**Theory of Group for Physics Applications**

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**Lecture - 28**
**Classical Groups - II**

Now, we have to be a little careful with the inner product.

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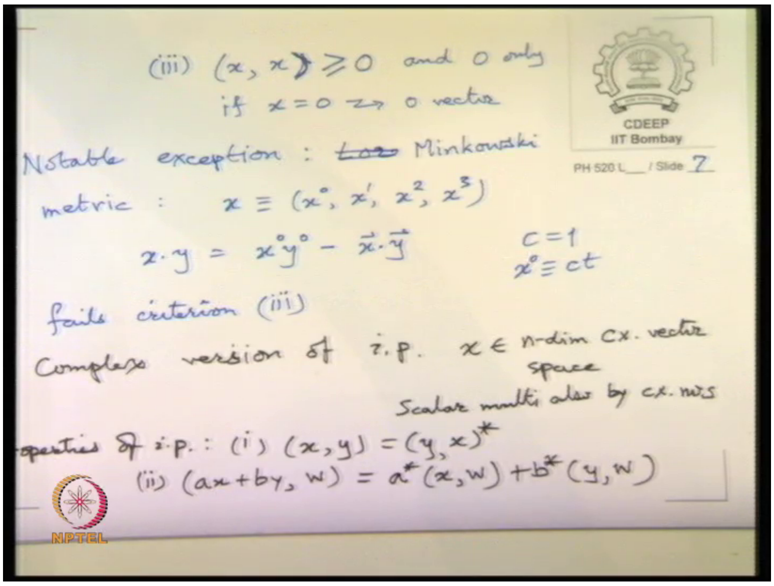


So, properties of inner product we have to say x y equal to y x star this is partly impose, so that the positive definite word ness; ness words ok. So, if you have x x it will be it is own complex conjugate. So, it will remain positive and so on. The second thing is therefore that if I put a x plus b y comma w, it will come out a star x comma w plus b star y comma w. So, in the second slot it will the property will look exactly like this, actually the property is exactly same.

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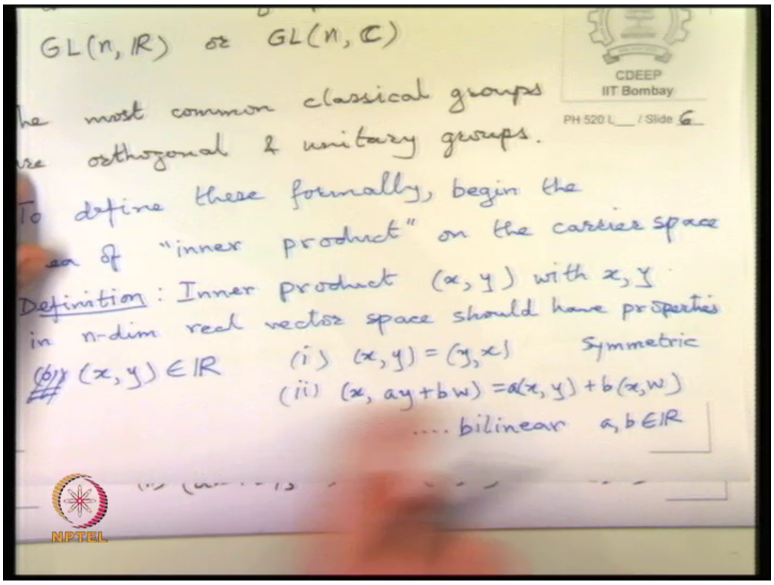


But be if you put it in the second slot, but because when you flip it you get a star that means, if you had done it in the first lot then you are to start the coefficients when you pull them out and of course the third property remains the same.

