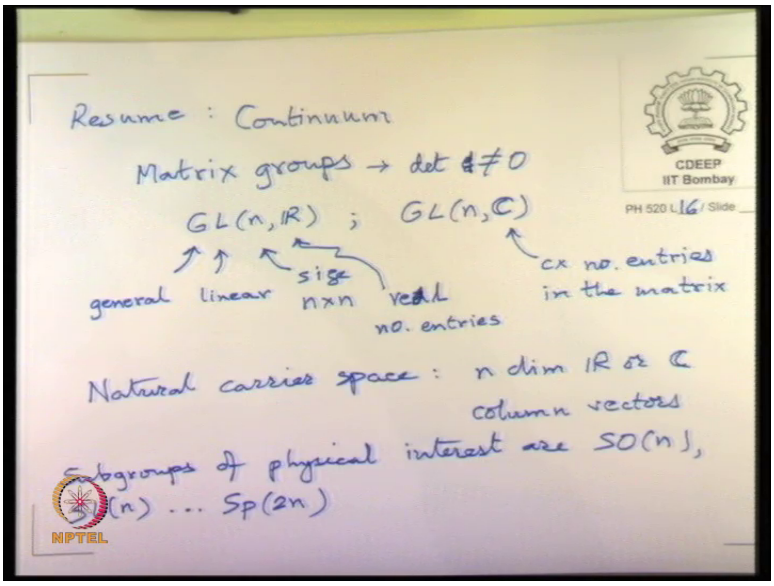
Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 29
Classical Groups – Topology - I

(Refer Slide Time: 00:18)



So, when we get to matrix groups, the statement was that we need matrixes of determinant not equal to 0 because that is the main point. Everything else is. So, very sorry determinant not equal to 0, that is what we need and so, the overarching groups biggest matrix groups are called G, L, n, R and G L, n, C; where the G does says general, that is s general as it can get subject to only this condition determinant not equal to 0.

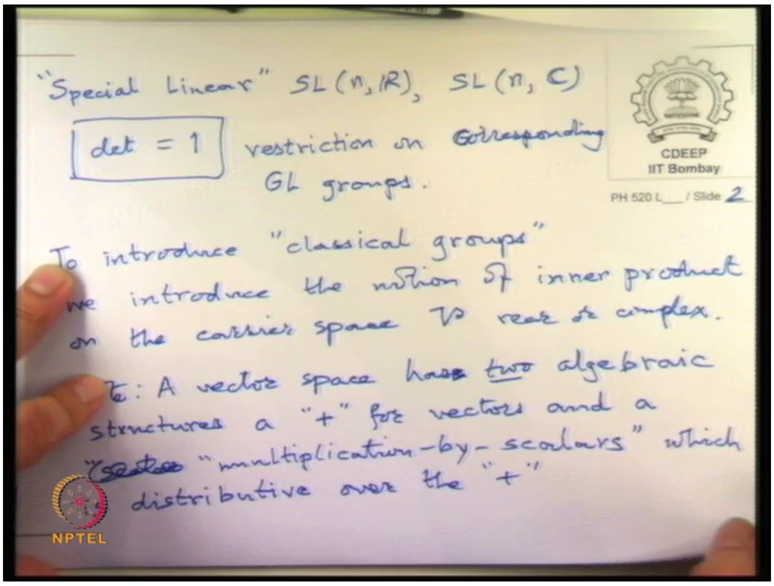
And L does means linear, n is the size of the matrixes n by n and R means it is over reals. So, it is a over a real vectors space or we can for this say real entries and here it is complex number entries in the matrix.

Now, these groups they have a natural carriers space on which they act is essentially n dimensional real or complex column vectors, that is the convection we could have taken them to the row vectors, but column is the convection and then the row vectors are called the dual that the row vectors are called as space dual to these. I did not use this terminology so far, but and maybe we do not it. So, do not worry, but the row vectors are not the primary space it is kind of a mirror image space of the column vectors.

