Theory of Group for Physics Applications
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Lecture - 30
Classical Groups - Topology – II

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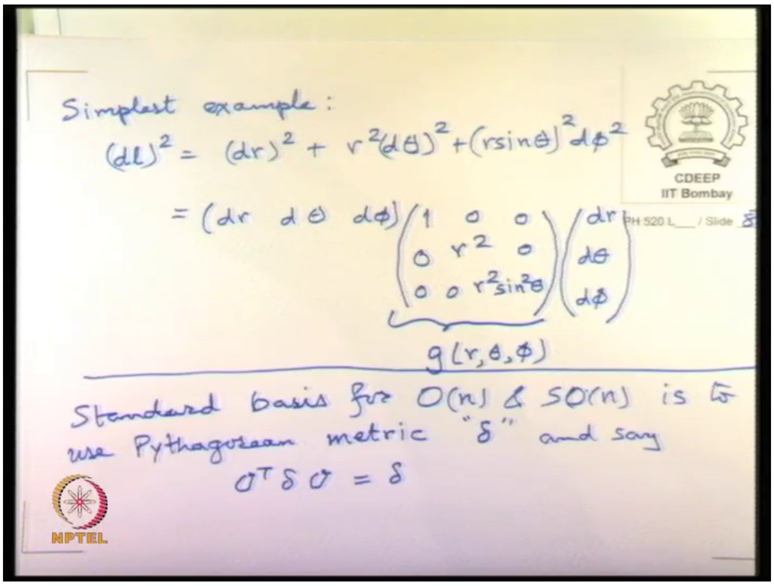

So, the simplest examples is we said dl squared equal to d r squared plus r square d theta square plus r sin theta squared d phi squared. And we can write this out as d r d theta row vector d phi, and then d r d theta d phi and 1 r squared and r squared sin squared theta. So, this is our metric g Gaussian matrix g; which itself is a function of r theta phi in principle, well it does not have phi in the present case, but it could be. So, this is why you need this Gaussian trick of introducing more general kind of matrices as the matrix is rather than just the Pythagorean one ok.

So, when we say that the inner product is left in variant by the O n groups we can think of all the 3 different ways the most general abstract way with simple things in terms of inner product, or we can think of the matrices specific matrices row and column matrices and write O transpose O in matrix language and say it is identity, or we can think of it as having a general Gaussian metric g even then we can essence the group we will get is the same. Because it keeps the that matrix invariant, but they so I should correct myself.

So, the SO n group is specifically the one that keeps the for O n. So, I may as well draw a line here and SO n is to use Pythagorean metric just call it delta and say that O transpose delta O equal to delta that is what the that statement written in terms of indices means is essentially that where we inside did not put delta.

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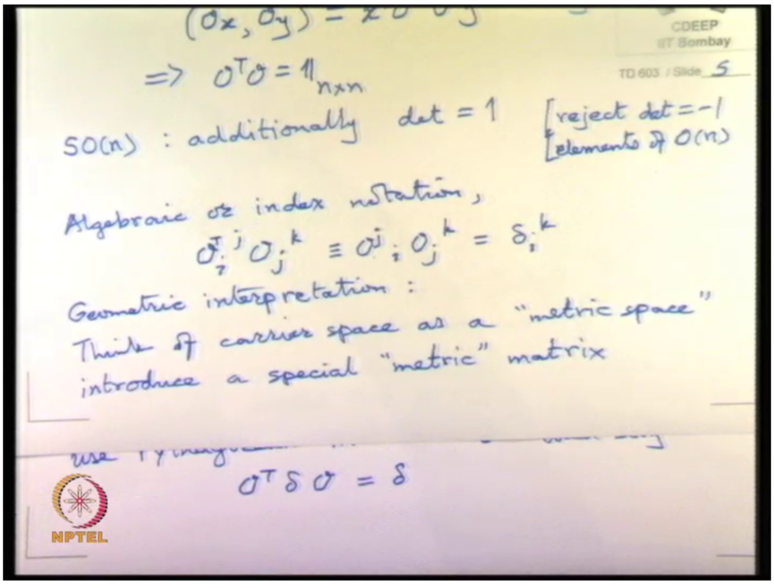

So, all this song and dance I did to tell you is that actually it is O transpose times the metric O keeps the same metric, so that is the geometric meaning of the transformation. But since we make the standard choice of the metric being delta it just looks like this, but it keeps that matrix invariant. Now that allows us to say what happens in the Lorentz transformation case.

