Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 03
Basic Group Concepts & Low Order groups - I

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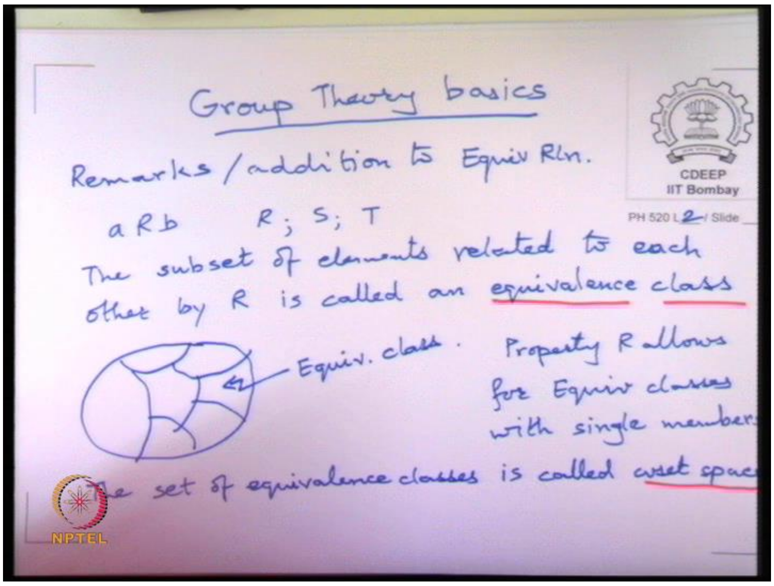

Today we begin with group theory basics, but before we go to it we need to clear up something that we did towards the end of the last class, which was about the equivalence relations. So the main so just to remind you what we said was that there is some relation, we specify and there it has to be reflexive symmetric and transitive. So those are the 3 properties and then we said that we proved a theorem in roughly not very rigorously, but we checked that this operation actually sub divides the group into disjoint sets, such that the union of those disjoint subsets becomes the whole set.

The one terminology I did not introduce last time was equivalence classes the subset of elements related to each other by R is called an equivalence class. So the set of subset of elements which are all related to each other forms one equivalence class. So what we drew this bag of potatoes we were drawing each one is essentially one equivalence class, and our statement was that the groups splits up into disjoint equivalence classes such that the union of those equivalence classes is again back the whole group.

And here the reflexivity allows for equivalence classes of single elements ok. So property R allows for equivalence classes with single members. So you may have something that is related to itself, but is not related to anything else in the group then that will become an equivalence class by itself, and typically we will see in group theory very often the identity element may end up forming in an equivalence class by itself under some property that you specify. So that will happen there is one of the terminology which will again encounter later, but sometimes we want to think of each of this equivalence class as one element. So the class essentially if you pick any one element from it represents all the others as far as that equivalence relation is concerned in that case this is a set is called a coset space.

So, this is a set of sets so we will see more about coset spaces later, when the equivalences are themselves more refined here our equivalence relation R is very primitive, but we will see more advanced kind of equivalence relations and they are the coset space idea becomes more important. So what we have drawn here is effectively the whole set and equivalence classes in it or equivalently you can think that we have drawn the equivalence classes which make up the coset space ok. So now we begin with the so called group theory basics, and what we will do is we start thinking of some specific examples ok.

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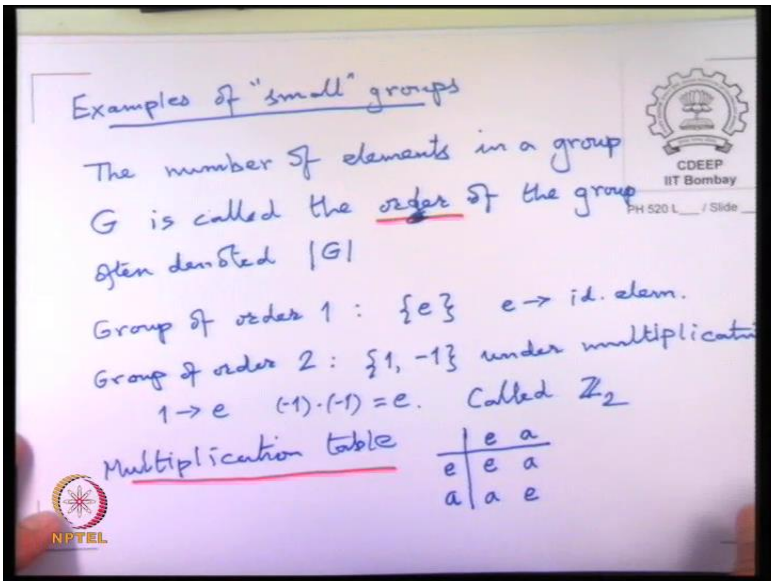

So examples of small groups so to quantify what is meant by small group we give a definition of the order of a group, the number of elements in a group is called order of the group. In a group and we will often write the symbol G, order of order of the group, and we often denoted as with |G| which is which stands for the group with modulus sign is used sometimes if we expressitly know the number of elements we may just write n, or whatever the number is but if not and we are discussing generalities and we may just write this.