So, this is the basic layout and that is where one begins, and what one shows. So, what one shows is that there are a variety of subgroups of the general linear group, which are of physical interest. So, SO n, SU n etcetera and also what are call symplectic groups S p 2 n. So, these are all various groups motive met they are all matrix groups, they are motivated by various physical consideration and they are subgroups of the big matrix groups G, L, n, R and G L, n, C on whom the only restriction is determinant not equal to 0.

So in fact, we next go to one class of groups which are not very commonly used in physics, but they at least one of them is used and it is not any of the once I listed that is call the special linear groups.

(Refer Slide Time: 04:57)



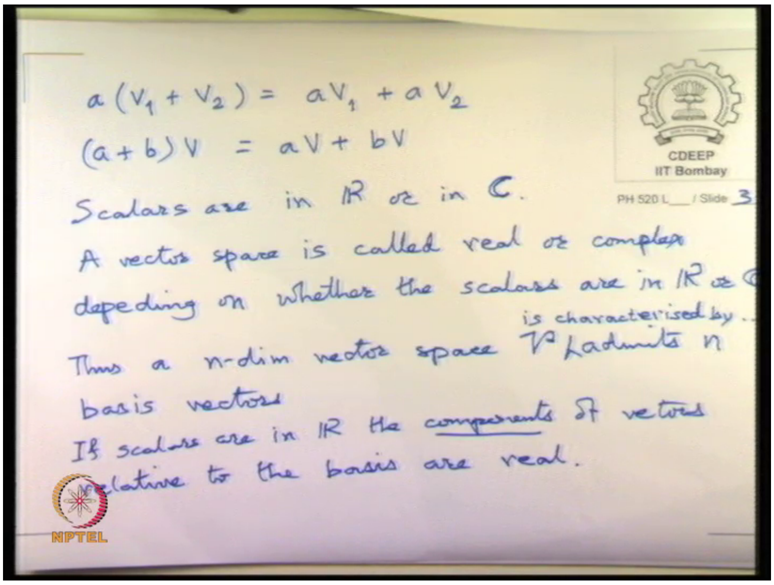
The only restriction these put on the G L, n, R and G L, n, C is that the value of the determinant is plus 1 ok. So, here the requirement is determinant equal to 1 restriction on corresponding G L. So, earlier the determinant only was non zero, but it could be any finite value, for any value in the complex to n O d l numbers, but now further we also require it could be determinant equal to plus 1.

So, we will see that sometimes these groups are also interesting to consider and in fact, they are very useful even for the Lorentz group. Next to talk about these groups SO n, SU n, S p 2 n it is useful to introduce the idea of inner product. We introduce classical groups, notion of inner product on the carrier space, which are some V real or complex.

Actually it is works knowing a particular extract mathematics fact that, we tend to think of the vectors space as real or complex by saying its components are real or complex.

But in general a vector is by itself not a number. So, the components is only a language in terms of which we think. So, in the abstract language, a vector space becomes real or complex depending on whether the scalar multiplication is by reals or complex. So, note that vector space has 2 algebraic structures, a plus for the vectors and a scalar multiplication or multiplication by scalars not correctly let us write that to (Refer Time: 08:55) some, which is distributive over that plus. So, we have the property that which we have a times.