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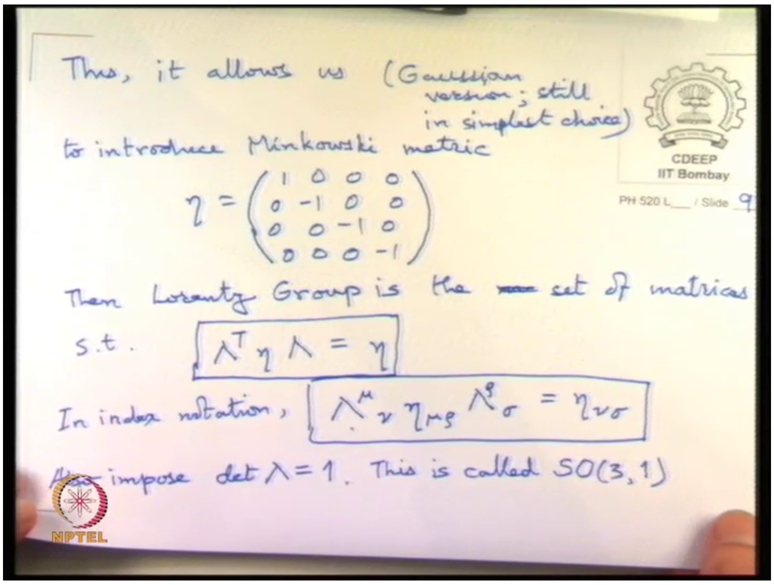

It allows us it meaning the Gaussian notation Gaussian version but still in it is simplest form to introduce Minkowski metric which is 1 minus 1 minus 1 minus 1 0 0 0. So, I am thinking of this eta is some kind of a g, but I still made it simple enough that there are only 1's and minus 1's in it and there on the diagonal.

But there is a sign change unlike the Pythagorean one and then we say that Lorentz transformations, or Lorentz group is the set of matrices such that lambda transpose eta lambda equal to eta. The lambda matrices of constitute the Lorentz group. Index notation; we say lambda mu nu I hope you like Greek letters mu rho lambda rho sigma equal to eta nu sigma.

So, this is how you will often see it written, but the invariant kind of statement is this, even that is not all that in why because we actually it is still a matrix, here we have written out all the indices. So, this is how it is usually written and the eta matrix is written with both indices down. This is the standard notation, people also introduce the I mean more advanced things are possible with it, but for our purpose this is what the group is going to be. This group because of the one sign opposite to the other 3 signs is called SO 3 1, special orthogonal group of and so after imposing also impose determinant lambda equal to 1.

So, this leaves out space in versions and operations like that which have a negative determinant mirror reflection things like that, also impose determinant lambda equal to 1;

this is called SO 3. 1; special orthogonal group with 3 positive signs, and 1 negative sign or 3 negative and 1 positive it does not matter, so it just called S O 3, 1 ok. So, next we do a little bit of abstract stuff again and then come back to this, or I should introduce one more group before we go on yes.

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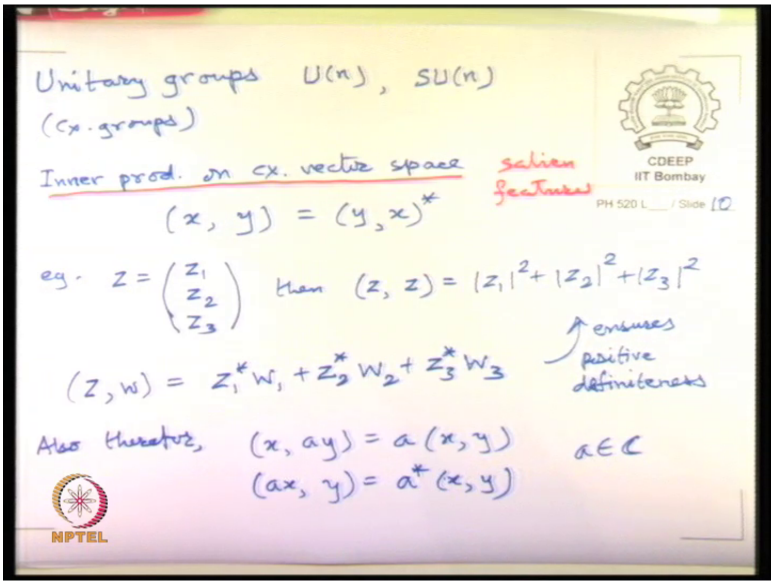

So, while we are introducing groups let us do two things more, unitary groups and symplectic groups, U n, and S U n, there actually C but the C is hidden because unitary generally means permission U dagger U equal, it is unitarity is already on complex matrices. So, nobody write C for them, but they are complex groups.