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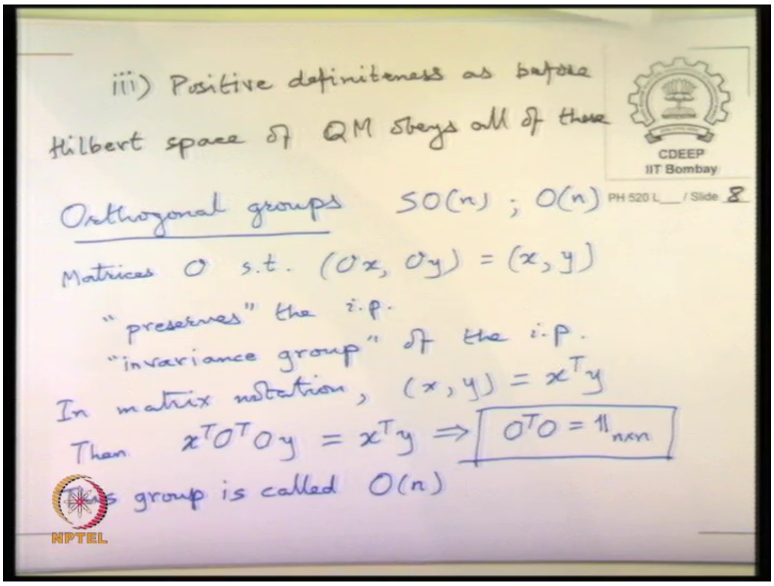


And the good news is that, the Hilbert space of quantum mechanics obeys all of this. If you want to know why it is called Hilbert space after it some complex linear space right, you have to read the notes. It has to do with the infinite dimensionality and the fact that you eventually you have to use the continuum. To define to define plane waves you have to define e raise to I k dot x, where you label the k is the parameter is the wave number, but k takes all the possible continuous values. So, to allow representing a free particle you have to include all those kind of basis states as possible vectors and there are carnalities that arise because, of that the same as the problem of Fourier series versus Fourier transform.

The Fourier transformer is the more complicated thing, then Fourier series, but you need not worry about it right now if you want to read there are discussion, so why it is called Hilbert space. But it is essentially a complex vector space with a continuum number of dimension, continuum number of vectors in it, the dimensionality maybe finite, but in the sense that it may be in x only 1 dim one dimensional two dimensional and so on.

But it is an infinite dimensional vector space as a vector space can be more complicated things happen, when you go to quantized fields like in condense matter physics and so on. There you have to consider in fact higher orders of infinities. Now coming back to this, we will come back to our classical groups and we define orthogonal sets orthogonal group I wrote that 2 soon, there is O n and there is S O n and this is what I will discuss now.

So, essentially what we say is the matrix matrice is O, such that O x comma O y remain same as x comma y the inner product is preserved by multiplication of all vectors by O, the inner product is preserved. So, preserves this i p or we call it the invariance group. In matrix notation we would have the inner product has to be written as x transpose y.

We said that our matrix are matrix our career space are column vectors right. So, the usual way of writing out the inner product is to take x transpose y, and then we can see what we need is x transpose O transpose, because transpose of I should have written that first ok, but you know your grown up people.

So, so therefore the property required is O transpose O equal to identity matrix. So, that is the defining property of orthogonal group and this group is called O n. The set of all such matrices; so obviously a subgroup of the G L n R because, when you take a general linear matrix, which has an inverse, but then restricted to satisfy this O transpose O condition, it is a subgroup of the G L n R and then in physics we have to worry about one more thing. This O transpose O can also have matrices that will flip the axis mirror reflect that axis. So, first we write out so for the specialization.

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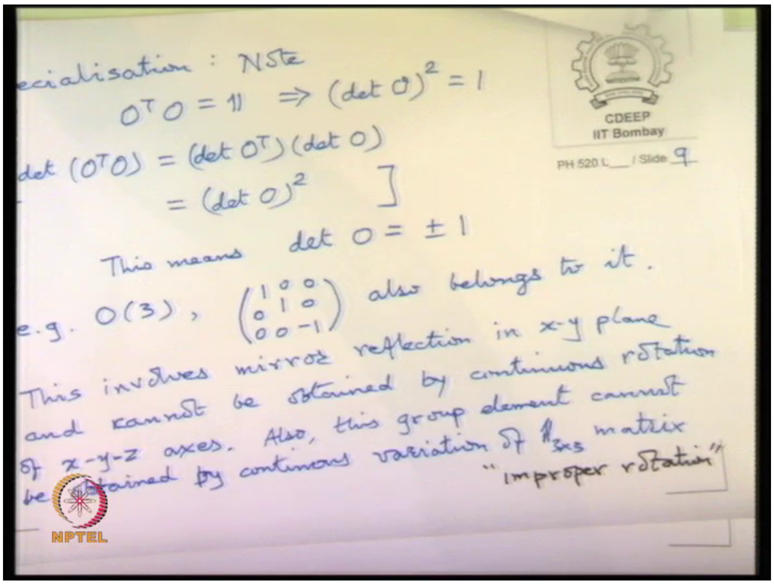


Note O transpose O equal to identity, implies that determinant of O squared equal to 1 right because, determinant of O transpose O there is a theorem of matrix algebra, that this says that this is same as that O transpose times that O and determinant of O transpose is same as determinant of O right.