(Refer Slide Time: 09:31)



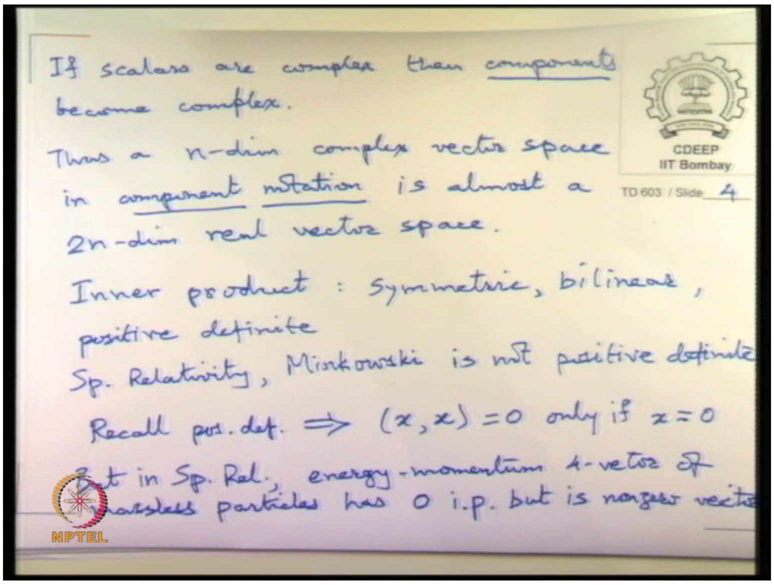
V 1 plus V 2 it becomes equal to a V 1 plus a V 2; it is also true that a plus b V is equal to a V plus b V. So, the scalar multiplication by scalar satisfies its quantity, these properties some sometimes in mathematics they generalize this set of scalars.

So, scalars are typically either reals or complex. Now the main thing that I was earlier saying a vector space is called real or complex depending on whether the scalars are real or complex. What this means is, you can have a vector space of dimension n at this point I will tell you nothing about whether it is multipliable by re[al]- scalars or complex numbers.

Then there it is just means that it has n basis vectors or vector space V is characterized by the fact that it admits n basis vectors at this point this basis vectors I mean either real or complex such as basis ; however, if I now say that the vector space is real then. So, if the scalars are in r then components of vectors relative to this basis are real that is what this means the components become real because you said the scalar multiplication is by reals.

The same n dimensional vectors space, if it is now if I allow complex scalars then its scalars are complex then components become complex.

(Refer Slide Time: 13:07)



But what this boils down to in practice is that secretly it becomes a 2 n dimensional real vector space because the complex numbers you can read all the real multiplication separately and all the imaginary part multiplication separately.

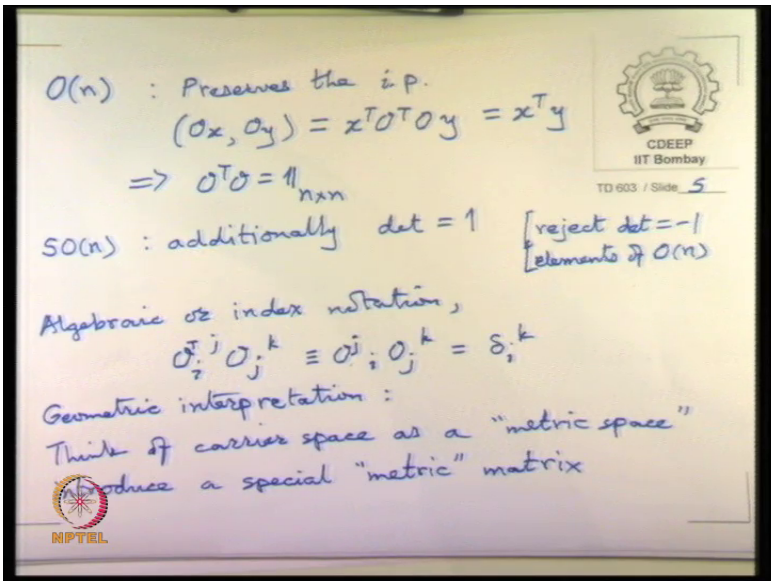
Thus say n dimensional complex vector space in component notation is almost a 2 n dimensional real vector space. So, we just keep this in mind, but remember that it is not that the basis we can complex because vector space is a vector space, it is neither belonging to real numbers or complex it is the scalar multiplication that makes it real or complex.