So, they act on a vector space that allows scalar multiplication by complexes and here we need to introduce the inner product for complex vector space which is which has the property that x y is equal to y x star. So, we need to do this to keep things meaningful because, so this ensures actually the positive definiteness.

So, for example, if I have a vector Z which is a Z 1, Z 2, Z 3, then Z Z will be equal to Z 1 star; so, it will be mod Z 1 squared plus mod Z 2 square plus mod Z 3 square right. Because this star effectively means that you have to construct the inner product Z W to be equal to Z 1 star W 1, plus Z 2 star W 2 plus Z 3 start W 3, and it ensures the positive definiteness.

Also we have that x comma ay is equal to a times x comma y, but a x times y is equal to a star comma x, y. So, your scalar so a belongs to complex numbers, the scalar by which you multiply your vector scaled your vector if it is whether it is multiplying the second entry or the first entry of the inner product matters.

Later on in quantum mechanics beginner defines anti linear transformations where it is the opposite it when it is taken out of the front factor it is complex if I done not if it is, so that requires with some very special thing that is required for time reversal operations, but this is good enough for the time being. So, these are the properties of the inner product. So, these are salient features and then it is easy to define what is unitary and what is not.

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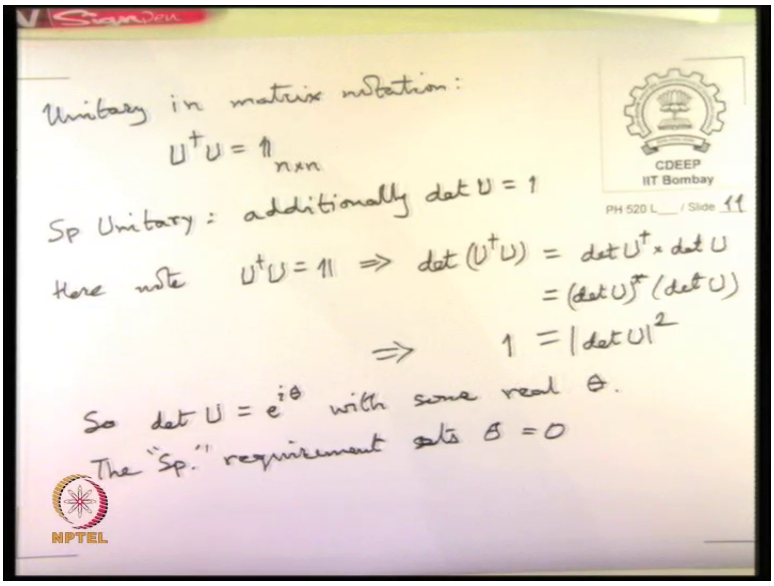

I mean what is unitary and what is special unitary simply says that in matrix notation is simply U dagger U equal to 1 identity matrix, and special unitary additionally determinant U equal to 1. So, here note U dagger U equal to 1 implies that determinant of U dagger U which by theorem of determinants is same as determinant of U dagger times determinant of U becomes star of determinant U.

Because when your dagger you transpose and your complex conjugate, but transposition does not change the determinant, but that starring will remain. So, the determinant will basically be the star of the determinant U and therefore, you get mod det U mod square;

this equal to determinant of the right hand side which is just 1. So, this implies 1 equal to det U square. So, det U is in general e raise to i theta with some real theta.

The special unitary requirement sets theta equal to 0, it has to be 1 only. In the real case it could be plus 1 or minus 1. The orthogonal groups without the S condition is plus or minus 1, the complex groups without the condition can be a whole continuous continuum of possibilities the whole circle. So, this brings us to a very broad distinction between various kinds of groups.

