Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 31
SO(3) and Matrix Exponent - I

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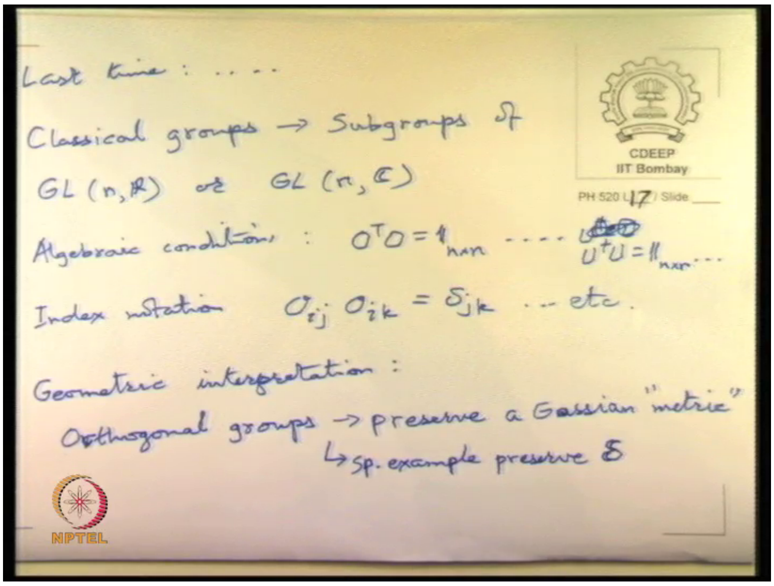

Which are subgroups of GL n R or GL n C and most of the classical subgroups have algebraic condition like saying that O transpose O equal to 1 etcetera or we had index notation which was that O i j then we O i j O i k equal to delta j k something like this etcetera because we had the unitary things also. So, if you want to I will write U dagger U equal to 1 U dagger U and most interestingly we also had a geometric interpretation which is for orthogonal groups. They preserve the Pythagorean metric or let us just say the preserve a Gaussian metric and a special example preserve delta metric. So, writing it as if it is some symbol metric delta a matrix which is simply Pythagorean.

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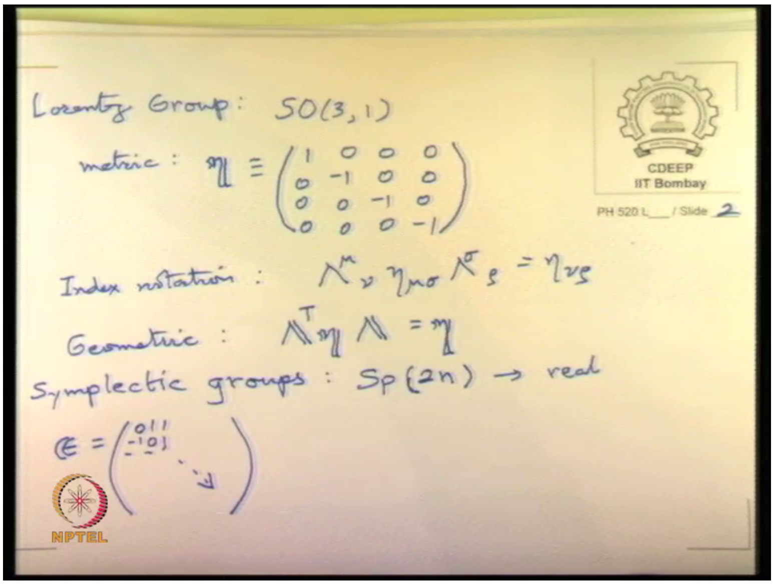

For Lorentz group we introduce the metric 1 minus 1 minus 1 minus 1; it is SO 3 1 in the algebraic sense it preserves matrix of 3 plus signs and 1 minus signs and we write the metric to be eta and the algebraic or index notation is lambda mu nu eta mu sigma equal to times lambda sigma rho equal to eta nu rho. So, that is the index notation for the Lorentz group lambda matrices or abstract matrix notation geometric statement is that these lambda matrices times eta sorry for it looking too (Refer Time: 05:28) but this and of course, we here there are many more certainty, it is not enough just to say determinant equal to plus 1 or minus 1, one has to worry about the sign of the time component versus signs of these little more interesting this group because one is phase inversion the other is time reversal and the discussion is little long.