Determinant of a b is same as the determinant of product is product of the determinates and further, the trans determinant of transpose remains the same as the determinant of the original matrix. So, det O transpose O is same as that O squared and on the right hand side determinant of identity matrix is 1. So, we will get that O squared equal to 1, this means that O can be plus or minus 1.

Now, if we take S O 3 S O 3 so sorry, so if we take O 3 matrix 1 1 minus 1 also belongs to it. But this is not a proper rotation you have to flip the z axis, so this cannot be obtained by rotations that can be continuously reduced to identity ok. You cannot get this by any rotation of the xyz axis that can slowly be transform back to identity.

So, this involves mirror reflection in x y plane and cannot be obtained by rotation. In other words it cannot be obtained as a continuous variation of the identity matrix. So, sometimes we call this improper rotation, is not a it is like a rotation, but not actually a rotation. So, we call it in proper rotation is it safe to write here ok. To avoid, if you want to avoid improper rotation sometimes you want to include mirror reflections not a problem. But if you want to avoid mirror reflections, then you call it then you put a refer the restrictions, the determinant there should be plus 1.

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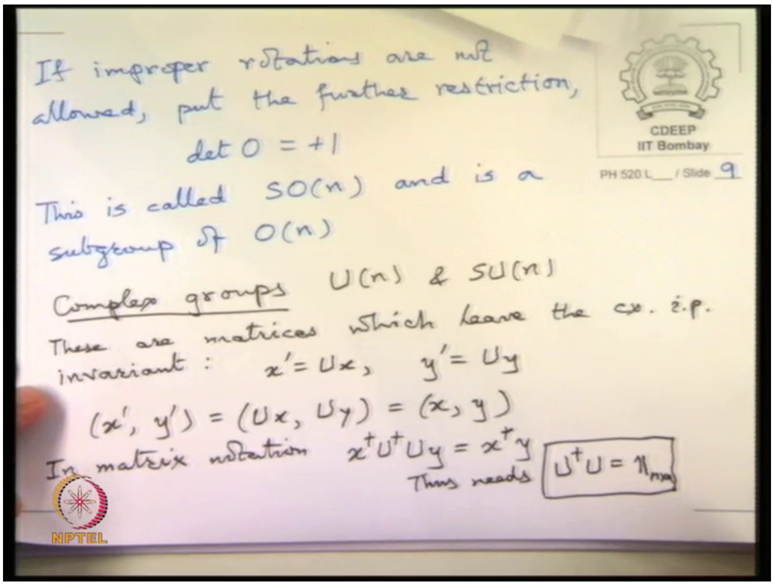


Now, you can see that this det O equal to plus 1 will be a subgroup of the O n. So, this is called S O n, next let us look at the complex groups. So, we have stopped writing in for g l we said comma r comma c here we do not because, by context it is always clear orthogonal groups are real groups and the unitary groups are automatically complex groups. So, here we it is a similar thing, it is leaving the complex inner product invariant.

Again in matrix notation we need x dagger U dagger U y equal to x dagger y because, the inner product in the matrix case will be equal to x dagger y. So, this means that U dagger U equal to 1, identity matrix what this means in the complex case on the determinant, you will get complex conjugate of the determinant.