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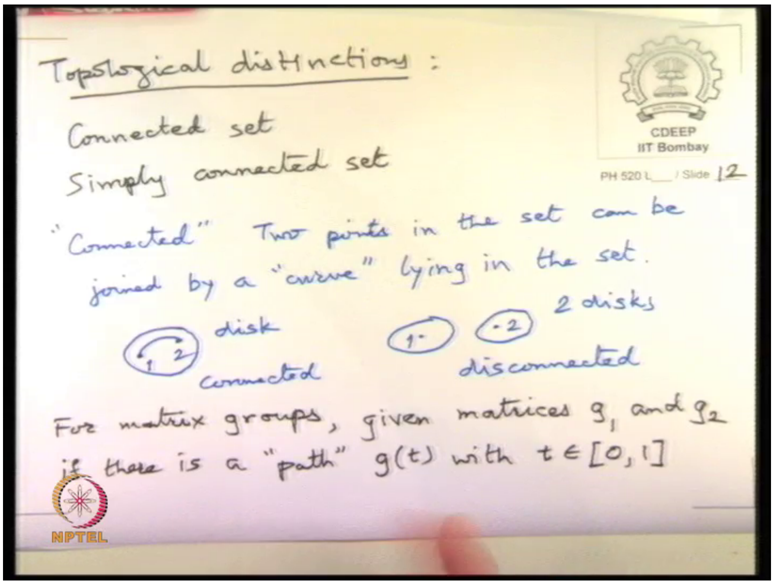

So, we now get to some topology, we will introduce the idea of connectedness and simply connectedness; I still did not introduce the symplectic group we will do it at some point ok. So, connectedness we simply say that so very simply speaking let me just draw some pictures first. Connected simply means that any two points within the set can be joined by a series of by a curve which lies entirely within the set that is all it means or actually two elements, but we are already thinking geometrically.

So, two points in the set can be joined by a curve lying in the set. So, if I draw this disk and I draw 2 disks; the disk is a connected set because I can always draw a curve connecting 1 and 2, but here if 1 is here and 2 is here; then I cannot connect them, so this is connected, this is disconnected ok.

So, this is the very basic intuitive idea and the to formalize we say that given elements g 1 and g 2 there exists a path parameterized by some parameter t. So, let us be specific to matrix groups given, matrices g 1 and g 2 if there is a path by a which we actually mean a continuous mapping of the interval 0 1 into g; g of t with t belonging to the closed interval 0 to 1, that is the meaning of this word path.

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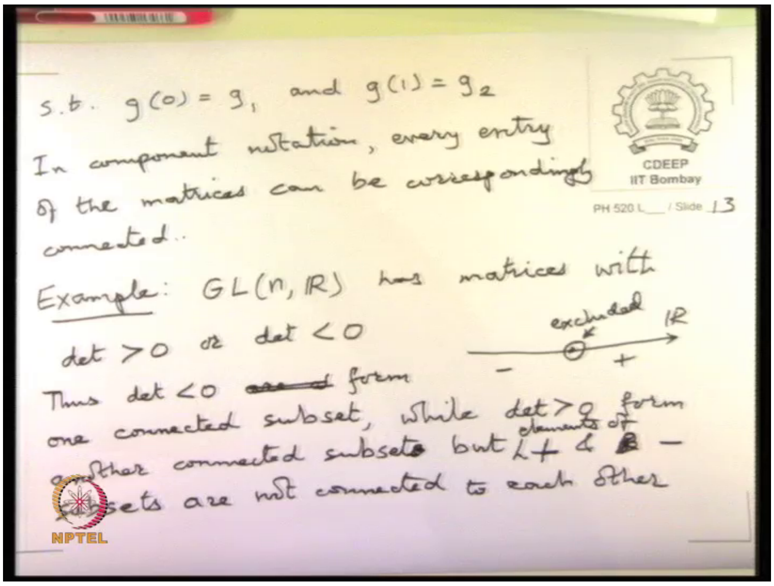

The word path means this map for from 0 1, I have created a map into the group such that at 0 it becomes g 1 and at 1 at becomes g 2 such that g of 0 is equal to g 1, and g of 1 is equal to g 2. So, this of course, as to be the if you write out the matrix as an array then this has to be true for every single entry, entry by entry you should be starting with this entry in this matrix and ending with entry in that matrix.

So, in component notation every entry of the two of the two matrices can be correspondingly connected I am just keeping it short. Now this immediately I mean so far we are just talking some topology, but we see immediately the application that if you take the GL groups, then the GL n R groups are not connected as matrices which determinant either greater than 0 or less than 0 right. Because we excluded determinant equal to 0, because determinant equal to 0 will not allow inversion, so there will be no inverse existing and it will not have a group structure.