So, for the time being we just write these things and finally, we did the symplectic groups which were S p 2 n S p 2 n and I think I did not write any R or C in it, we just call it S p 2 n for the time being we will not be interested in S p 2 n C. So, automatically R and we introduced (Refer Time: 06:35) epsilon which is of the form block diagonal plus and minus 1. It runs along the diagonal.

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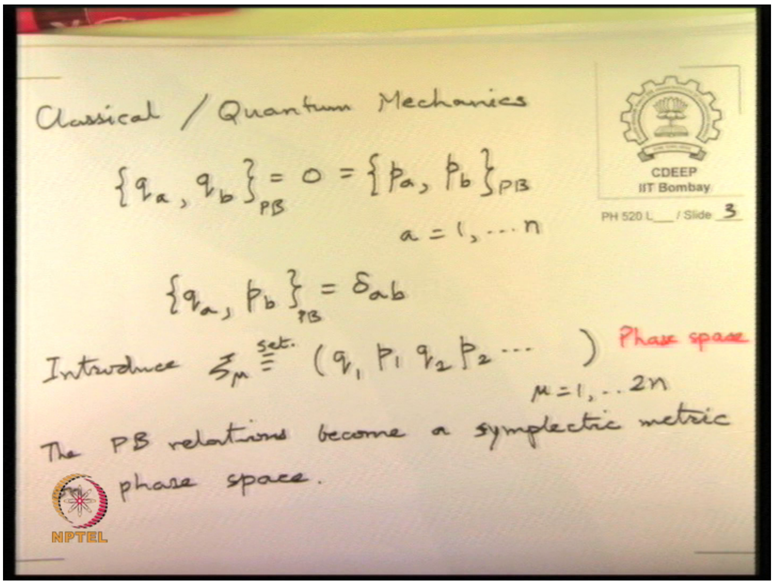

The classic example of this is that in classical mechanics well it could be either because structurally they are both the same. We have either the Poisson brackets which we say q a q b equal to 0 equal to p a p b a equal to 1 to n, but we also have the requirement that q a p b are equal to delta a b.

So, this amongst to introducing psi, which is equal to which is from the list, so as a set ok set equality equal to q 1 or p 1 or q 2 or p 2 and here mu will then go over 1 to 2n. So, make these the new coordinates q 1 p 1 q 2 p 2. Then essentially this Poisson brackets so, maybe I put here a this for clarity because curly brackets are used by lot of people. So as Poisson brackets so the Poisson bracket relations essentially become a symplectic metric on this so called Phase space.

And canonical transformations are the ones that preserve Poisson brackets. So, the canonical transformations are essentially they form the symplectic group. I think we wrote the detail last time in symbols, I will not repeat it just now.

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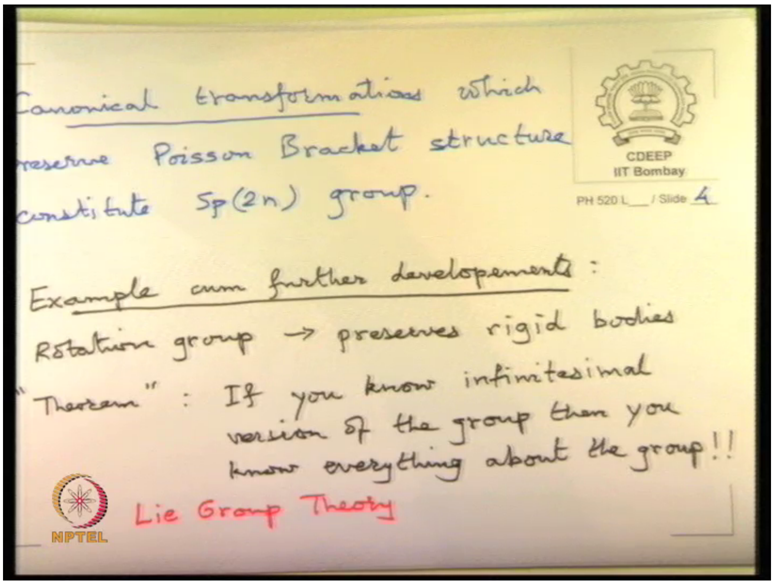

