Theory of Group for Physics Applications
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Lecture – 26
Preliminaries about the continuum – II

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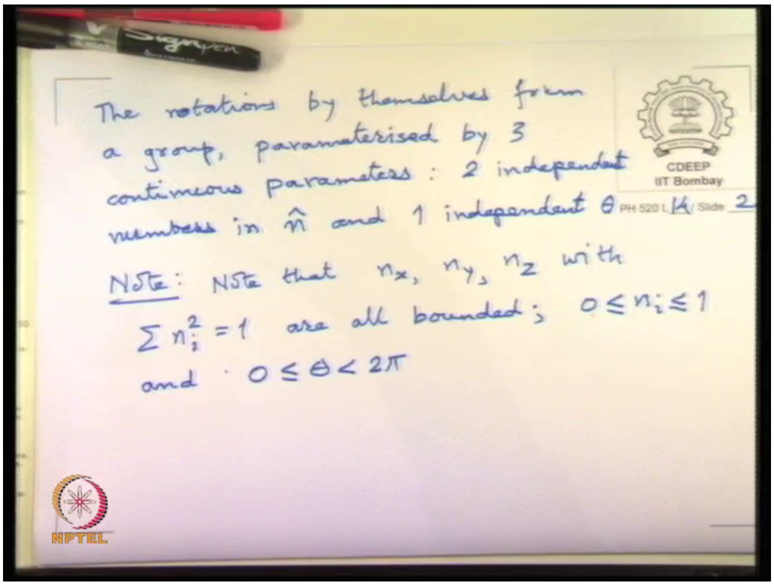

So, since we are mentioning examples, let us next take the example of a Lorentz group.

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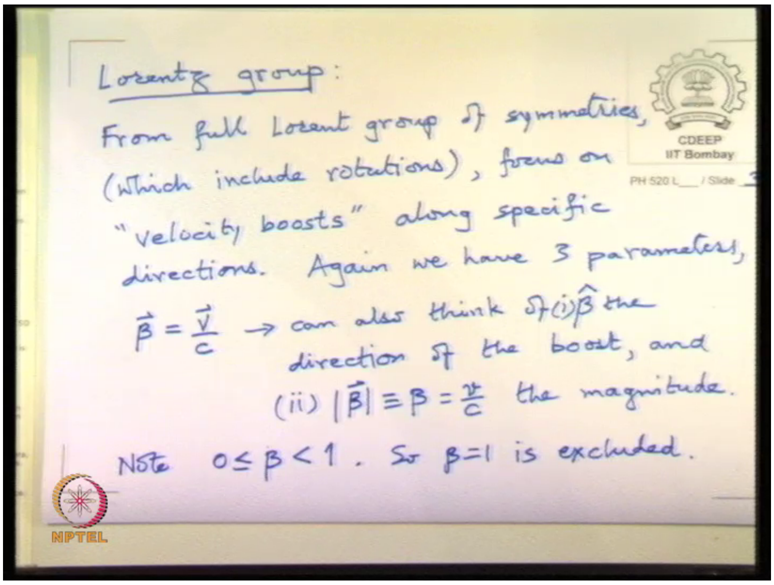

And let us first restrict to, well, in this case also we can say that any Lorentz group can eventually be made into we just want boost, ok. So, from the full Lorentz group of symmetries which include rotation focus on what are called velocity boosts along specific directions. So, again we have 3 parameters. We say data, so we define beta to be equal to the Newtonian velocity V divided by C. And we can think of this as beta cap the direction of the boost and modulus beta just written beta typically to be equal to V by C the magnitude.

Now, here also the beta cap obviously, is a unique vector. So, there are 2 degrees of freedom in it and the magnitude of beta we know as we tell everybody restricted to remain less than one because V has to remain less than C in any relativistic transformations. By the way nobody said that you cannot begin with an object with velocity greater than speed of light because that we have never observed one, but here we are saying that you cannot boost anything by an amount greater than C.

