Theory of Group for Physics Applications
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Lecture - 25
Preliminaries about the continuum - I

So, to begin with this lecture the things we wanted to do was to wind up the previous discussion of discrete groups and from now on we will start with continuous groups. So, let me first write over here.

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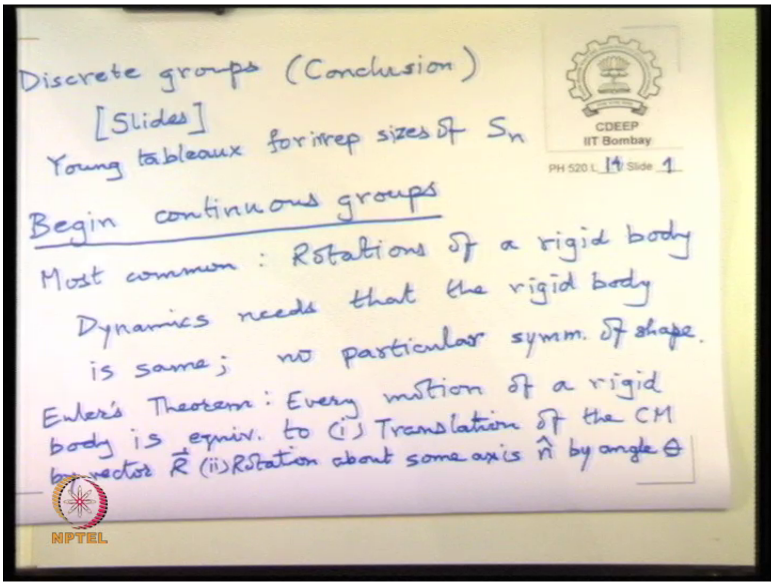

So, I have some slides for this which basically discussed Young tableaux for irreps of for reps sizes of irreps sizes of course, S n the symmetric groups. So, there is some method to get sizes of representations of the permutation groups and we will just go over this the tail piece of the discrete group discussion.

We did go through well I think we will cover it; I have prepared some slides to do this. The orthogonality main consequence which is that sigma p i square sigma p i square or l i squared equal to 1 that is the number of classes.

So, we saw that the number of classes is equal to number of irreps and the sum of squares of so, there as many classes as there irreps and sum square of the numbers sizes of these is equal to the size of the whole group.

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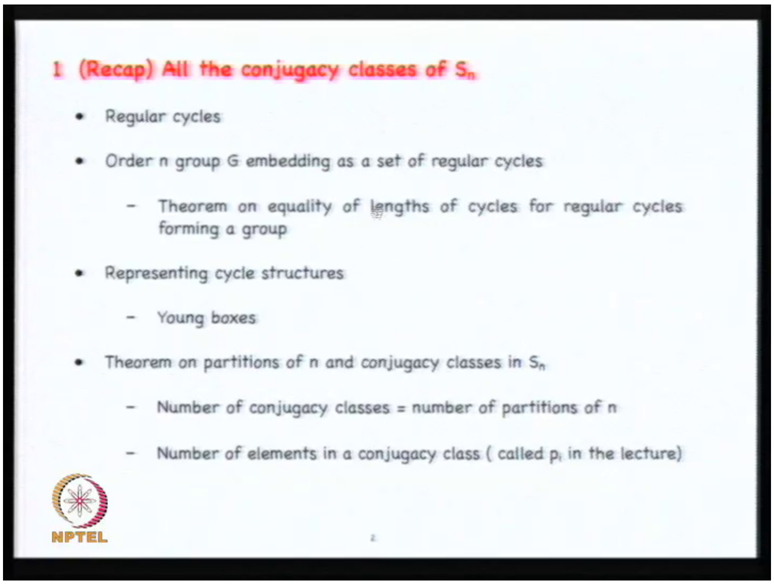

So, this title is carried over from the previous all the conjugacy classes of S n, but it just recapitulating some of that. We had done regular cycles and order n group can be embedded through regular cycles and representing cycle structures through Young boxes that is what we will do today. And we had theorem on partitions of partitions of the number n and the number of conjugacy classes n.