So, the fact that it has group structure means that you are excluded 0, but; that means, that because the determinant is just lying on the real line and you have excluded 0. So, these are plus side and minus side and 0 is excluded. So, there is no way of going from one side to the other continuously.

So, all the matrices lying on the left side are connected to each other and all the matrices on this side are connected to each other, but not continuous, but not to each other. Thus determinant less than 0 are all from one connected set, one connected subset, while det greater than 0 form another connected subset, but the two subsets are not connected. I should have said plus and minus to each other rather elements of plus and minus subsets ok. So, the G L n R is a not connected set which side with will you like to be on they are both equivalent right they are both mirror images of each other, but which of these will form a subgroup. So, the one on the which contains the plus will be the subgroup.

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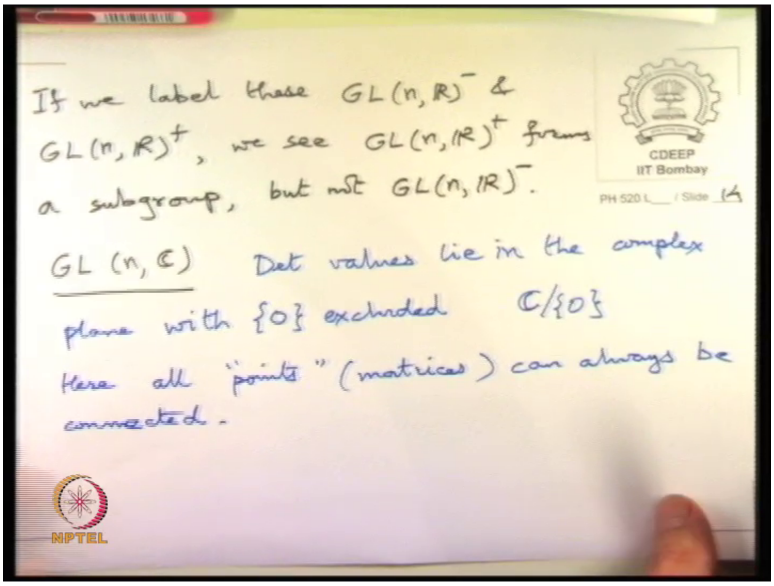

So, if sets are called G L n R minus and G L n R plus respectively, we see G L n R plus forms a sub group, but not G L n R minus although they are isomorphic to each other. But the other one will have a negative of the identity which we do not accept the identity. So but now let us look at the G L n C the story is different. What happens here? The determinant is a complex number.

So, the determinant values lie in the complex plane with 0 excluded. I think sometimes mathematicians write something like this, C with 0 excluded only one point excluded; see how much excitement can be heard by removing just one point out of the plane, remember the point has not even any area. So, if you exclude 0, then you can make it into a group and therefore, the determinant values lie, all over the complex plane.

Now this set is all connected because you are allowed it is not that line, when you have a line; and you remove a point it becomes two disconnected pieces, but the plane with one point removed does not become disconnected. So, here say here so here all points which are actually matrices can always be connected since we are physicist we will not ask for a proof for it right. In complex plane you can connect and it fit to anything, but in mathematics you have to prove it ok. So, G L n C is not connected, it is connected whereas that one the G L n R are not connected.

But there is the next distinction simply connected versus connected. So, simply connected means that if you draw any circle in the set and you can shrink the circle to 0 such sets are called simply connected. So, the simple thing to say is that if you take the corridor outside and the classroom together as one set then it is not simply connected.

Because you can get out of this door and enter through this door and that circle cannot be shrunk to a point you cannot continuously shrink. If you are only inside the classroom you take any circular trajectory you can always make it shrink to 0, but if you include the outside corridor not a larger thing we do not know, but if you take the outside only the outside part, you can enter through one side and leave through I mean leave through one side and re enter through the other then you get a non simply connected trajectory.

So, we will see in that the G L n C although they are connected they are not simply connected because that point 0 has been removed. So, there will like the so there will exists two classes of curves. A class of curve which does not go around the point 0, those curves can also be shrunk to a point and the class of curve that go around 0 and then cannot be shrunk to a point.