If you are any particle that has velocity less than C the only boost you can possibly make are up to C, but never exceeding C. But what is even more interesting is that note that 0 less than or equal to beta is less than 1, beta equal to 1 transformations do not exist it is never possible to boost a particle of nonzero mass to actually travel at the speed of light.

So, was the beta is a continuous and bounded parameter. It is actually not compact we will that is what we are going to define actually I gave this examples to do this, to show the ideas of compactness and non compactness.

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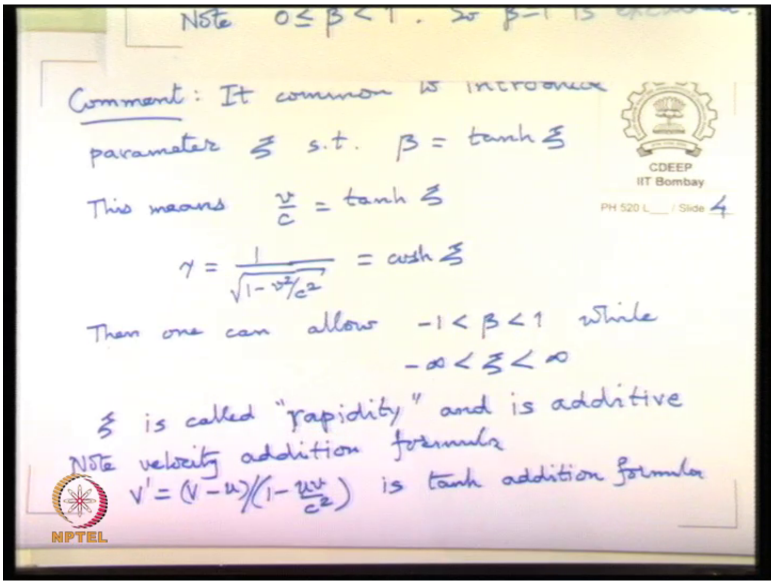

But just in passing I want to tell you that there is another way that people often write this. Sometimes one introduces, introduce a continuous parameter that takes infinite range of values. It is common to introduce parameter xi such that beta equal to tanh xi. And in this case we let beta take values from minus 1 to plus 1 and xi from minus infinity to infinity. So, this means that V by C equal to and the gamma factor turns out to be cosh xi.

So, with this allow let us say minus 1 less than beta less than 1 while minus infinity less than xi less than infinity. So, that xi can then take free range from minus infinity to plus infinity. But of course, xi equal to infinity or minus infinity is not allowed and xi sometimes called rapidity and is additive, while beta is not.

You remember the velocity addition rule it is some complicated rule, but now you can see what that rule is. So, if you wrote out v and u in terms of xi 1 and xi 2 then you will see that it is the same it is the total is tanh of xi 1 minus xi 2 that is tanh xi minus tanh xi to divided by 1 minus tanh xi tanh xi, ok. So, it is essentially tanh hyperbolic addition formula. So, that was new way comment because I thought that you it can be useful in other areas.

But main thing is that better is the parameter that is that looks what bounded restricted, but the point is that it cannot reach the limit point and so there is a big difference between theta well. So, we might think that theta and beta are similar, but there is a very big difference there is really no harm in say dropping 0. So, we can say this or we can say 0 less than theta less than or equal to 2 pi.

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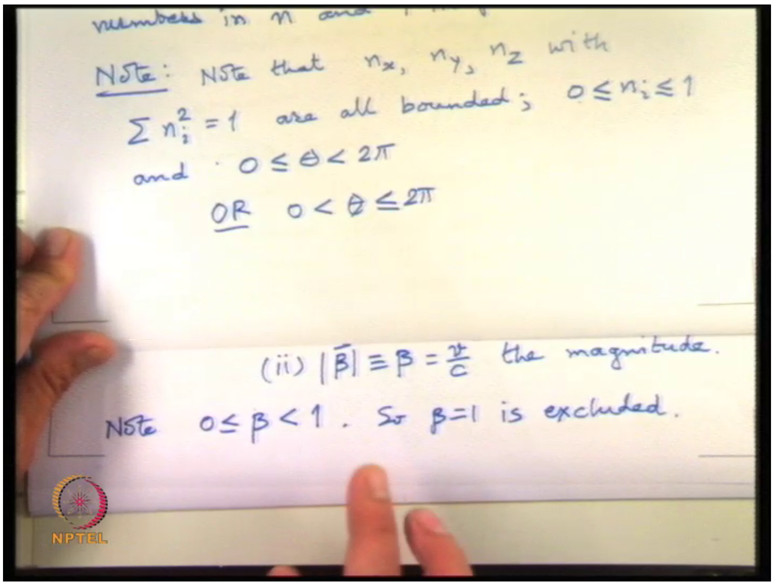