So, we had seen that the number of conjugacy classes in S n is equal to the number of possible partitions of the number n because it amounts to just saying how many cycles structures you can create.

So, each cycle structure is one conjugacy class right and therefore, counting the number of cycle structures give you one conjugacy class. So, we had proved that conjugacy class conjugacy the cycle structure does not change under conjugacy operation and therefore, each cycle structure represents one conjugacy class and so, if we can count the number of such conjugacy classes; that can be done simply by seeing how many ways you can partition n because that is the way to create cycles.

So, number of conjugacy classes is equal to number of partitions of n for the group S n and the number of elements in a particular conjugacy class is we had designated it p i while we discuss the theorems about orthogonality. We said some over character p i times character of that class squared etcetera.

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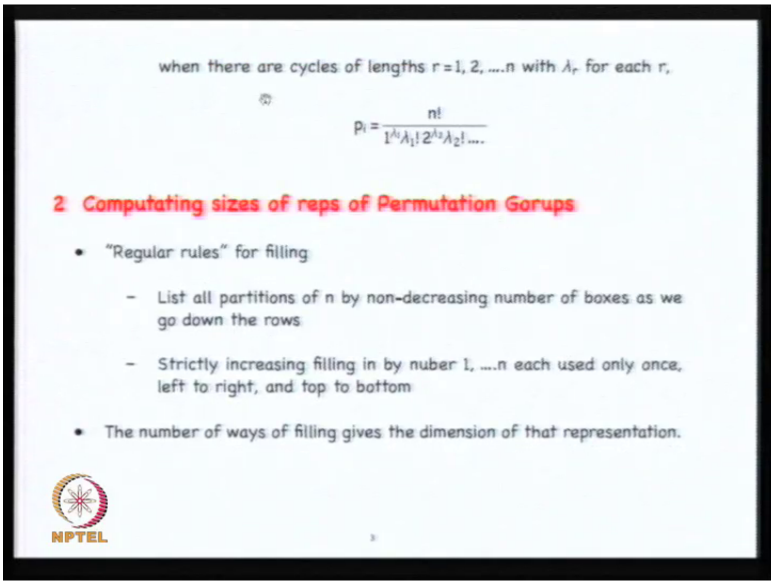

So, that p i can be calculated as follows. Suppose I have a particular class which content cycles of length 1, 2, 3 maximum, it can have a cycle of length n and suppose there are lambda r cycles of size r; So, 2, 2 cycles or 5, 2 cycles or 3, 2 3, 2 cycles whatever.

So, lambda r is the number of cycles of size r then the size of that particular the number of elements in that particular conjugacy class is given by n factorial divided by the number of cycles of size 1; if it is lambda 1 then it is lambda 1 factorial times 1 to the lambda 1, 2 to the lambda 2 times 2 factorial etcetera.

So, this formula we had not actually derived, but it can be seen it is not very difficult to check this formula from just the definition. So, now, I go on to thus the second part that representing cycle structures through Young boxes and we will see that since the number the one conjugacy class basically stands for one irrep. There is a correspondence between number of irreducible representations in number of conjugacy classes.

So, first we represent the conjugacy class using Young tableau and then we say how to calculate the dimension of the irreducible representation associated with that particular class. So, I am sorry for this word regular occurring too often, but this is what the books write.

So, to remain in consonance with them I have written this; that two books I have referred are Hamermesh which really has everything although it is written in a very I mean it is because it has so, much material I think he did not have time to be to systematic about the presentation. So, things are really running into each other; So, Hamermesh book and Dresselhaus book.

So, Dresselhaus books book does not talk about Young tableaux at all, but she has another way of calculations permutation, but the sizes of irreps, but we will do this ok. So, the regular rules for filling are that.

