Theory of Group for Physics Applications

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Lecture - 27
Classical Groups - I

So, today we are going to continue with the Continuous Group and I put up some notes. So, the things we wish to now talk about are continuous groups.

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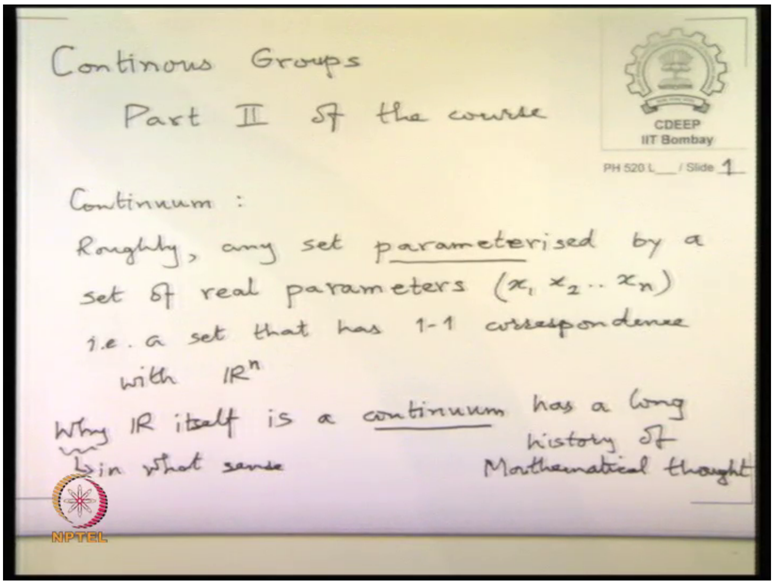

So, last time we talk about continuum groups last time we began with discussing about what is the continuum. So, we need a set and we need to understand what is meant by a continuous set; very roughly speaking any set parameterized, parameterized and this is of course, a special word by a set of real parameters. We will temporally call them something x 1 x 2 x n we are just using the Cartesian notation.

In other words a set that is that has 1 to 1 correspondence with R n. So, last time what we try to do was actually discuss in what sense very sorry in what sense the real line itself is a continuum. So, fundamentally if you accept that the real numbers form a continuum everything else that you can put into 1 to 1 correspondence with it is a continuum, but why R itself forms a continuum has had a very difficult intellectual history.

Now, we take it all for granted, but it is a important history. So, by why we mean in what sense mathematical thought. So, it is not philosophy as in philosophy, but actually mathematics. So, this we went over last time essentially saying there is a progression from fractions which do form a set such which looks like you can always find a new fraction between two existing fraction, between two integers you cannot find a new integer always that is the only and next integer or a previous integer.

But given two fractions and just focus between 0 and 1 even the fractions between 0 and 1 have the property that between any two of them there is a new one ok. So, you because you can take sum of the two and divided by 2; so, you get one that is in between despite the fact that there are therefore, an infinite number of them. They turn out to be in 1 to 1 correspondence with integers with natural numbers ok.

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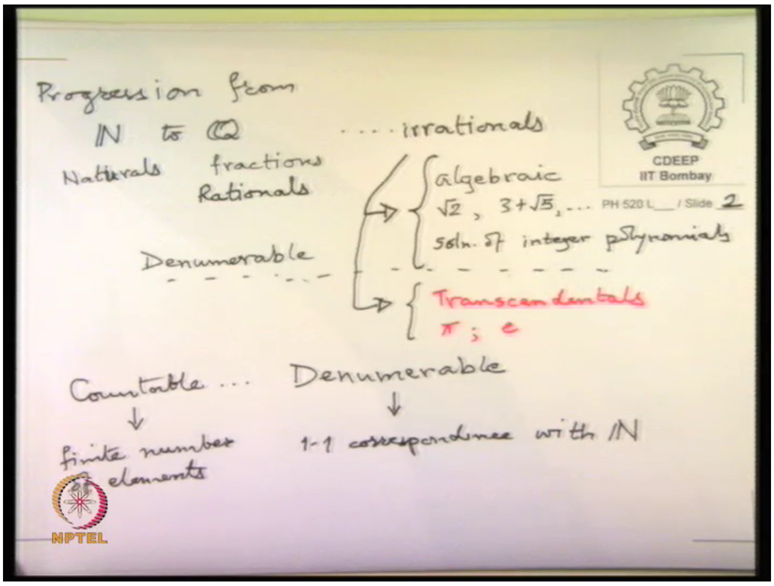

So, very roughly speaking natural numbers to natural to ratios fractions to what are called algebra. So, these are called Rationals that is what I am I looking for rationals to irrationals which forming two class; one class is called the algebraic irrational.