This leaving out there one of the end points is only to avoid double counting, but there is no harm saying it reaches 2 pi, but that is that cannot be done for beta beta can never reach equal to 1. So, there is a fundamental difference between the two. So, now, with this kind of introduction from some familiar things, let me begin by saying a few formal things some general things about the continuum, ok.

So, in just to repeat what we have now going to do is have groups whose elements are not listed by an 1 2 3 up to order of the group, but they are continuously parameterized. For every value of a parameter like theta you have a group element. So, you cannot be writing a multiplication table because there is a continuum of elements to write on the two sides, but you can define them as functions you can define the multiplication table as a function of the parameters.

But, so we will go through some of the preliminaries that are important to know one of the ideas. So, just as we went through algebraic preliminaries about homomorphism and so on and so forth. For continuous groups we will see some of the continuum preliminaries.

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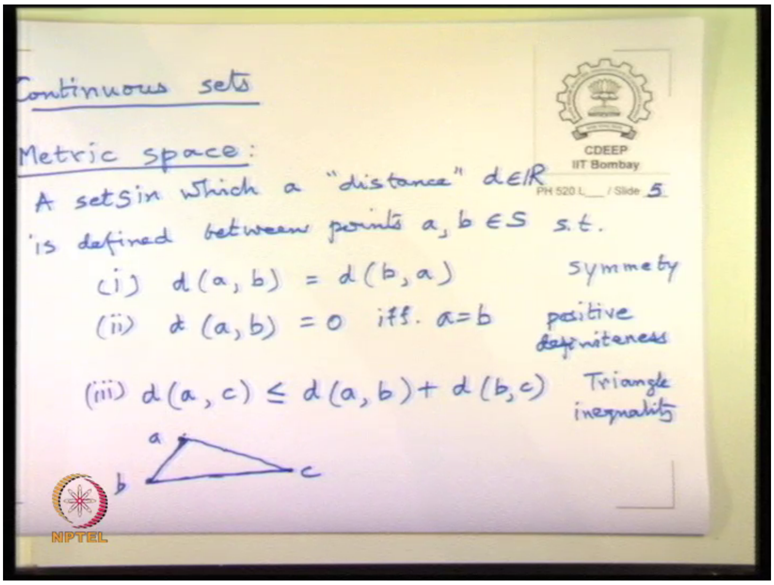

One idea is the idea for metric space. Metric space is a set of points a set in which a distance d is defined between points a and b set S in which such that and there are 3 varies simple axioms of the distance formula. One is that d distance between a and b has to be equal to the distance between b and a, it is symmetric. And it is positive definite if and only if a equal to b. So, this properties called symmetry, this is called positive definiteness and the third thing is that it should satisfy the triangle and equality d a c is always less than or equal to d a b plus d b c right.

Given a b and c, these are some arbitrary points you do not want to end up giving some rule such that the usual triangle in equality is violated the distance between a and c has to be always less than some of these. It can become equal if b comes and falls exactly somewhere in between a and c then there is an equality, but it can never be less ok. So, this is called triangle in equality. So, metric space is a, you can think of any abstract space you like some set of points no other structure presumed except that there is a function d which is a number. So, d is a number d belongs to R real numbers, ok.

So, there is a real number R which measures distances. It need not be a continuum at this point it is just any set of points. So, you can define a metrics space; like this now.