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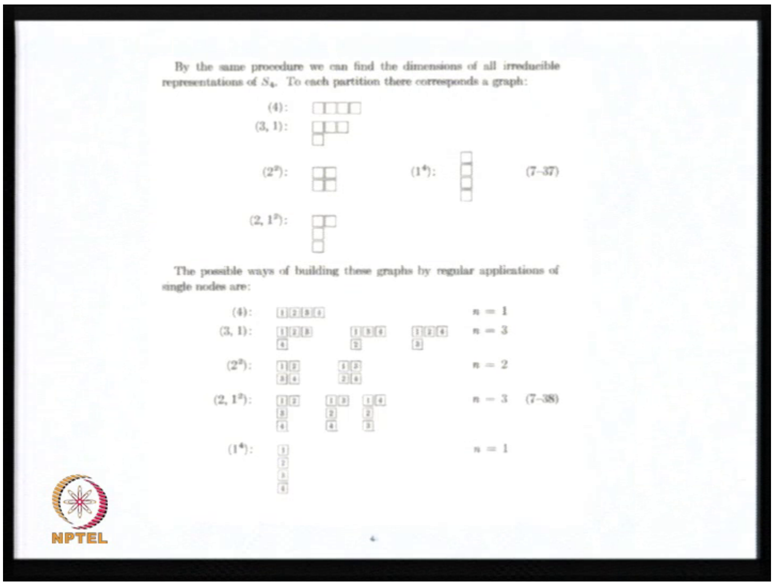

So, let us first look at the example; suppose I have to deal with this figure I can you see the figure.

So, suppose we are dealing with permutation group S 4 ok. This S 4 it is permutations of four different objects then what you do is you start drawing 4 boxes and then draw them in various order and then fill them in with numbers 1 to 4 ok. The rules for filling are as fall as its I am going back to one transparency the regular rules for filling are; List all the partitions of n by n by non decreasing number of boxes as we go down the rules.

So, take the total number n of boxes and start stacking them sideways and then also downwards, but such that the upper row has at least as many boxes as the next row. So, this is a non-decreasing number of boxes as you go down I mean in each row. So, this is a row with 2 boxes. So, there can only be 1 and 1 below or this is row with 2 boxes below can only be 2 when you get so, you begin with 4 in the row.

Then suppose you reduce it to 3 then put the 1 here leftmost then suppose you took 2 in the first row. Next you can put 2 then you took 2, there is another way when you have taken 2 boxes in the first choice to stack 1 and 1 in the next 2 rows. But suppose finally, you take only 1 box in the top row then you can only put 1 1 1 1 below ok.

So, this is the way to put the boxes then fill this up by numbers 1 to n in strictly increasing order in each row ok, top to left to right and also top to bottom. So, here first letters just check that this is according to the system we saw write. The number of boxes in subsequent rows is strictly non-increasing ok. It can be same, but not increase. So, then we get these 4 patterns for 5 patterns for S 4.

Now, we start filling with numbers 1 to 4. In the first row there is only one way to do it in a strictly increasing order 1, 2, 3 and 4 then in this pattern which is called 3 coma 1. We can fill 1, 2, 3 and 4 or we can fill 1, 3 and 4 because it is increasing, but we can put 1 and 2.

Similarly, we can put 1, 2, and 4 and fill 1 and 3 showing vertical as well as horizontal directions the filling is strictly increasing; then let us look at this it is 1, 2, 1, 3, 2, 4 or 1, 3, 1, 2, but this also has to be 3, 4.

So, you could not have put 1, 4 and 3, 2 because 1, 4 would be increasing downward, but 3, 2 would not be. So, you can only fill in strictly increasing order and use is number only ones. So, then this gives this pattern there are two ways of doing this; then we go to this format. We can put 1, 2 then we can put 3 and 4 below there is no other option. If we put 1, 3 then we can put 2 and 4 and if you put 1, 4 then we can only fill 2 and 3.

Similarly, when we come to the final format we can only go 1, 2, 3, 4 there is nothing else you can do. Now, the point is after filling out count the number of ways you could fill out, show in the case one for this pattern there was only one way of filling. So, we write that n equal to 1. There is only one way of filling it.

Then we write one there are three different ways of filling this pattern 3, 1 and so, we call it n equal to 3 and this one is 2 and so on, ok. So, this gives 2 this has 3 ways, this is 1 way. So, this gives the dimensionality of each of the irreps there are 4 sorry 5 irreducible representations. Remember that partitions of 4 or 5 so, that there are that many conjugacy classes.