Now, we come to our inner product idea and we discussed it already last time, it has those three properties of symmetric bilinear and positive definite. Symmetric bilinear and positive definite and we have to add the rider that when we come to special relativity the Minkowski metric does not have the third property.

In fact, so, I remember positive definite means that x, x the inner product of x which itself equal to 0 only if x its itself 0 vector. So, always emphasize this, this 0 is 0 vector this 0 is the number zero, but nobody bothers distinguishing them, but you should know that the different. So, the positive definiteness idea is that it becomes 0 only if x itself is 0, but in Minkowski space the direction of propagation of light or the energy momentum carried by light is a 0 magnitude Minkowski magnitude vector the 4 vector of photons of massless particles has 0 magnitude, but is non zero.

So, we will remember that distinction and carry it on later, I now wanted to go back to that 2 things that we wrote one is algebraic form and one is geometric way of thinking.

(Refer Slide Time: 17:46)



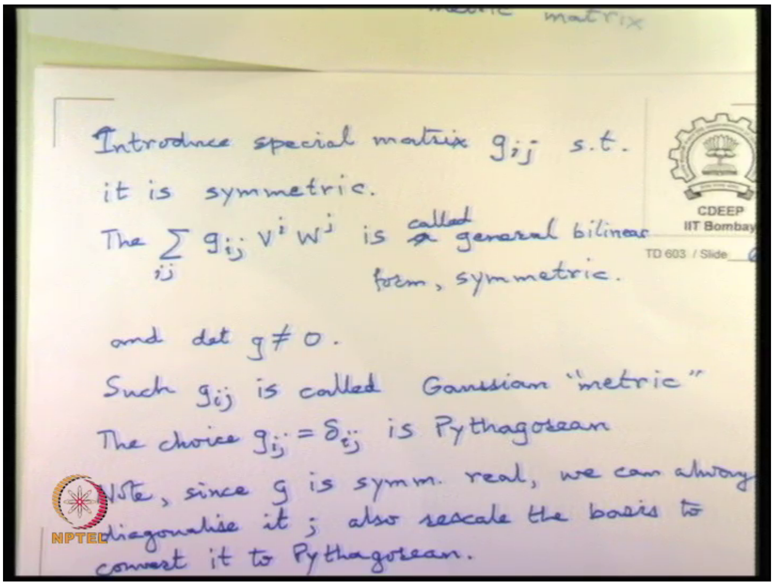
So, we introduced O n is basically one that preserves the inner product, which in matrix languages x transpose, O transpose, O y is same as x transpose x x transpose y this requires that O transpose O equal to identity matrix of size n by n and then S O n additionally as determinant equal to 1. So, we check last time that since O transpose O is equal to 1 the determinant has to be plus or minus 1.

So, if you restrict yourself to then you. So, reject determinant equal to minus 1 members of elements O of O O n. There is a there are 2 other ways of writing this out which is one is a geometric way of thinking and the other is algebraic detailed way of writing. So, the algebraic or index notation we write O i j O. So, I have to write O transpose sorry; now I have to really be careful what will do is write O transpose i j and then I have to multiply O j k that is matrix multiplication, but that actually same as O and then we will put j here and i here O j k equal to. So, this is the transpose of that, but I remember that this index is up and this is down this is what slight complication is there because of the way we are dealing with this and this notation is useful when we go to the Minkowski space.

So, I will just define it like this and so, this should be equal to the identity matrix which now becomes delta i k for identity it actually does not matter where i n k go, but. So, this is a index notation for stating the same thing and then there is a geometric interpretation. So, we earlier defined inner product essentially algebraically although we did not write an indices, we just gave the algebraic properties which should be symmetric by linear positive definite if the geometric way of thinking about it is that I actually I am defining a distance a metric on the career space, and introduce a metric a special metric matrix I am sorry to say this, but that is what it is. So, the metric is the abstract motion and matrix is of course, one of the arrays array matrices.