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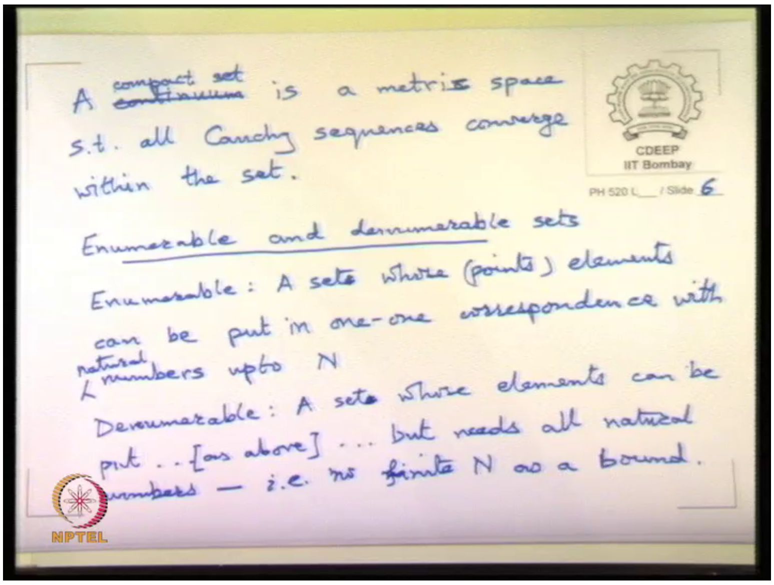

The next statement is that a continuous space is such that it satisfies it is a metric space such that all Cauchy sequences converge within the set. So, I actually over stated I am not I should not try to define the continuum, but we will say a compact set not continuum. What is the continuum is a separate story ok. You go you progress from integers two fractions and then to limits of fractions to get irrationals you have studied this somewhere Dedekind cuts or so.

So, there is some formality that was there is a distinction between fractions all though you can always find a new fraction between any two fractions were taking differences. So, you can keep filling a points in the real line, but all the fractions are countable by a process called cantor diagonal process.

So, you can always write the numerator on the rows and denominators on the columns and then you can always list them. So, you can enumerate them. So, let me begin by saying what is enumerable and denumerable. So, enumerable means a set whose points or elements can be put in one to one correspondence with numbers up to some large numbers up to some number N, natural numbers.

Denumerable on the other hand where it can be put in one to one correspondence with the integers ok, or a set whose elements can be put to into, so as above ok, but needs all natural numbers, i.e. not bounded by no finite N. Now immediately a logical distinction arises between enumerable and denumerable sets because if it is enumerable then it has just specific number of elements in it. However, large whether it is the number of rupees in Indian economy or declared or not declared it is still number that can be counted, and the denumerable one on the other hand is essentially as said that is infinite it has no upper limit, ok.

The funny thing about denumerables is that their subsets can also be infinite. So, if you take every odd number it is clearly a subset, but to enumerate to list it you will have to again use all the natural numbers. You can take multiple of, you can pick any prime number you like and you just say multiples of the prime number p those say.

So, the size code size of the set is sort of undefined for denumerable you can subsets proper subsets will have will can be porting to one to one correspondence with the whole set right. The odd numbers is the proper subset of the all the numbers, but you can put them in 1 to N. In fact, you can count you can tell it is 2 and plus 1. So, it is labeled by N.

So, there is a correspondence between odd numbers and all the numbers, whereas odd numbers is only a proper subset that can never happen for enumerable sets. Its proper subset is necessarily smaller than the set itself whereas, the denumerable sets have this funny ambiguity.