These things this kind of thinking developed only around 1900s and later it is it is called topology, it one of the pioneers was (Refer Time: 30:35) and he was also thinking of trajectories in the phase space and the various categories of trajectories that can happen and so on. So, some of the classical generalities of classical dynamics were also couched in this language.

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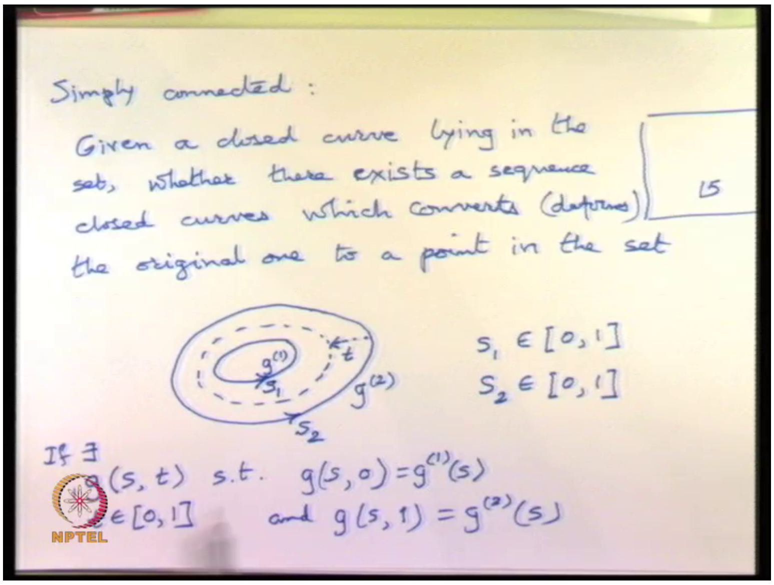

So, yeah so this started with those kinds of reasoning and it is today part of topology. So, the idea of simply connected is that given a close curve lying in the set, whether there exists a sequence of close curves which converts the or deforms the original one to a point and there is some formal way of saying all this which is right.

So, simplest way of thinking about it is to I will not write out everything because it just is a bit lengthy and not so illuminating. Suppose we draw two such things two curves and put a parameter t 1 on S 1 on this and put parameter S 2 on this. So, both S 1 and S 2 go from 0 to 1; that makes each of them a circle, but now we also introduce a parameter t at which we construct intermediate circles.

So, if there is a path which we saw earlier what we did not right it out; we yeah we wrote we wrote this g of t yes, so for matrices we wrote we wrote g of t. So, now we say similarly if there is g of S and t such that g of S comma 0 which of course, t belonging to again 0 to 1 such that t of g of S comma 0 becomes equal to g 1.

Now, I have to put 1 somewhere else g 1 of S and g of S comma 1 becomes equal to g 2 of S. So, I call this curve g 1 the inner one should be g 1 and the outer one is g 2. So, I have a curve g upper index 1, and it is put in bracket because it is not really important as a number it just a label whereas these are important as numbers. So, this is as labeling one possible curve whose parameter is S 1, so g becomes equal to you can I do not want to model this up. But there is a more general map with two parameters S and t with t running 0 to 1, S running also 0 to 1; said that if I ran over all the parameter values I will be covering this whole annular region ok. Now these two are said to be deformable to each other.

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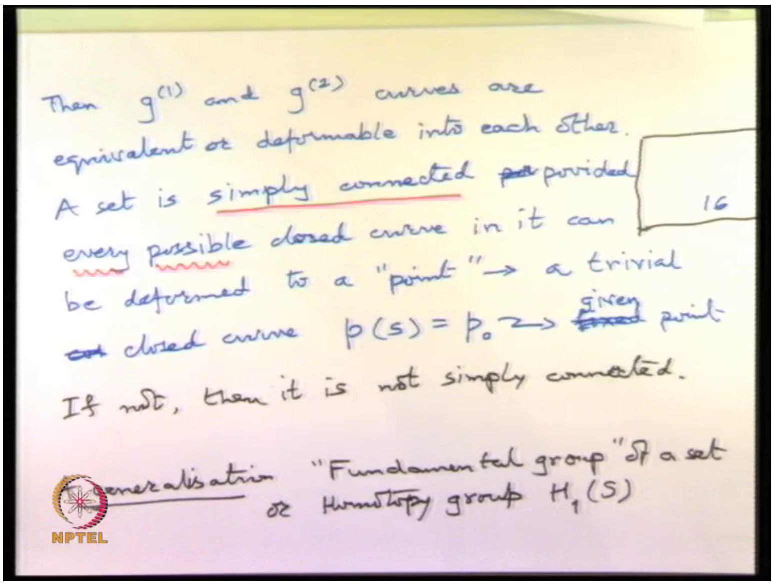