But therefore, there are that many irreps and now this list is giving the sizes of the irreps. So, there is a more formal way it is called Frobenius formula which actually give some algebraic answer. But this Young tabular method is a simpler way of realizing the Frobenius formula in a pictorial way.

So, here we find that there are 5 irreps and their sizes are 1, 1, 3, 3 and 2; one particular thing to note about permutation groups is specific to permutation groups is that you will have this conjugate patterns. You will have 3, 1 then you will have a 2 and 1 squared, but if you see pictorially. So, it is written 2, 1 squared because 2 in first row than 1 and 1. So, it is 2, 1 squared this thus correspond to 2 cycle, 1 cycle, 1 cycle this corresponds to 3 cycle and 1 cycle.

But the 3 recycle, 1 cycle and 2 cycle and 2, 1 cycles are actually geometrically visually flipped rotated versions of each other, ok. This if you flip, if you rotate it by 90 degrees well whatever mean sorry, this if you rotate let us say clockwise and then flip mirror image then you will get this.

So, this pattern and this pattern are conjugate to each other. The first pattern and the last pattern are also conjugate to each other, I can just rotate it clockwise by 90 degrees I get this; such conjugate patterns has same sizes and the 2, 2 squared pattern goes into itself.

So, it has no it is self conjugate. The other thing we can now check is that 1 square plus 3 square plus 2 square plus 3 square plus 1 square that should add up to S 4 size, right; So, 9 and 9, 18 and 4, 24, 25 23 and 24, 22, 23 and 24. So, we have for the S 4 group all of the irreducible representations and their sizes listed by this Young tubule group method.

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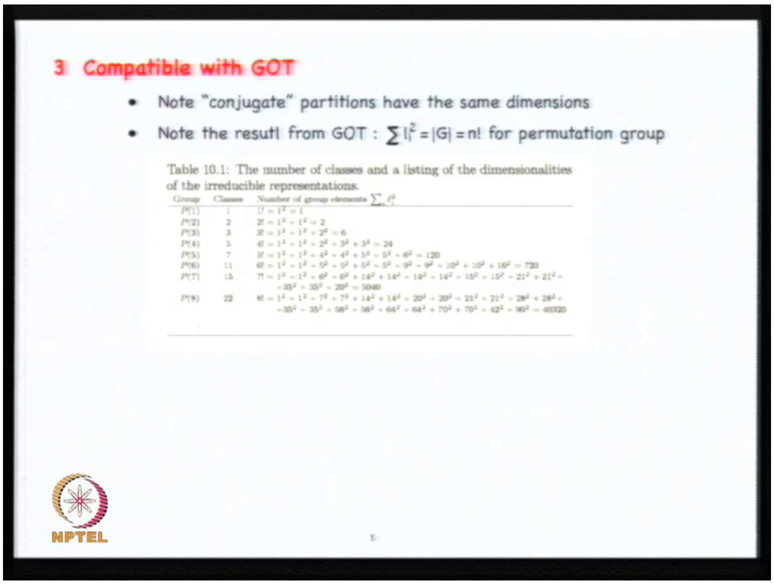

So, I have just written compatible with GOT basically this got I just listed; some of the actually I should have written p i squared. The number of elements in each conjugacy, no this is the dimensional it is of each of the irreps squared is equal to the order of the group which is n factorial for permutation group.

And this what I already said the conjugate partitions have the same dimensions, where conjugacy is in this pictorial sense of 3, 4, 3, 1 and 1 and that same pattern rotated becomes this no, but they are not in same conjugacy class structures different there, there independent conjugacy class.

So, again I have put conjugate on in inverted commas Hamermesh book calls them conjugate box patterns, but it is dangerous because we are using conjugate in many places. But it is only this geometric conjugacy, it not that it means an it is just an observation that if you do apply these.

So, firstly I am we are not going to prove give the proof of why this process works; we are just giving this as a recipe. I should have return here I will just update this file before uploading it. So, this is called the Frobenius theorem where Frobenius explicitly gives proof of how to count the sizes of the irreps of S n of the permutation groups.