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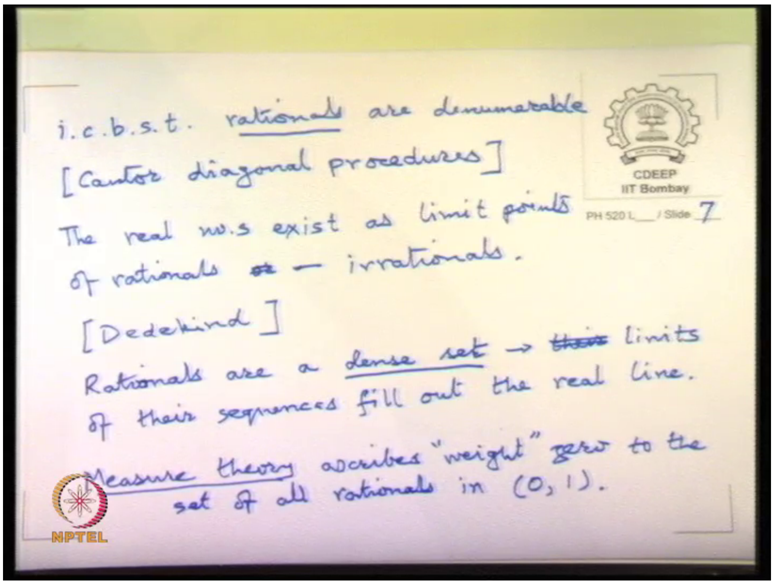

Now it can be shown that rationals are denumerable ok. You can have between 0 and 1 you can have any number of fractions you want they are infinite, but they are actually listable, it is possible to list them put them into one to one correspondence with natural numbers. The real numbers, so this is called cantor diagonal procedure the real numbers are exist as limit points of rationals, ok.

So, there can be limit points which are themselves rational numbers some sequences may converge to rational another rational number. But there will exist limits that are not themselves limits of sequences of rational numbers which do not converge to a rational number and then new points you get this way are the irrational numbers ok. So, rationals examples, so these are called irrationals.

So, irrationals arising many different contexts as solutions of some algebraic equations or some problem that you pose or for example, pi the ratio of circumference to diameter and so on; So, do have numbers which are not themselves rationals, but any such can be always obtained as a limit of some sequence, ok.

So, their limits of sequences sometimes this sometimes describe to Dedekind the point is. And so once you include these you have filled out the reals completely there is nothing else left. So, one can prove that there are; so rationals are dense, dense set in the sense that their limit points fill out the real line. So, sequences of their limits oh god, limits of the sequences and yet the number of irrationals out number is the rationals, ok.

So, how you define the number of them is the difficult thing that is something called measure theory and measure theory tells you that the although the rationals are dense, so dense that you can always reproduce everything else by other sequences they are a set of measure 0 ok. If you really try to escribe some weightage to how much of them is there it is 0 and it is the irrational that really make up the whole real line the real weight of the real line.

So, all rationals in 0 to 1 in the interval 0 to 1 ok, and so the story of course, repeats in every interval, but that is what it is. So, these are some of the amusements about this and it goes on and down when can Cantor and Dedekind talked about these things I think one of these two was like ridicule by others. They did not read his papers, they banned his publications, they said he was going mad or that you are saying things that were against religion and against humanities some (Refer Time: 27:59).

So, they created quite a riot at the time they came up with these ideas. But later we rely on them very heavily to make sense of what we are talking about. So, the continuum that we talk about is therefore a conceptually of being step out of the discrete sets, ok. So, going back a set is compact I said something a little in precise there.

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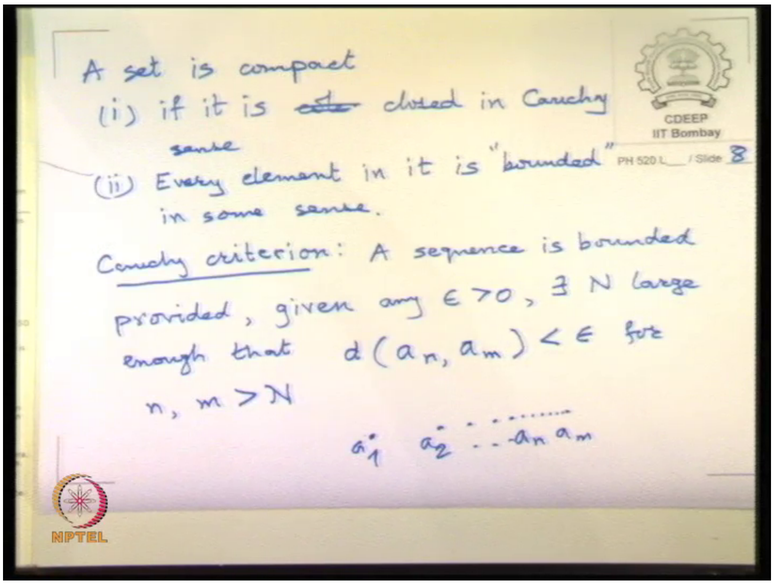