So, these are your square root 2 and 3 plus square root 5 or whatever because this can all occur as solutions of polynomials with integer coefficients. The polynomial should not have arbitrary real numbers as coefficients and if they have any rational coefficients you can always convert them to integers by multiplying out by their LCM. So, these are solutions of so called integer polynomials. It turns out that the fractions plus the algebraic irrational are all countable.

So, I will leave some space here and write over here countable and sorry I meant to say denumerable. So, this is finite number elements a set is countable if it has finite number of elements 5, 15, 59,000 so, all countable. Denumerable means in 1 to 1 correspondence with naturals. So, indefinitely large, but you can put them into 1 to 1 correspondence with natural numbers.

The funny property of denumerable says that their proper subsets can be in 1 to 1 correspondence with them; usually proper subset means you leave out some elements then you can never create a 1 to 1 correspondence in any finite set. But a denumerable set 1 to infinity where you can leave out the first 3 elements and start counting 4 as first, 5 as second. Again you have a 1 to 1 and correspondence with the original set.

So, that is this strange property denumerables, but they are still under control in the sense that they are listable as natural numbers all the odd integers, all the integers that are multiple of 3. So these are all subsets they are proper subsets but they are in 1 to 1 correspondence with the original set. This is called denumerable and the algebraic irrational said well as the Q's the rational are all denumerable this is an amazing result, but that is true. So, up to here, but then there is still remain funny creatures called transcendentals.

The most common examples are pi and e the base of natural log and all the other examples of transcendentals would come only from values of special functions like your Bessel functions and then you define all kinds of functions; that can map out the whole space of real. You will find this next category of irrational which are not solutions of algebraic equations, but their solutions of some different there are values of solutions of differential equations.

They will occur in the list of values of items in list, but whatever the in the set of solution set of and in fact, does functions are therefore, call transcendental functions sometimes. Turns out that the real lines real weight its real content transcendental ok, because all these are countables and you can define a way of giving weightage to an interval such that all the countables all the denumerable have 0 weight.

But then the real line still has the interval 0 to 1 still has a weightage 1 and that weightage come entirely from the transcendental. So, funny as it may sound this is how the continuum is defined. The most important property of the fractions related to the two irrational together is that the rationals are dense in real. So, then we defined. So, to understand the relation between these two classes of numbers we needed to define with this last time metric spaces.

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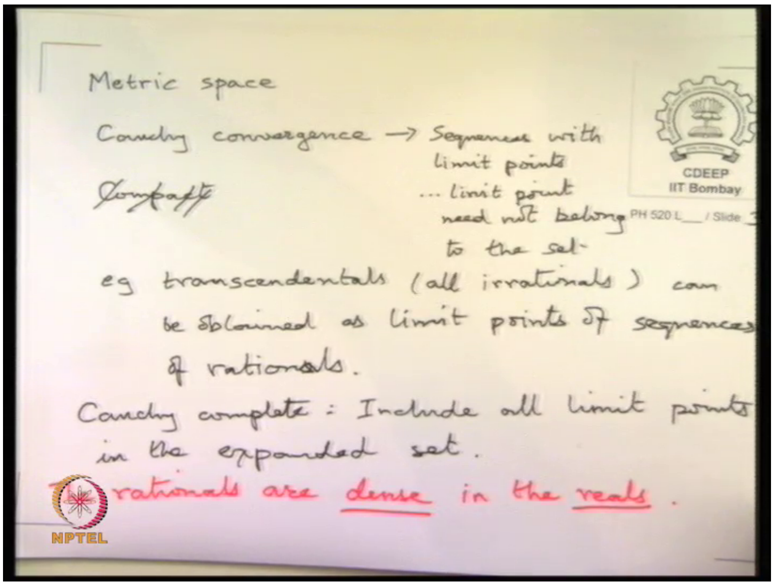

So, that there is a sense of distance between two elements and then the Cauchy criterion, Cauchy convergence.

So, just think of the interval between 0 and 1 and it is a matrix space because you can display the distance between them is subtraction of 1 from the other the greater from the smaller. And the Cauchy convergence means that there is a limit point, the sequence is convergent in the Cauchy sense if the difference between the sufficiently far away parts of the series is all close to together, close together and then we say that a space is compact.

So, Cauchy convergence does not necessarily mean the limit point belongs to the same set, all it means that the limit the exists, ok. Limit points that are the main content of Cauchy sequence, but limit need not belong to the set. So, let us keep this below and the classic example of this is you can get pi you can get e as sequences of rationals.