So, constitute S p 2n. So, that is what we were doing and this gives the example of the canonical transfer of the symplectic group we did last time.

So, now I will move on to something much more practical and direct before we come back to slightly more formal things. So now, we begin with SO n or SO 3, example cum further developments. So, let us begin with rotation group. So, sometimes we also call it rotation group instead of SO 3, when we actually have the physical meaning in mind. Rotation group preserves the rigid length I do not know whether I wrote it here or not I spoke something like that.

So, you have preserve a Gaussian metric, but specifically it preserves the Pythagorean length, the distance between points. So, it rotates rigid rods into rigid rods. So, preserve rigid bodies. What we are now going to do is try to understand some interesting aspects of this group which is to show that if you know the infinitesimal version of the group then you know everything about the group.

So, this is a very powerful statement and I put it as a theorem in quote max right now because I am just saying something that is intuitively understandable, but not a precise statement. This is essentially what is called Lie Group Theory; this is at the heart of and that is essentially the second half of this course. It is Lie group theory and we will start with SO 3 as an example and SU 2.

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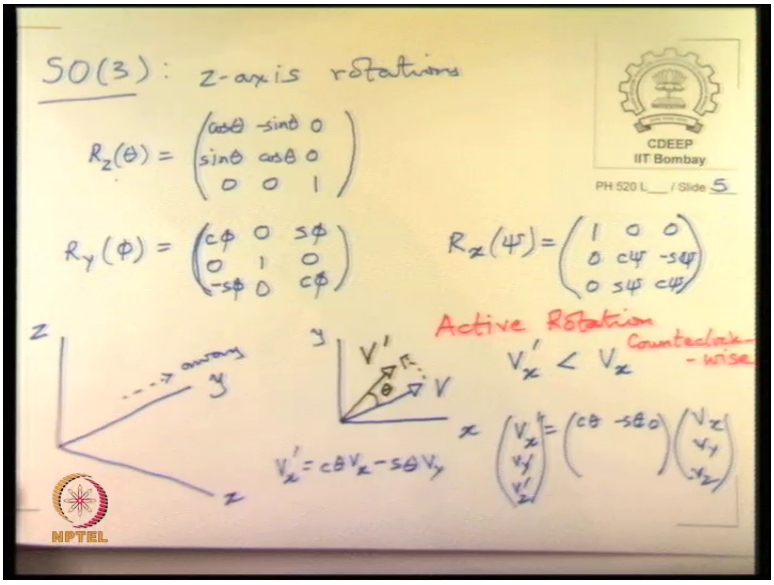

So, we start with SO 3, a typical example is rotation around the z-axis. So, z axis does not move. So, that remains 0 0 1 and here we introduce we write it out in terms of an angle minus sin theta sin theta and cos theta and so let us also so, let us call this R z theta. We can similarly write out R y phi and then R x psi this is all completely temporary notations I mean the angles we are using. So, what should we be writing here? Can you fill this out? That main point to remember is that by our conventions if we have minus sign in the upper diagonal for R z, we also written minus sign above the diagonal for R x, but for R y we have minus sign below ok.

So, this happens because of choice of right handed axis essentially choice of right handed axis. So, once you choose x so, several points I want to make here of mainly of conventions and how we think about these. So, so we should have a right hand screw rule working, so, we can put x y and z. So, that is the correct orientation of course, in this figure it is difficult to see thus this is going away from us ok.

