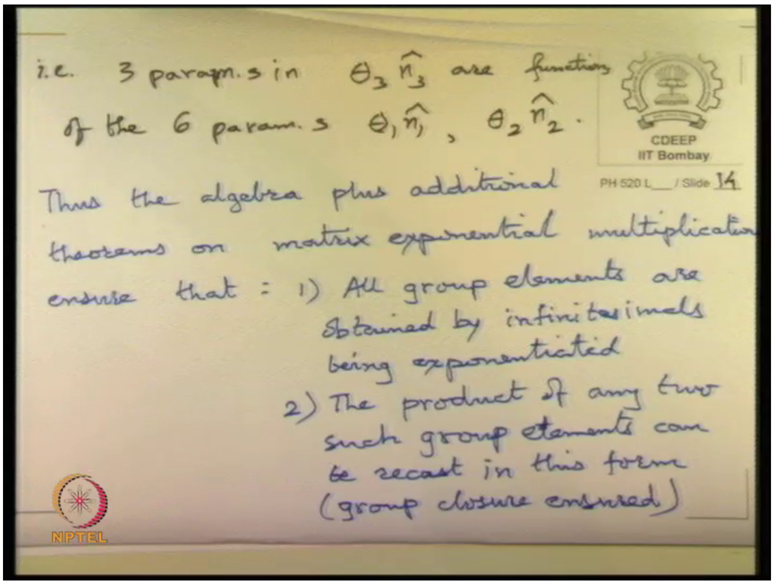

Parameters in, but this kind of formula is available and the existence of this algebra ensures the closure of the group and it also ensures that any group element can be accessed by exponentiation, elements can be recast in this form, making group multiplication manifest, group closure property, group closure ensured, in this form good. So, we have covered quite a bit of the ground, I could had a little bit in terms of geometry of this.

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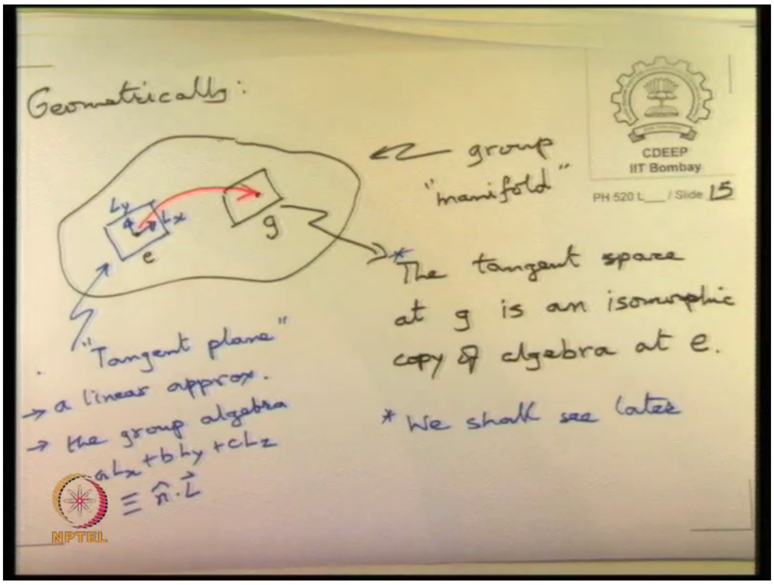

Think of the group manifold, all the points in the group, the group is the continuous group. So, it is parameterized by continuous parameters. So, it is some kind of a continuum. Of course, if you take 3 by 3 matrices, then there are a it is an orthogonal group. So, there have what 6 independent entries and so, it is some six dimensional manifold.

But the point is that suppose this is the identity in it so, this is our group manifold and I have not defined manifold either, I will tell you in the minute what that means, but you just think of thus set of all the points that constitute the matrix group SO 3, then you draw a local tangent plane, linearize it linear approximation to it, which is also a the group algebra.

So, you can erect in it these unit vector these vectors L x, L y etcetera I will draw only 2. So, you know that in when it is infinitesimal there is a matrix available L x in terms of which all x rotations, but we saw that I can now do n cap dot L. So, this whole plane is parameterized by n cap dot L and what our theorems are telling us is that, I can map it to any other place I like. In the neighborhood of g and the group structure essentially remains the same.

Again I can write it in terms of exponentials of well. So, I can now reach g by using the tangent space at this and the 2 tangent planes are isomorphic. Isomorphic copy of algebra at e, did not prove all that, but this is what geometrical it will come out. We have to show that after doing the exponentiation, you can still write L x, L y, L z again in this location which have it is just going to be a similarity transformation, we will do sum of it a little bit later ok. So, I just want to get to another important practical group, but this is what roughly the picture emerging is.