You can obtain them in the limit I mean like pi v says is 22 by 7 first approximation and then you can keep growing the fraction until it approaches by as well as you like, but by itself is not a rational.

So, for example, transcendental and all of all irrationals can be obtained as limit points of sequences of rationals. So, the limit point need not be itself a rational, but it exist what we do now is that we include all the limit points. Take a set consider all Cauchy sequences and whatever the limit points of all the Cauchy sequences are include them in the original set that set is called Cauchy complete, ok.

So, just to think back a little bit what we have done is well when you talk about limit again you are talking of infinite things you know, but they are denumerable the sequence is a denumerable number of terms in it. So, we are trying to grasp then non-denumerable by some well defined logical processors of denumerable and that is the great achievement of this whole exercise.

And after that the important thing is that the rationals are dense within reals; every single real number can always be obtained as a limit point, what one can be sure of is that there is nothing else left out ok. So, I will not spend more time on this you can read the notes, but this is really the underlying logic. This is all the legal fine print behind everything else that we discuss later.

And if there are you can coming do not be surprised that every once in a while there will be some exceptional case that will be thrown up and this is why people have tried to plug all these logical holes and you never know you may still come out with something new. And when you do you have something to go back to, to check whether all the previous reasoning led to that or not or you do you have something completely new.

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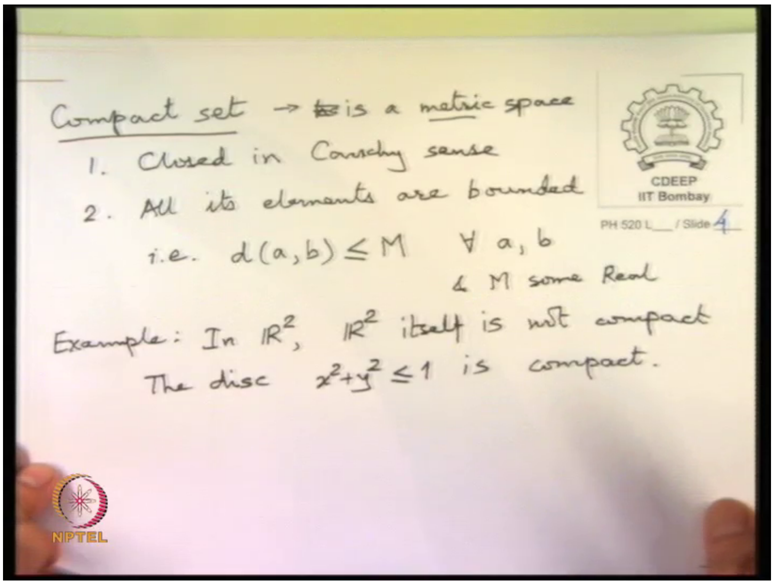

So, now we write down finally, the idea of compactness. It is closed in Cauchy sense. So, it obviously has a metric. So, is a metric space closed in Cauchy sense and to all its elements are bounded. One way to say it is that given any two elements the d, the metric that is existing d a b is always less than or equal to some M. The distance between them is bounded to remain and sure to remain less than some M, for all a and b belonging to the set and for M some real number. So, such a set is compact.

One classic example is let the disc let us saying the in the plane, the disc is compact whereas, the plane itself is not compact. The unit disc with what shall we say x squared plus y squared less than or equal to 1 is compact, ok.

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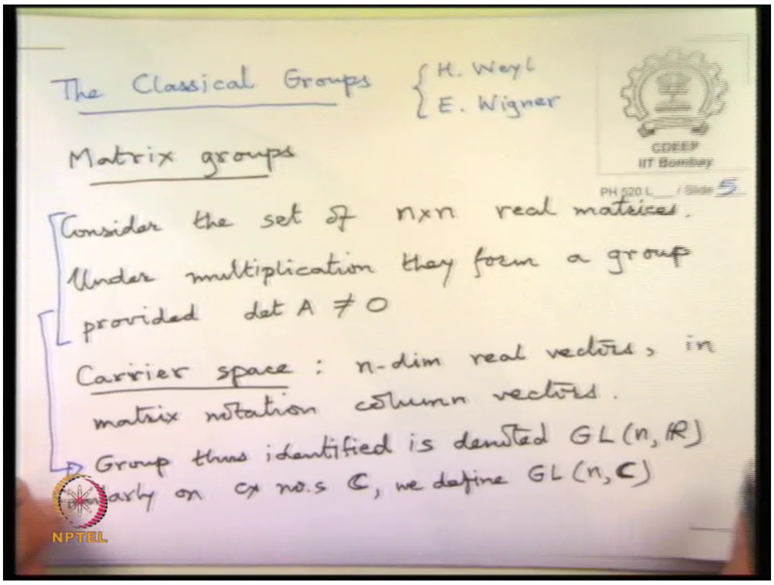