But when you begin to use this Young tabule you can see the relation of this box that arrangement to this box arrangement and then you will find the this their sizes are same, that is all. There is nothing else that at this point can be said, probably if we followed the whole Frobenius proof then we would see why the sizes of these come out the same sizes of irreps come out the same.

So, this is a table taken out of that Mildred Dresselhaus book table 10.1: The number of classes and listing of the dimensionalities for the irreducible representations of various permutation groups from P 1 up to P 8 and so, we just did this P 4 which had 5 conjugacy classes and one checks that 4 factorial is equal to 1 squared plus 1 squared 2 squared plus 3 squared plus 3 square.

Similarly, if we did P 5 there would be 7 conjugacy classes and you will see the because of that geometric conjugacy between the box layouts these numbers are repeated there are two 1's there are two 5's well here there are four 5's there are two 9's there are two 10's and one 16.

So, one gets this total numbers 720 for P 6 and so, on. Of course, we are not going to these are not realistic we do not really need them, but I thought it is good that at a glance this table has been given. The next thing so, this is basically this for this method you should know. I did not want to include too much enumeration in the up to mid sem, but now we will include it in the tutorial sheet and you should know this.

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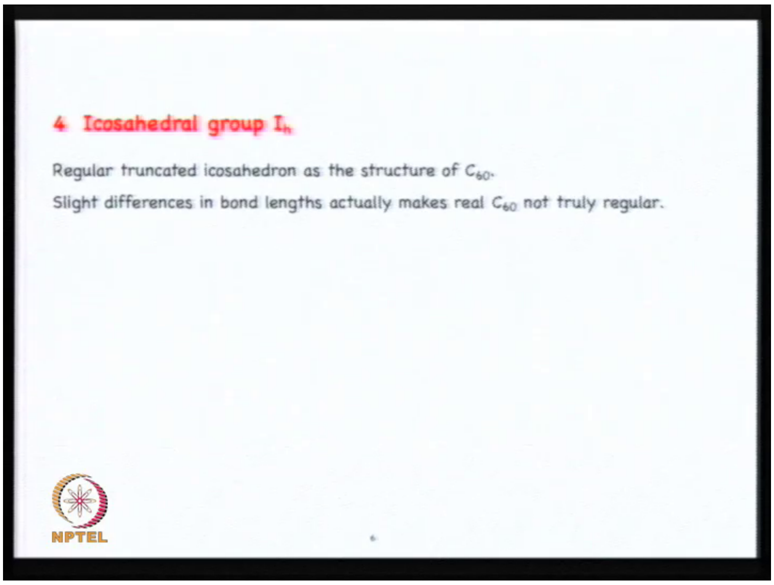

The only comment as I was going through this Dresselhaus book was that we also have and this is an interesting group. So, there is an icosahedral group per designated I h, icosari refers to 12 I think 12, sorry 20 headed group. A 20 sided regular polyhedron and this is the regular truncated icosahedrons.

So, let me show the structure.

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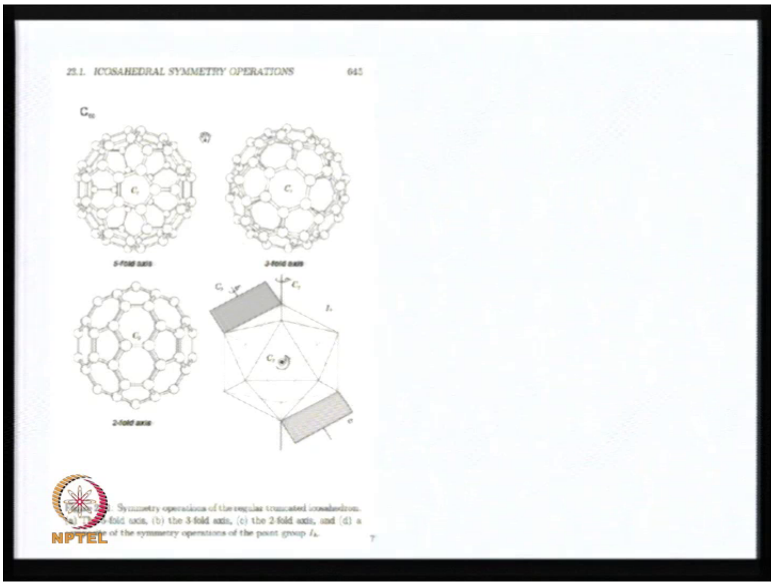

It is the structure of carbon 60 and it so, that is why it is important and actually this author of this book Dresselhaus is famous for her work with carbon nano-tubes and C 60.