So, this is called order of the group let us think of very simple group the simplest group of order 2, well the most trivial one is group of order one this is a group that consists of the identity element right, there is nothing very profound about it if you can probably think of some examples, but it certainly is a group an element that multiplies that under group combination gives back itself its square is same it is itself then it is its own inverse of course, So that is the simplest to the possible group.

The next possible group is group of order 2 and here we can think of a simple example of {1, -1} under multiplication. So the group operation is the usual multiplication, so we can see that $(-1)^2$ so one is the identity obviously, and $(-1)^2$ is so 1 i.e. (-1) (-1) is its own inverse and so on ok. So this is the simplest possible group this group is sometimes called Z_2 as Americans call it with this script Z as you know the script Z is used to denote integers all integers from minus infinity to infinity including 0 and if you write something below then it usually refers to a class modulo that many integers.

So, this group {1, -1} under multiplication is called Z_2 you can also well i, so this is one example of the simplest group, and we can draw table for it which is going to be technically useful concept and device. So we may as well underline it and we draw it like this. We list the elements here e and a.

So now I am being a little more general and just saying that there are some 2 elements either identity has always to be there, and there is one more element which I call a. What can be the possible multiplication table $e^2 = e$, e a = a, a e = a and $a^2 = e$. So this is a group multiplication table, now the point is that here I did not write 1 and (-1) now the table is more general than the specific example we had and so we will soon see how the generalisation is made to an abstract group.

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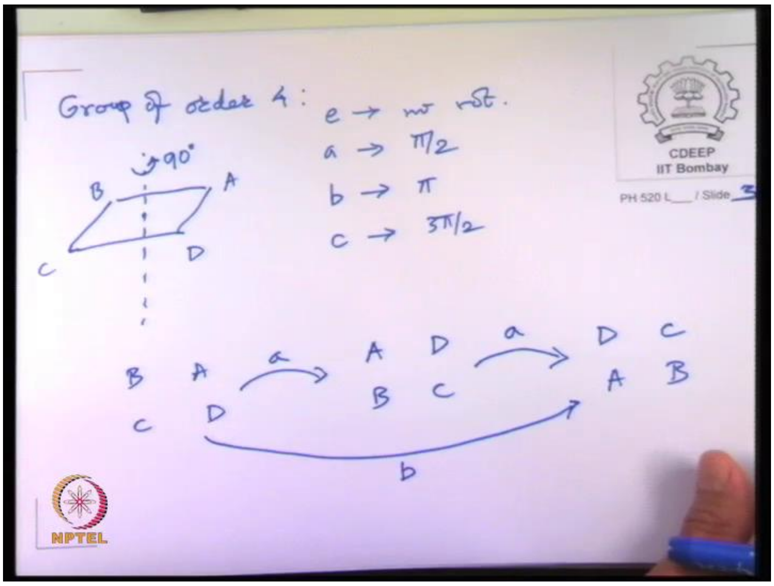

But this is a very specific example, there happens to be no group of order 3. So we next have group of order 4, here let us think of a simple example of square, which we rotate about the axis perpendicular to it by 90 degrees at a time. So if we have a square. So let us say that if I rotate by 90 degrees. So of course, identities no rotation and let us say a is 90 degrees or $\pi/2$, b is π and c is a $3\pi/2$. So this is a very simple example of a slightly more difficult group, we can see that it is satisfies all the group axioms, because it is closed all the operations are closed you combine any 2 of them you reproduced some another operation. So remember that the configuration of this square, you may even try to remember what the. So call it ABCD ok.

So if you perform any operation then the configuration will change, but the square is just a generic square; so the ABCD label is only for remembering what happened to it. So we can see that the configuration will change, do not mix up the configuration with the operation ok. So if I have ABCD, let us draw smaller picture if I do operation a, it becomes A B goes here, C goes here, D goes there. If I do another operation it is still operation a, but now it becomes DCAB. So this configuration now is rotated with respect to the original by full pi, but the operation just rotated it one at a ones a time. So the configuration is that you have in the space on which the group is acting, is to be distinguished from the operation of the group. So of course, we can write the whole thing together that this is nothing, but our element b. So we learned that a squared is b now in general when somebody suggests you any this is all very simple.