Now, we come to classical groups, the continuous groups. Finally, with all this preparation we come to what are called classical groups. This nomenclature is due to Hermann Weyl and Eugene Wigner. So, Wigner has I think Wigner has a book called Group Theory and Applications to Quantum Mechanics. This is the first person to write out the whole book. So, as you may know Eisenberg who proposed the matrix mechanics, did not know matrices until after he proposed them.

So, physicist education of mathematics did not include matrices in 1920's. So, you can imagine that group theory was far from out of that education and so, physicists continued to be surprised all the time how smart mathematician where; because when they got to a problem and they found the method of calculating. They found that mathematicians already had developed the detail mathematics of it.

So, that happened to Wigner and Weyl also; Weyl was himself a mathematician, but to Wigner found that everything you needed to do had been done 50 years before and so, Wigner later wrote an article called the unreasonable applicability of mathematics to physics and so, we are doing this course because of this very unreasonable applicability of group theory to lot of physics. So, they called these groups classical you will understand in what sense; the other terminology for it is Matrix groups.

So, we begin by talking about Matrix groups. What do we mean by this? Take n by n matrices and consider them as group elements as a group set of n by n. Let us begin with real matrices under multiplication, usual matrix multiplication they form a group provided what we need four properties; closure we know product of any matrix is a matrix. We know the matrix product is associative so, that is not a problem. The identity matrix exists only you to put one on the diagonals, but we also need the inverse.

So, if the matrix is 0 rank I mean if its determined is 0 then you cannot invert it. So, you need to put the condition that determinant of A is not equal to 0. So, it is obvious that this that forms a group. The set of all real n bind matrices that are invertible forms a group in the usual sense and this is going to be our prime and essentially the entire example of what continuous groups are; the only continuous groups we will discuss are all the matrix groups.

So, there is always a very clear coat realization. Remember we kept talking in group theory about career space and realization of the group. In the continuous group theory we are primarily going to talk entirely through the realization itself. But if you go to all the sophisticated mathematics book, they say that there is life beyond that there are continuous groups that are not necessarily matrix groups. But we are not going to really have to deal with them and that is why we are going to swear by Weyl and Wigner and live with the classical matrix groups.

So, to begin with let us next say. So, we already said we are going to talk about the continuous groups entirely in terms of the realization which is as matrices. The carrier space is the n-dimensional vectors in matrix representation column vectors. So, that is going to be the carrier space. The group that we identified the most general group is called therefore; general linear group.

And I talked about the career space because that makes it clear why we are calling it general and linear. General means if they are all the possible matrices that have determinant not equal to 0 and they are essentially linear transformations on this careers space. Similarly, you can do the same exercise with complex numbers. Introduce complex and dimensional vectors as career space, complex and band matrices whose determinant is not 0 and those are called GL n C.

All the classical groups that Weyl and Wigner introduced are going to be subgroups of this GL n R or GL n C, where GL n R is itself a subgroup of GL n C by setting imagery parts to 0.

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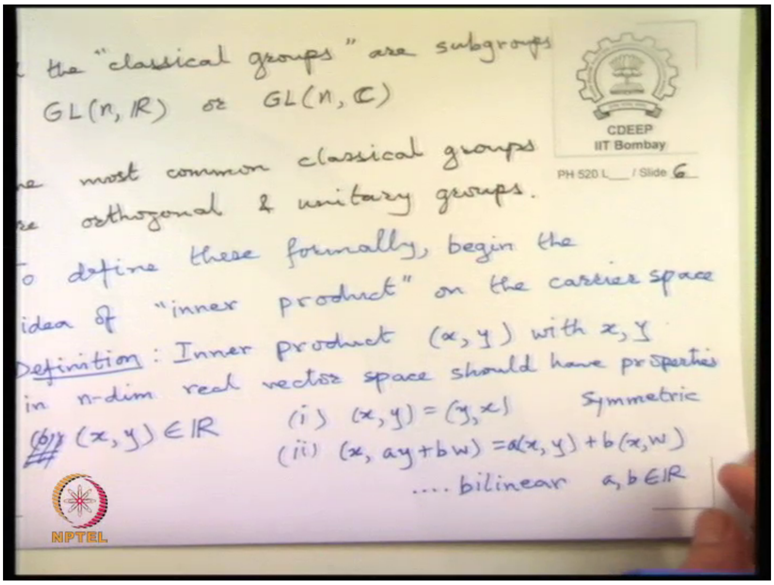

So, now let us begin with one of the classic classical group which is the orthogonal groups, and to discuss. So, the most common classical groups are the orthogonal and unitary groups. Since, you already would be familiar with this in some ways we drop the words, but now we develop formally what this means; we begin with the idea of inner products.