So, I introduce a special matrix, which is let us call it g in one or 2 gauss and because it is become standard notation.

(Refer Slide Time: 22:08)

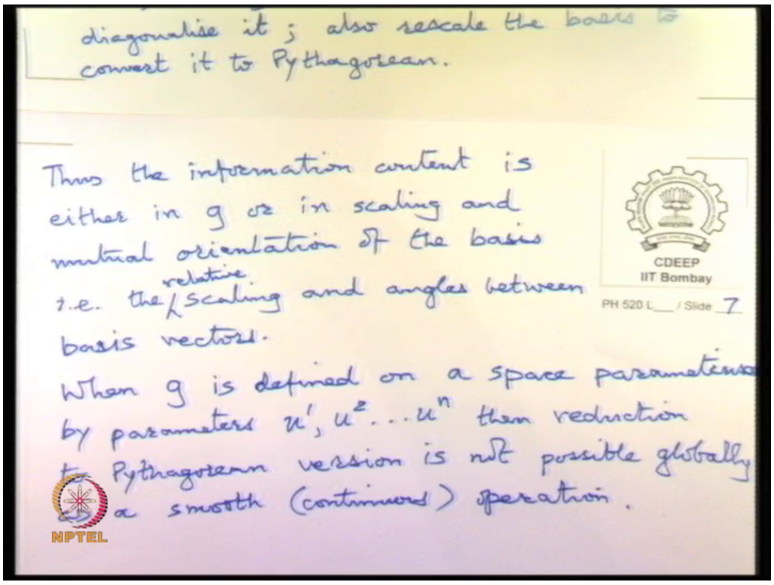


So, introduce special matrix g and let us call it g i j such that such that it is symmetric then g i j, v i w j with summation over i j is a general bilinear form is called this is the one of the old nomenclatures. So, it is called general bilinear form which is symmetric and we also required that determinant of g is not zero. So, that it does not have some null row vector.

So, such a g i j is called a metric in Gaussian and its geometric interpretation is that it actually kind of stretches and strains the vectors. So, the let me write one more thing the choice g i j equal to delta ij is called Pythagorean and is the simplest. In fact, a g as introduced is a symmetric matrix, a symmetric matrix can always be diagonalized and a symmetric matrix by transferring its diagonal whatever scales that whatever eigenvalues there are you rescale the basis vectors with inverse of square roots of those, you can make the matrix completely identity matrix. So, by g is symmetric real matrix, we can always diagonalize it also rescale the basis to convert it to Pythagorean.

So, in a sense the information content in both is quite similar or can be shifted back and forth between the choice of your basis and the choice of coefficients in g, where it comes useful is where such a metric changes from one point in space to another. So, for example, if you use space spherical polar coordinates, then the metric is r square sorry 1 then r square and r sin theta squared whole squared. So, if as a function of the coordinates the metric keeps changing then this comes in useful.

(Refer Slide Time: 27:44)



Thus, is either in g or in scaling and orientation of the basis.

So, now I have already made you accept what is geometric about it. It is actually just stretching and the angles between the basis vectors, relative scaling and angles between basis vectors. So, you might say what is the advantage then always use Pythagorean, but the answer is that when g is defined on a space parameterized by some parameters that is a repetition, but anyway say u 1, u 2, u 3 then reduction to Pythagorean may not be globally possible not possible as a continuous operation; if you want to do it as a continuous function. So, the Pythagorean version reduction to Pythagorean version is not possible globally as a smooth operation smooth or continuous operation.

So, our simplest example is the curvilinear coordinates we use, but the curvilinear coordinates used in most physics problems like electromagnetism and so on are still such that the local frames are orthonormal, r theta phi also the r cap vector theta cap vector phi cap vector are mutually orthogonal that much we retain for simplicity, but when you get into more complicated spaces you may be forced to choose even more complicated matrix.