So, this is the cleanest picture to see here, you have a hexagon here and another hexagon another sorry hexagon, but it is it has a pentagon as its neighbor on this side. So, off the 6 sides 3 sides this one, this one and this one can you series. These 3 have pentagons as neighbors and the other alternating ones have hexagons as neighbors.

This used to be the football pattern I do not know, they keep changing the football patterns, but some football patterns have this pattern on them pentagons and hexagons. And they are stitched together in this particular way and if we see the environment of a pentagon, let us say here you can see this pentagon here.

So, this pentagon has hexagon, hexagon, hexagon, hexagon, and hexagon. So, the pentagon has all hexagons and all 5 sides whereas, the hexagons have alternating pentagons and alternating hexagons.

This itself is of course, not a regular polyhedron, but it can be got by a chopping of the vertices of the irregular icosahedron. So, and this is the structure that and if you count the total number of vertices there are 60 vertices on this regular figure well not regular, but the symmetric figure and amazingly enough carbon actually forms a structure like this.

They one comment made in this book is that ideally geometrically these sides are all the same ok. The hexagon sides have a particular length and the pentagon lengths are exactly the same, but it need not be because it is actually from a regular polyhedron your chopped off vertices.

So, if you did not chop them off exactly at the correct point you can get slightly different sizes and the author says that the sizes of the hexagon bond lengths are slightly smaller than the pentagon bond length. So, it is not truly highly symmetric structure in C 60, but approximately is it is 1.2 verses 1.4 angstrom something like this.

So, there is a difference, but approximately people are going to treat it as this and you can deduce a lot from that; you can incorporate the small inequality as a departure from symmetry. First you drive the symmetric thing and then you can treat the next thing as a perturbation it. So, now, this figure also shows the various symmetry axis, if you stand on top of a pentagon like a desktop figure then you have a C 5 axis that you can draw pointing down from you right.

So, there are C 5 rotation points then there are C 6 rotation points that are here sorry C 3 I am sorry; we already discussed that because this hexagon has to be rotated into this hexagon. So, if you are in if you are on a pentagonal face then you have C 5 you have 5-4 rotation axis.

If you are in a hexagonal face then you have only a threefold rotation axis because there is an alternate of hexagon pentagon pattern. So, this is C 3 as you can see finally, there is a C 2 access shown. This requires a little carefully looking at the whole figure this is passing through the centre of this bond ok.

So, if you look at a hexagon, hexagon, pentagon, pentagon, pattern then this particular bond which is between the vertices of 2 pentagons, but is a shared side of 2 hexagons. If you stand on top of this bond and through its centers and axis; then you can see that the pattern can be rotated C 2 180 degrees there is a symmetry of pentagon here, pentagon here, hexagon here, hexagon here.

So, there is a C 2 access and there is some other C 3 axis shown here which looks a little more complicated. So, one can probably pick up the actual buckyball as it is called and see all the details, but its character table is given here um.

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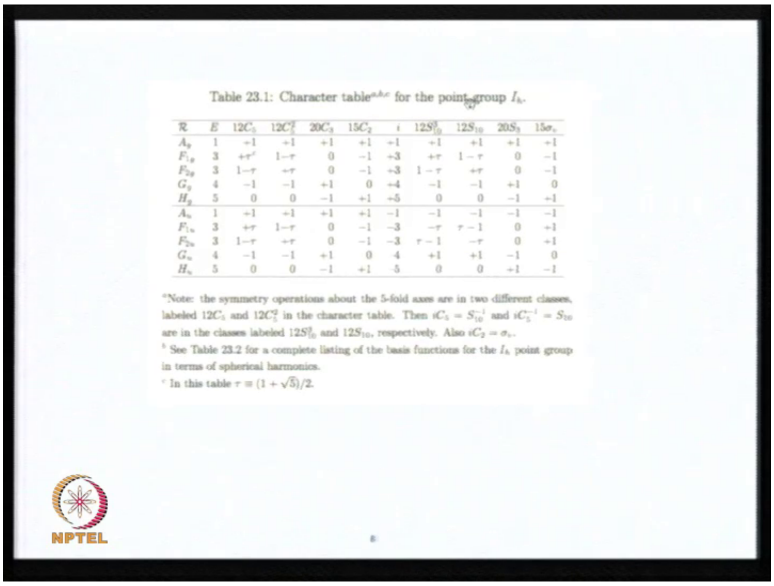