And then we say that a set is simply connected provided or deformable into each other. A set is simply connected provided every single every possible closed curve in it can be deformed to the trivial close curve to a point, some fixed point given point some p 0 given point; for all values of the parameter it is just the same point. So, if every single, this every possible is a important thing; every possible closed curve in the set should be reducible to a point then it is called a simply connected set. And if it is not then it is not simply connected.

In fact, in topologies somewhere interesting things are done based on this, which is that when you do not know too much about a space or do not want to become too specific. So, topology studies point sets in their most generality without imposing too much structure like length, and angles, and anything. So, if you just want to know the properties of a set you like casting a net, you throw all possible circles you can throw into it and check how many of them can be shrunk and then there can be non shrinkable once, but independent non shrinkable once.

So, if you take the surface of a torus then you can of course, draw little curves on it which can all be shrunk to a point or I mean you can draw even a big one, but such that it is shrinks, but you can make a one that goes around one of the circumferences which cannot be shrunk, because it is stuck like this, but you can also draw a circle that runs like this and that also cannot shrink because it is stuck around this circle.

So, there are two in-equivalent circles which cannot be shrunk to a point. By studying all the possible circles that cannot be shrunk to points or cannot be deformed into each other you can classify the space. So, this is called and actually the beautiful thing about it is it actually form has a group structure of it is own. So, it is called homotopy group and that captures the, it is called the fundamental group of the set. So, maybe later on we will come to fundamental group.

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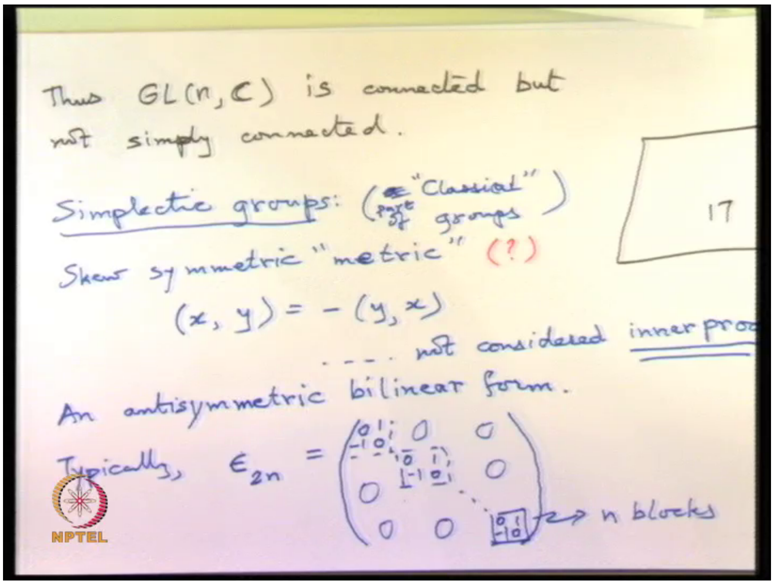

Generalization or homotopy group H 1 which should suggest to you that you can also define H 2, H 3. You can classify a n dimensional space by casting spheres S thus you know spheres of size n dimension n into it and see whether they can be shrunk or not. So, that essentially captures the topological the connectivity aspects of n dimensional spaces even when you do not care about anything else ok.

So, for the time being that is so we saw that so did we assert that therefore, the G L n C is not simply connected ok. The next thing I will do is what I missed in the middle which was introducing the symplectic groups; classical not a classical groups part of this is the case when the metric is a funny one it is not diagonalizable, but it is Skew symmetric metric.

And now this metric is actually very doubtful, do you really want to call it a metric. Because when it a skew symmetric it means that x axis 0, it is it has the property that x y is equal to minus y x. So, it has no magnitude notion in it actually because x will always be 0. So, it is not called a not considered an inner product, it does not qualify to being called an inner product, but it is still a metric in some general sense it is a bilinear form. So, this is an anti symmetric bilinear form the ones that classifies inner product are symmetric bilinear forms and this is an anti symmetric bilinear form.