But in general when you are suggested an example, it is best to enumerate the elements you have, and then try to work out the whole multiplicative multiplication table and you may have to sometimes even make a little model, you have to cut out a little square out of a piece of paper and then see what is happening.

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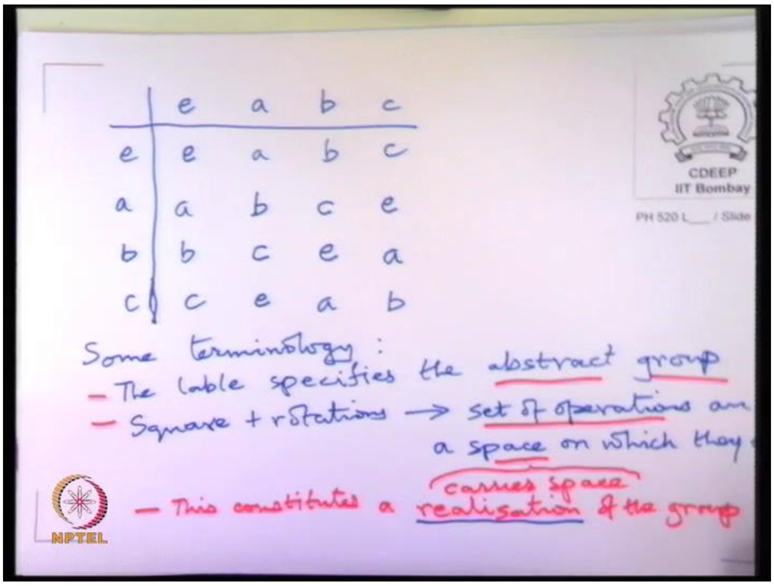

In this case things are very simple so, we can just write out the multiplication table; so e a b and c. So this of course, is e, and we can see that the top row and top column actually just repeats the labels above. Now a squared as we just as we just checked here, a squared is basically b; so we get b, if we already have b which is a π rotation and then do additional π/2 rotation will get 3π/2 rotation. So this is c and then we are back then we should have you can see now the only thing left is e so; obviously, π/2 rotation, 90 degree rotation combined with a 270 degree rotation gives you back identity.

So, that is how this row goes and similarly if we start with b followed by a, then we should get a c right and 2 b's is going to be identity, because b is a π ration and b times c will be equal to a, because b is π rotation and then you do additional 3π/2 rotation you will get a π/2 rotation. So that is equivalent to a finally, as far as the 3π/2 goes, this time a will be equal to identity right 3π/2 an additional π/2. And then 3π/2 with the π will produce an a, and two 3 π/2 rotations will produce π effective. So it will to produce 3π, but everything is modular 2π. So it will produce a π rotation which is b. So this is the multiplication table for our group.

We are ready to say a few things we say that we introduce some ideas. When we introduce an abstract group this multiplication table by itself is completely abstract. If somebody hands you this and says this is my group, all you do is check in at that it does not conflict with any of the group axioms and then you can accept it as a group. But what we do is we say that, the table defines an abstract group the square we had is the square along with the operation geometric operations on it we call it a space on which it acts some operations and the space on which acts.

So, our square plus rotations is actually set of operations and a space on which they act. So this is the generic property of any group. You will in all the if you have just given the multiplication table that, is enough to define the group in practice you will want some realisation of it. So this is called the carrier space on which those things act now this is called this whole package in number 2 second dash is called a realisation. Finally, we have the notion of representation, which will have lot more to do with later on.