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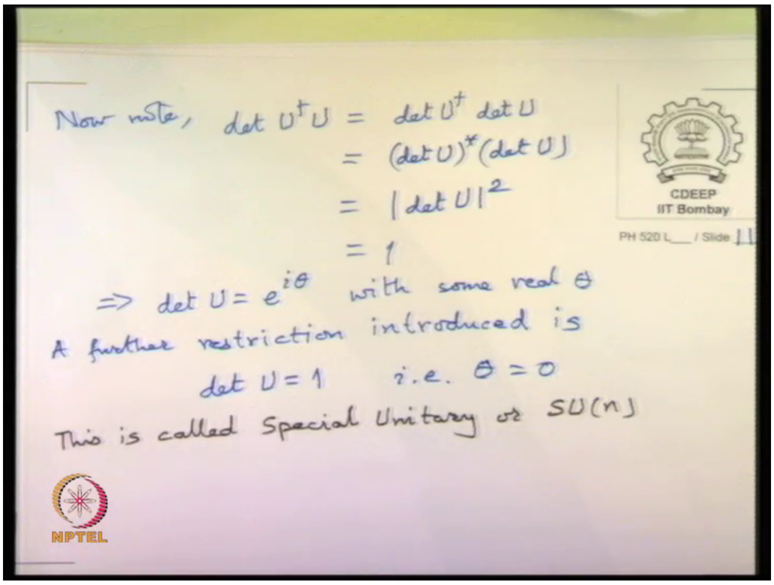


So, this is modulus of mod that U mod square and this is set equal to 1. So, this means that determinant U is some e raise to i theta, it is the phase so it mod square should come at 1. So, this is a much bigger ambiguity than in the orthogonal case where you just got plus or minus 1, but here the determinant can lie on the unit circle to restrict this demanded determinant U equal to 1 i e theta necessarily equal to 0.

This is then calls special unitary group, in quantum mechanics we actually have this the only thing that we need for norm of wave function to remain the same is that they have to be unitary. So, in quantum mechanics ok, so we have to backtrack a little bit, what is the significance of this G L n R are in geometric sense or specifically of. Let us say the S O n and U n O n, S O n, U n sun what do these things do, what they really do is carry out a change of basis. So, we said that the mattresses act on various vectors x goes to O x, x goes to U x, think now x to be 1 of the basis vectors one of the unit vectors.

Then all that we are saying is that we have rotated all the unit vectors the same way by this transformation U. So, it amounts to just change of basis, it is the this operation. In fact, the even the general linear transformation it is essentially a change of basis, but it also allow stretching the basis it does not leave the magnitude 1. Whereas, the orthogonal so we can write this down I want to do 2 things before we close one is this and the other is the slightly algebraic notation. So, maybe we do the algebraic notation first and then we come to this.

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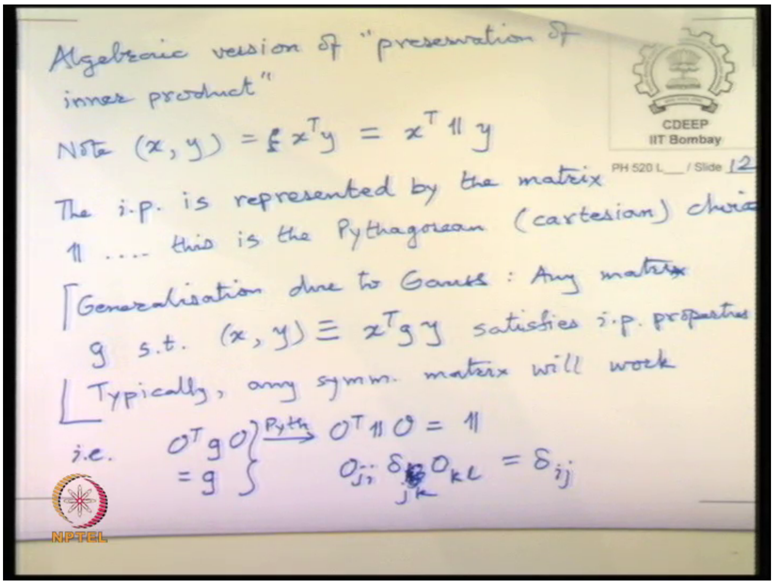


Algebraic version of preservation of metric of the inner product I am sorry. Note that for equal to x transpose y is same as x transpose identity times y, this is what it actually means. We want to think of the inner product a little bit more generally the inner product is actually a matrix by itself.

This is the Pythagorean choice well and if it is uni Cartesian, we can more generally define any. So, a generalization due to gauss any matrix g, such that x y defined by x transpose g y you insert this matrix g. But it should obey all the i p requirement that is all you do not say it is identity matrix you can put any matrix that you like so typical it will turn out it has to be a symmetric matrix that is all.

So, typically all symmetric matrices, so this is the generalization due to gauss and he calls it the metric tensor, the white is tensor and all is a different story. But what we want to point out is that in fact, the inner product all it needs is some matrix g which is symmetric and which will say which is such that x transpose g x will remain positive definite.