So, if y is going away into the paper then this forms the right handed system and then there is a issue of active versus passive rotations ok. So, this writing minus sign for R z is one convention, the other convention is to write plus sign and minus sign for R z. Of course you have to do the same thing for R x and then for y it will flip. But let us think of what is active rotation and I am actually doing it in front of you. So, let us see what we get? So suppose we restrict to x y axis and take a (Refer Time: 17:06) vector. So, we rotate a vector not the axis.

So, the difference between so called active rotations and passive rotations is whether you rotate the axis or you rotate the object. So, if I have this vector V and if I rotate it to a new position by an angle theta, then what I find is that x component. So, I do not want to clutter the diagram too much, but there is V x V y there is V x planes. We can just say V and V prime ok. This vector is V and it is rotated version is V prime. Then we can see that V x prime is less than V x; this is how I remember things ok.

So, I rotate then V x prime is less than V x. Now I look at this matrix the R x matrix and R scales the x component going to reduce or increase and sorry yeah. So, rotating about R a R z, so I look at this matrix and ask whether the R x component is going to reduce or increase. It reduces provided I put a minus sign here ok, so because I will never put here V x right. So, V x prime is going to be found from cos theta minus sin theta 0 acting on V x V y V z. So, V x prime is going to be equal to cos theta times V x and minus sin theta times V y and the example vector we have taken is all in first quadrant. So, all components are positive to begin with.

So, this is going to reduce the V x component and therefore, this is what I call active rotation, I have rotated a vector and one more matter of convention of course, it is an active counter clockwise rotation, somehow we have all accepted that counter clockwise is the correct thing. Apparently in England you can actually biclocks that (Refer Time: 20:11) counter clockwise which is which would be good to have on the wall because everyone confused.

So, but somehow counter clockwise is standard. So, active counter clockwise rotation would lead to this choice of signs. Now let us look at the infinitesimal version that is where all the interesting story begins and I am sure you are familiar with all of this.

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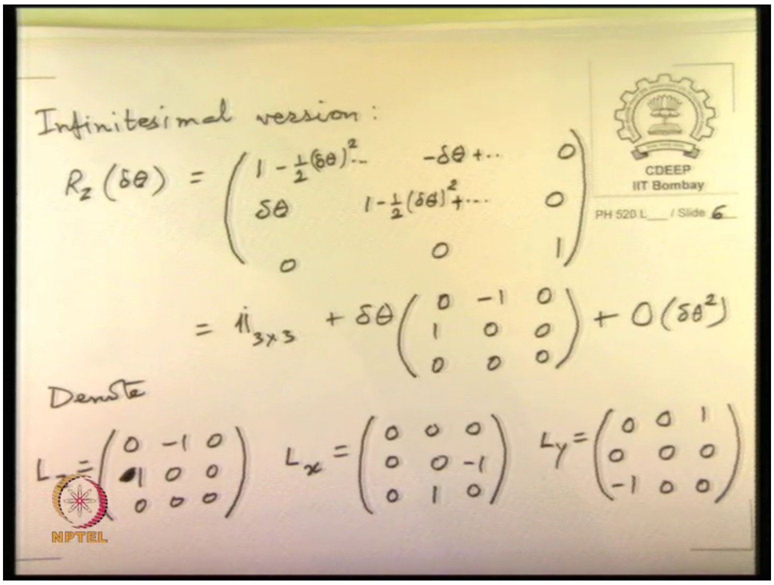

So, so R z if I have a small angle delta theta it becomes equal to 1 minus a half delta theta squared right dot dot dot then the sin theta which is a minus sin theta becomes minus delta theta plus dot dot dot 0 1 then delta theta here, this remains the same 1 minus a half delta theta squared dot dot dot and 1 0 0.

So, if we ignore the quadratic things then it is the identity 3 by 3 identity matrix and then plus delta theta times 1 0 minus 1 1 0 0 0 0 0 0 plus ordered delta theta square which we are going to ignore. So, the infinitesimal version; suggest this matrix which has no numbers in it, it is just 1's and minus 1's and it is anti symmetric like this.