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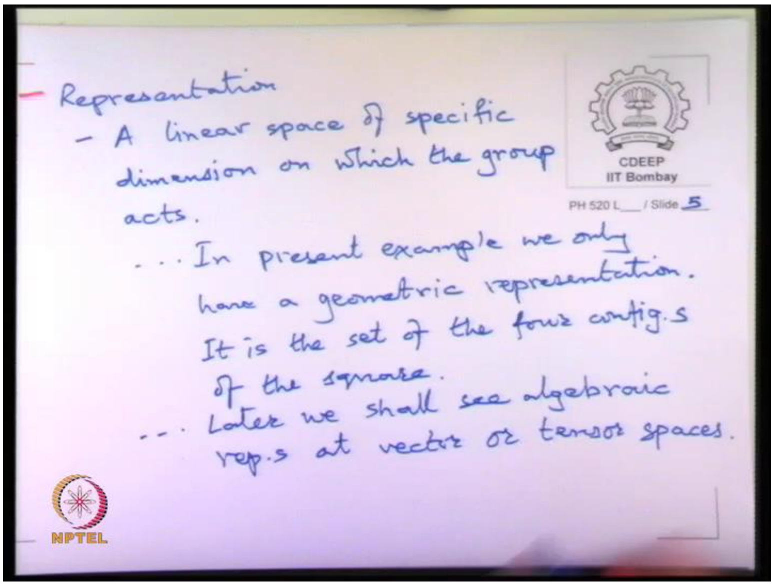

We can say that the set of rotations actually is. So when we say representation, we mean a particular size vector space, a linear space of specific dimension on which the group acts. Now here we are not actually laid out the linear nature too much, but you can say the 4 vertices or that geometrically we have only a geometric representation so far; in present example we only have a geometric representation, which consists of the 4 configurations it is the set of the 4 configuration. So that is our representation space here.

So we will have more to say about this a little bit later, I should just to warn you that occasionally the way people use language then mixup realisation and representation because they are very close. Realisation is the whole package, some space on which the group acts. So that whole package is a realisation of the; it is like a concrete realisation. Whereas representation is just the base space the configuration of the base space that actually can be written more or less like a vector or a tensor like a linear space. So we will have more concrete algebraic representations we will see later representations as vector or tensor spaces ok.

So we are going hand in hand setting up some examples, but also setting up some terminology, the next we come back to specific terminology. So next we discuss the possibility that. So we can say generators.

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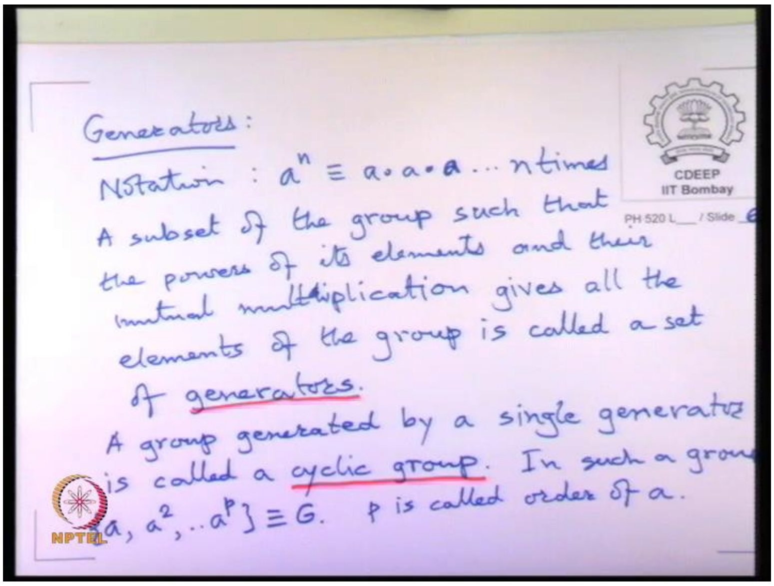

By generator we mean some set of elements, which when raise to some power multiplied by itself or among themselves generate the whole group ok. So we already saw a trivial example I have a 90 degree rotation. If I keep doing 90 degree rotation several times I actually recover all the 4 elements of the group. So the idea of a generator is. So first we say some notation $a^n = a \bullet a \bullet a \bullet a$.......n times. So if there is a subset of the group such that the powers of a, it is elements and then mutual multiplication. Multiplication gives back all the elements of the group is called a set of generators. The simplest case is when you have just one generator. A group generated by a single generator is called a cyclic group.

It is like a cycle, it is like you multiply several times you get all the group elements and then you start again right. In cyclic group it must happen that, you have one element you start raising it to powers a, a times itself, a times itself thrice a^3, a^4 eventually it has to become identity because the total group size is finite and we have closer. So if you multiply any number of times you must get back group element, but the total number of elements in finite. So eventually a^p, should reproduced the identity element.

So in such a group $\{a, a^2, ..., a^p\}$is the whole group and p is called the order of the element a, a which is the generator and of. So I I forgot to add that. So where a^p is equivalent to identity, then p is called order of the element a is of this generator a. And we already saw that example because our group of this multiplications this rotations of the. So this particular example essentially the 90 degree rotation generates all the other elements. So it is a generator and this is a cyclic group of order 4.