What we have proved so, far is that using the algebra, near the identity I can reach any group element. And we have also check that if I reach group element g 1 and the group element g 2 I can multiply them together to also reach it directly by another exponentiation that picture I am not drawing here, but that this happens is yet to be proved ok. The other important so, yeah the other important group is SU 2.

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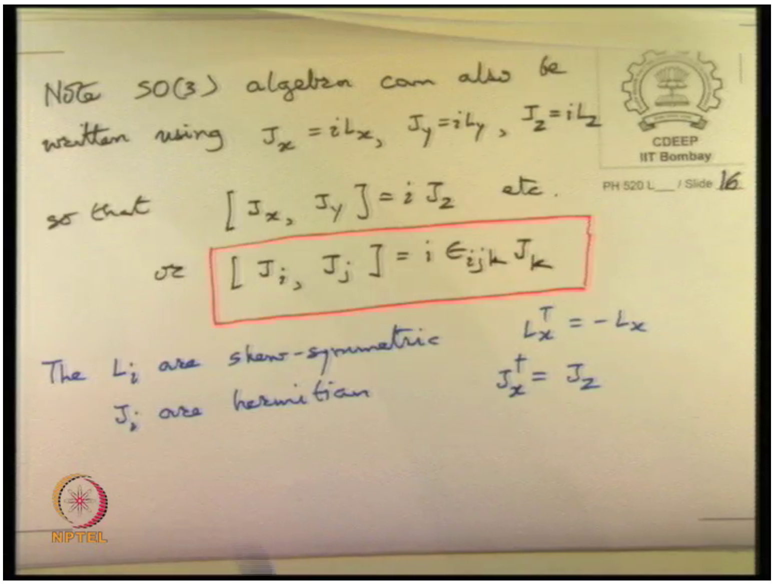

So, note SO 3 algebra can also be written using j equal to J i times L. When we were simple minded we try to get out of complex numbers by picking out the real parts. Now, that we get more advance we supply and i to make it complex. So, that then J x, J y equal to i times J z etcetera, or we say J I, J j equal to i times epsilon i j k J k. So, just by complexifying an actually it is; obviously, real algebra, but I have now complexified it. It has certain advantages. So, the L i are skew symmetric or anti symmetrics.

But the good think about the J i is that they are Hermitian right. So, L x transpose equal to minus L x, but J x dagger is therefore, going to be equal to J x because, there will be minus sign from transposition of the real part and minus sign from the i being complex conjugated. So, I will just get J dagger equal to J. So, what is nice is that exponentiation of Hermitian matrices gives you unitary matrices. So, it gives now I will remind you a old theorem we proved for finite groups, that any group any finite representation can be converted unitary, any finite size representation can be made unitary unitarized using the tricks of Schur's lemma.

So, here also if we do this then we obtain a unitary representation of the orthogonal group. And that takes us to the so, what is the time 12:20. So, that takes us to SU 2 group.

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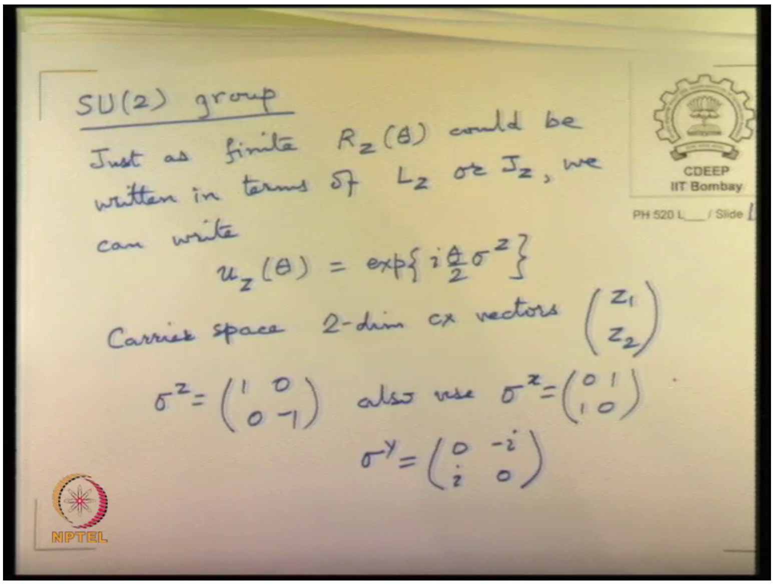

And as per as SU 2 group is concerned again, we have to analyze matrices of this type. So, just drop the answer by saying that just as finite R z theta could be written in terms of L z, or J z we can write u z theta a unitary matrix, to be exponent of i times theta by 2 times the Pauli matrix z. So, we I have made up big jump, we have switch to carrier space 2 dimensional complex vectors. Let us say z 1, z 2 and sigma z matrix of course, as you know is 1 minus 1 and also we will use sigma x matrix, which is 0 1 1 0 and sigma z sigma y.