So, those requirements is all it means typically all symmetric matrices we will satisfy this and if you want ultimately you can make them determinant one symmetric matrices. So, you we can now think of this as preserving the Pythagorean metric 1. So, this can also be written as O transpose O equal to 1, can also be written as O transpose g O in Pythagorean notation or in Pythagorean choice it becomes O transpose O equal to 1. So, equal to g that is what it actually saying and in Pythagorean notation it becomes this, so it preserves the metric delta ij. So, we would have written O ji delta ij O, so sorry ik so j j has to try to this so k j O k L right.

I wrote so I meant to write the transpose of O ij so it is O j i, but it is the this index which was actually here tried to this. So, j tries so it should be j k I am sorry j k jk jk O k L equal to delta I l. So, that is another way of thinking about what the O matrices have to do.

And similarly we can write out for the complex matrices. So, the with once we write like this, we can write out the Murkowski case.

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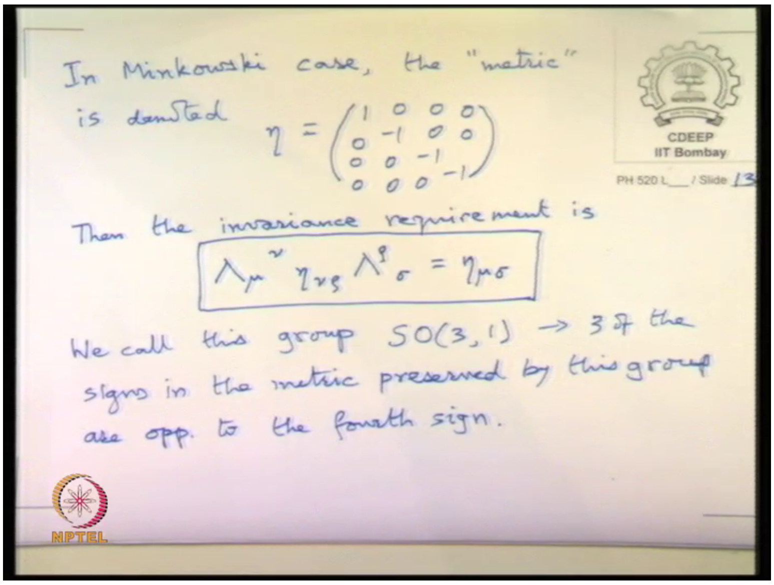


The metric is denoted eta and is equal to 1 minus 1 minus 1 minus 1, this has now become the standard choice in advanced physics. So, in the old days they were taking minus 1 here and taking plus 1 0, because it will written the usual Pythagorean think for the 3D. But overtime this has become preferred 1, so this is preferred because the energy momentum energy will have a positive sign because, this is the 0 th component that is the difference. But this has become the standard choice and in this case, so this generalizes the Gaussian restriction of positive definiteness, but now it is indefinite that is the only difference and then the requirement is.

So, this is another way of writing out the Murkowski inner product, this group you should check here some notation here because, so the placement of indices is somewhat important I will correct it next time. But this group we call S O 3 from comma 1, this means 3 of the signs in the metric preserved by this group are opposite to the fourth sign.

So, there is the old theorem call Sylvester's theorem, which says that the number of signs of the Eigen values remain the same and the transformations. So, regardless of what basis you choose you will have 3 of the Eigen values negative and one Eigen value positive and or 3 positive and 1 negative one negative either way is fine, but they are opposite that is all it says.

So, multiplying the eta overall by 1 sign does not change this relation and there were many versions in operation people actually at one point used to write I the imaginary unit and plus 1 plus 1 plus 1 plus 1 plus 1 plus 1. So, that it gives the usual length squared and that I save you the embarrassment of admitting that there is a minus sign that actually it is indefinite.

So, it looks like Pythagorean but eventually people stop being afraid of minus sign and started writing the minus sign explicitly in the metric. So, in fact Pauli has an article written in 1920 where he introduce the I ct notation to avoid writing and minus sign. But eventually people realize that actually the structure of the group changes because, of the 1 sign opposite the representation theory changes.

So, I think we will stop here what was I going to say it was about some geometric thing right. So we will do it next time.