Similarly, we can check that. So, this we call l subscript z 1 0 and we can also write out l x and l y. So, you can try to write it in your book; to the x rotation then all this will be 0; it should have same sign as l z and l y will have 0 here. So, we designate our matrices l x l y l z which capture the infinitesimal rotations in the y z, z x and x y plane respectively ok.

So, it is I think it is important to write this because later you will see some interesting matters of convention that come out of this.

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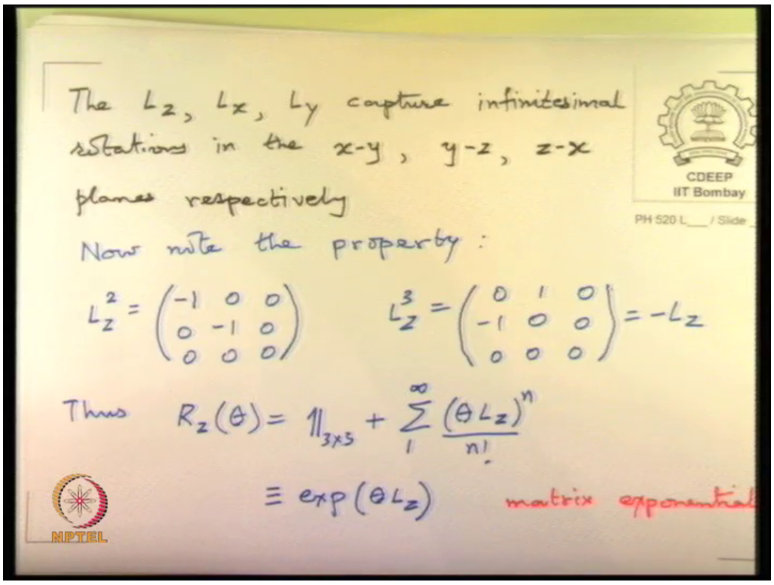

These the capture infinitesimal rotations in the x y, here the order of this x is important ok. So z x y, x y z and y z x; now if you restrict yourself to only z axis rotations or the x y plane rotations then the l z matrix is enough to capture everything ok.

So, now we note the property well we probably need to look at it a little bit; what is l z square? This row into this column equal to minus 1, this other things are 0. This row into this column is also minus 1; what about l z cube? Well these are minus signs so it is like minus the identity matrix l z square. So, if I cube l z then I have to take l z square times l z. So, essentially put minus signs on that. So, it is 0 plus 1 minus 1 0 0 0 0 0 0 which is equal to minus of l z, but now you are all experts to note that this is exactly what the trigonometric series does, square cube and with minus sign and so on.

So, so R z theta is actually equal to 1 plus theta times l z to the n and minus 1 to the sorry so theta times l z to the n over n factorial ok. So, what am I saying? So, well so if you look at the full R z, it has cosine thetas and sin thetas. So, what I am claiming is that this property of l z that it oscillates in signs, it will become l z; l z cube becomes minus l z. So, l z to the 4 will become just l z because l z square is any way equal to minus 1 and 1.

So, l z to the 4 will become sorry plus signs here. So, in the upper corner the alternating even powers are plus and minus ones in this corner and the odd powers are the same as l z again fluctuating in sin as you go 1 3 5 etcetera right. So, if you just raise this to enough powers with theta supplied you will just recover cosine and sin theta series.

So, R z theta is just equal to power series theta l z to the n over n factorial and a 3 by 3 identity matrix added. This so knowing the infinitesimal l z so this we sometimes denote symbolically as exponent of theta times l z; where this is now a matrix exponent and the exponential of a matrix is defined by its power series and the power series is well defined because it just matrix multiplication n number of times.

So, this is a matrix exponential. So, at least as far as rotations in any one plane are concerned they are just exponentials of corresponding infinitesimal rotations.