This is the only one you have to remember which some care. There is minus i in the upper right corner. So, we introduce matrices sigma x, sigma y, sigma z and I claim that exponentiating them exhausts all of the group SU 2.

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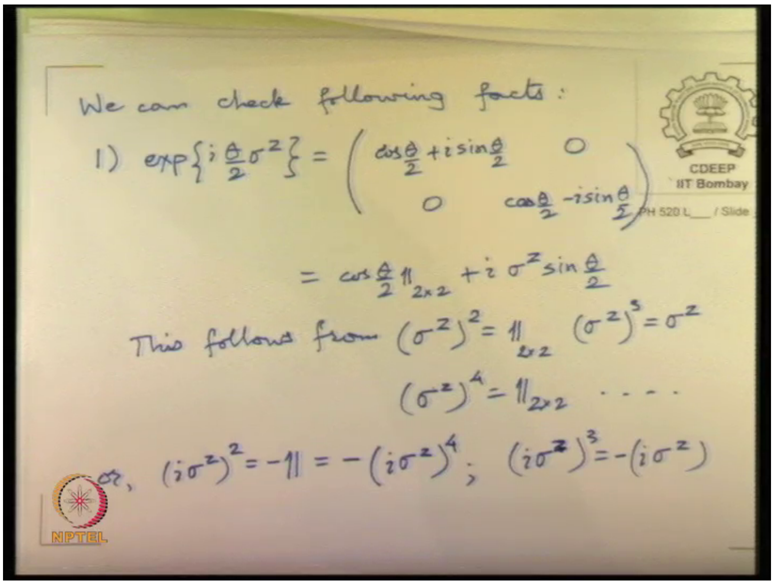

One is that exponent i theta by 2 sigma z, will turn out to be equal to equal to cos theta by 2 plus i sin theta by 2, this is easy to prove because, sigma z square is identity matrix and sigma z cube is equal to sigma z and therefore, sigma z to the 4 is identity again.

So, this again gives you cosine and sin series, where cosine series can be will have coefficient just 1 because, it will be either even. So, sigma z to the 4 is again back to identity etcetera. So, we get that all the even powers, we will just have identity matrix and, we will actually have the fluctuating part because of the i that comes only in the yeah. So, because of this and because of the fact that I am actually taking i times sigma so, or i sigma z square is equal to minus 1 equal to minus of i sigma z to the 4 ok. And i sigma z cube equal to minus times i sigma z.

So, we get the fluctuating signs because of the i introduced whereas, the Pauli matrices themselves are just not giving any signs in their powers signs come from the i, but once you do that you get cos theta and sin theta series like this. And the other factor was going to right was that the algebra is exactly the algebra we wrote.

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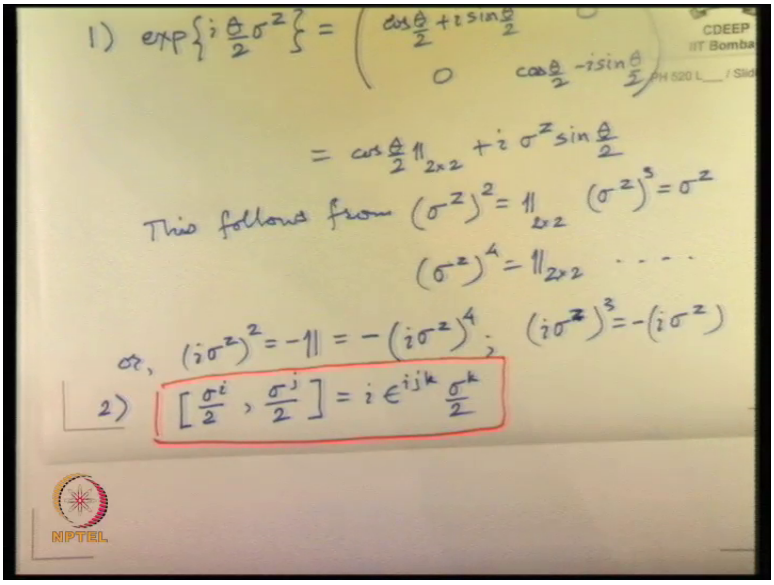

So, we will just end here, for the time I can may be write over here and 2 is that they obey this algebra sigma i, but you have to put this crucial factor half sigma j over 2 is equal to i times epsilon i j k sigma k by 2 which was exactly the algebra obeyed by the j version of rotation group. So, we will stop here today and resume with SU 2 groups and Lie algebras later.