So, you will see all your familiar things as we go along, familiar ideas as we go along remember that. So, anyway we will come to it; orthogonal meant o transpose o equal to 1 and why that is an interesting property to impose we go through this little formality.

So, we say that inner product x, y with x and y belonging to belonging to n-dimensional real vector space should have property. Of course x, y should belong to that is the 0th property should give a real number, but more importantly so, what I mean is it is not a property. So, it is part of the definition.

So, inner product is a operation that takes two elements of the career space and gives the maps them into a real number. Property 1 it is symmetric. So, you will see a analogy to the metric spaces we are defined; this essentially is like defining distance d between two points in a matrix space. There of course, analogies the d essentially will turn out to be the distance between n points of your stick vectors.

Number 2 is symmetric bilinear and positive definite. So, bilinear is where metric space is a property metric space does not have; in metric space we simply said that Schur's inequality. The triangle inequality it is actually the triangle inequality in the more general sense called Schur's inequality. So, the second property is bilinear which is saying x plus ay plus bw equal to a times x, y plus b times x, w. So, this is called bilinear.

We showed here the linearity. So, a and b are real numbers because you have a vector space, vector space is where you can multiply by real numbers so, a and b are real. This property is called bilinearity, because what we wrote out is linearity in the second slot, but the property 1 says you can flip it around. So, it must also apply to the first slot.

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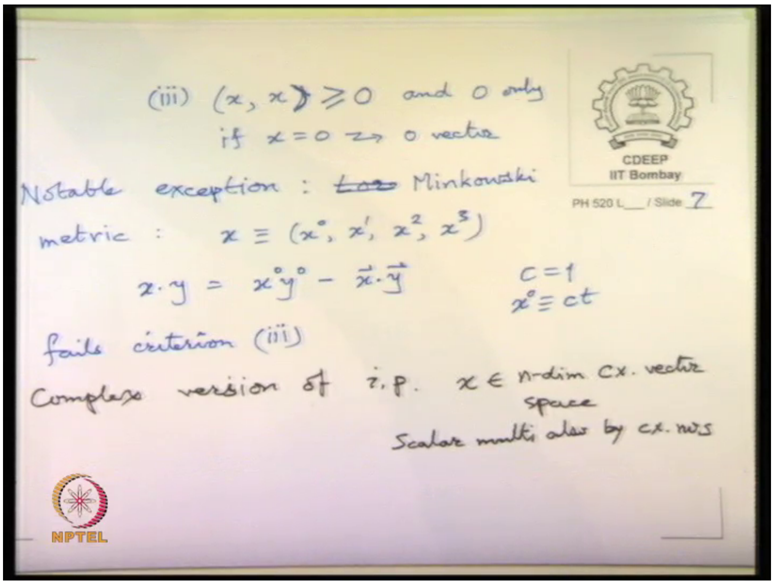

So, you have bilinearity and 3 is positive definiteness which is that x, x if you put the same thing into the bilinear function. This has to be greater than or equal to 0 and 0 only if x is the 0 vector that is real number 0 this is vector 0.

So, these are things familiar to us about dot product or inner product or scalar product a dot b a b a square plus b square plus 2 a b cos theta, but we already have one great counter example to this when we get to special relativity because the Minkowski inner product fails test number 3. So, mathematician said figured everything out in 19th century, but we had something to give the back to them as a gift ok.

So, something very fundamental to physics but which fails their criteria number 3. So, some of the theorem they proved assuming these three will fail. Well, notable exception Minkowski metric x equal to x 0, x 1, x 2, x 3 and x dot y equal to x 0 y 0, but minus x dot y and we are said c equal to 1 usually x 0 is c times t. But we know that the Lorentz invariant inner product of 4 vectors as this relative minus sign. So, it is not positive definite so, fails criterion 3.

But it turns out that still a lot of the things work ok, as far as the group itself is concerned, but we do get interesting things when we get to group representations and we will see some of that in subsequent lectures.

On the other hand, next thing is the complex version inner product where the x belongs to n-dimensional complex vector space and in this case the multiplication, the scalar multiplication is also by complex numbers. So, that is the main difference when we get to complex vector spaces.