So, a set is called compact provided it is closed in Cauchy sense and to every element in it is bounded of the set is bounded actually ok, bounded in some sense ok. So, we will give the specific example. Firstly, the Cauchy criterion as that a sequence is bounded provided given any epsilon sorry for this epsilon delta thing I hate it, but that is how everybody writes it. So, given any epsilon greater than 0 there exists N large enough that a n, that d of a n comma a m is less than epsilon for n and m greater than n.

So, does it is make sense it is suppose I have a sequence which is going like this 1 2 3. So, a 1, a 2, a 3, a n, a m somewhere. The difference distance between any subsequent points is progressively getting smaller necessarily getting smaller that is what it means.

So, you tell me any number as small as you like I can always find a index value n begin of. So, that the points have got closure than that bound that you supplied epsilon, so such a sequence convergence clearly at some point. That limit point may not be part of the sequence, but the limit point should be; so the closed in Cauchy sense means that the limit point of the sequence should belong to the set. So, I would not be too wrong if I said the continuum essentially satisfies that.

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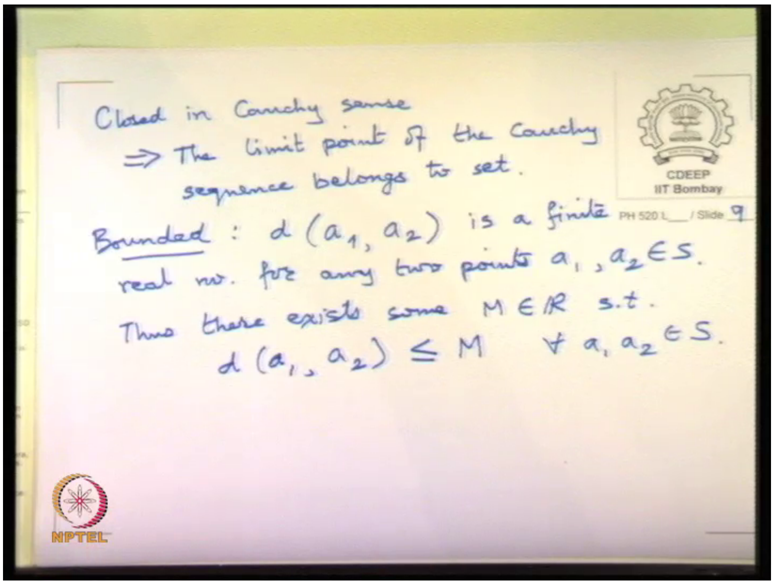

The second point was that every element is bounded. So, you can have limit points converging, but now think of this suppose I take the interval from 0 to 1, I can check that every Cauchy sequence it convergence to some point within the set including the real lines, so real interval.

But I stack up all of these and try to make it the whole real line not just the interval. Of course, it is also closed in Cauchy sense, but it goes on and on. So, there are there are numbers larger than any number you can think of we do not want that. So, we wanted to remain bounded, ok. So, the boundedness you can say in the given, so, the whole discussion is in a particular metric space.

So, you can say that d of any point from any other point is always finite, is a finite number for any points any two points a 1, a 2 belonging to S. Thus, there exists some n such that d a 1, a 2 is always less than or equal to M for all a 1, a 2 belonging to S. So, that is the idea of boundedness.

So, I think we have covered the very basic notions that I want to use while we discuss further. So, later on if we this is a language we will use as and when needed because we do not need to worry about these points later. So, by themselves these are not part of the core course, but this language is required for understanding rest of the course, ok.

So, we will stop here today.