So, what happens in this case is that so typically the metric in this case we call epsilon it has 2 by 2 blocks; n such blocks, but each block is 2 by 2 and unit it reads 1 and minus 1 and everything else zeros.

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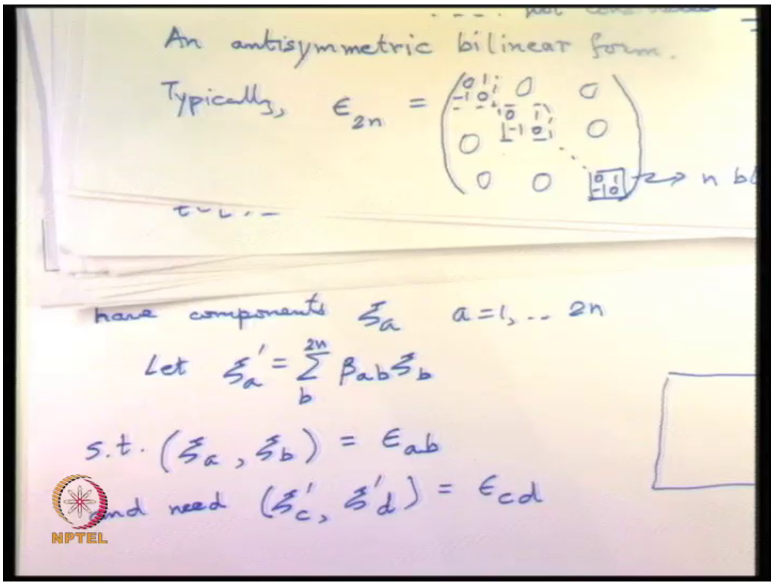

This is one version of writing. There are other versions where people will would plus I mean put entirely along this of main of diagonals minus 1's and plus 1's, you can recast it, so that it goes all along this way. But this is a most convenient one because it remembers which element is related to which so ok. So, this is what the metric looks like and this is what it is left invariant by the elements of this group. So suppose so in what did we call, algebraic or index notation? What is the time? Ok.

Let the vectors be carrier space have components xi a with a equal to 1 up to 2 n, we need even number because we have to you know make it anti symmetric in each block. Then what we need is the and suppose we let xi a prime be equal to some transformation matrix beta a b xi b sum over b up to 2 n. And what we need is that we are given xi a, xi b; the yeah this inner product is equal to epsilon a b and need xi prime c, xi prime d also equal to epsilon c d. So, what this does is that so the point is that these betas I can pull out of this.

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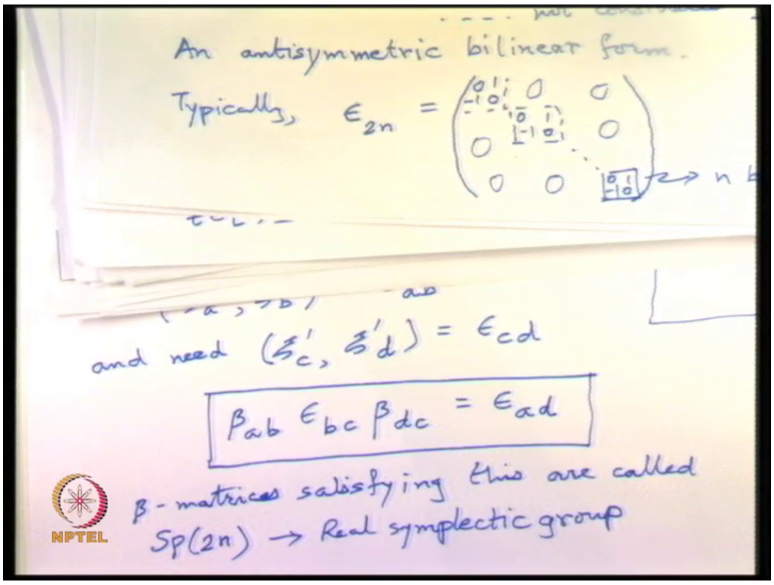

And what I will find is that beta I will see it by looking in my notes, because time is a little short. So beta a b epsilon b c and beta d c equal to epsilon a d; so this is the requirement that the beta matrices have to satisfy, so beta matrix is satisfying this are called S p 2 n Sp twice and S, and with real and it is called symplectic group. And by default dimensions are even number, so we will stop here today. So, we will see that the Sp 2 n groups are groups of invariance of canonical transformations.