Table 23.1: Character table[a,b,c] for the point group I_h.

$\mathcal{R}$	E	$12C_5$	$12C_5^2$	$20C_3$	$15C_2$	i	$12S_{10}^3$	$12S_{10}$	$20S_3$	$15\sigma_v$
A_g	1	+1	+1	+1	+1	+1	+1	+1	+1	+1
F_{1g}	3	$+\tau$[c]	$1-\tau$	0	−1	+3	$+\tau$	$1-\tau$	0	−1
F_{2g}	3	$1-\tau$	$+\tau$	0	−1	+3	$1-\tau$	$+\tau$	0	−1
G_g	4	−1	−1	+1	0	+4	−1	−1	+1	0
H_g	5	0	0	−1	+1	+5	0	0	−1	+1
A_u	1	+1	+1	+1	+1	−1	−1	−1	−1	−1
F_{1u}	3	$+\tau$	$1-\tau$	0	−1	−3	$-\tau$	$\tau-1$	0	+1
F_{2u}	3	$1-\tau$	$+\tau$	0	−1	−3	$\tau-1$	$-\tau$	0	+1
G_u	4	−1	−1	+1	0	4	+1	+1	−1	0
H_u	5	0	0	−1	+1	5	0	0	+1	−1

[a] Note: the symmetry operations about the 5-fold axes are in two different classes, labeled $12C_5$ and $12C_5^2$ in the character table. Then $iC_5 = S_{10}^{-1}$ and $iC_5^{-1} = S_{10}$ are in the classes labeled $12S_{10}^3$ and $12S_{10}$, respectively. Also $iC_2 = \sigma_v$.

[b] See Table 23.2 for a complete listing of the basis functions for the I_h point group in terms of spherical harmonics.

[c] In this table $\tau \equiv (1+\sqrt{5})/2$.

So, now that we are experts at reading character table we are not scared at with all these label we are not scared because they are not supposed to remember them um, but well because. So, these are some notations which I am not emphasizing in this course here we are more focused on the mathematics of the group theory and applications.

If you go to the relevant chemistry or molecular physics atomic physics course you will or solid state physics I am sorry not atomic; then you will learn the corresponding professional ways of denoting, but this basically is listing the conjugacy classes and this is listing the various irreducible representations.

So, sorry these are the conjugacy classes the cemetery operations about the so, there are notes well you can read them that is, ok. So, she is explaining there is a element I which is an inversion centre. So, there are 12 C 5's and 12 C 5's square operations. So, these are the independent conjugacy classes then there are 20 C 3s, the 20 hexagons in it and 15 C

2s those special things passing through bonds and so, on. So, these are all the conjugacy classes and all the irreps.

The corresponding statement is that this group has 120 elements, but it is not. So, icosahedral group I h is isomorphic to one of these permutation groups, but they are not the same ok. So, that is the important thing and so, you have to enumerate the I h separately. It just accidently has the same number of size as a one of the permutation groups; so these much for this discreet groups.

We will end with this nice BuckyBall example and practical things with it may be in the exercises. But the main thing new method that we are emphasizing in this class is this way of calculating the sizes of irreps of the permutation groups. This is not applicable to any generic group, but to permutation groups we can use Young tabular method to do calculate the sizes, ok.

We go to the second half of the course which has to do with continuous groups. Simplest example of continuous group is the rotation group, here we do not care too much about the symmetry of the rigid body. So, long as it remains the same body. So, we could have arbitrary shape, but it should not be stretching and deforming, but under rotation it remains the same thing.

So, from the point of your mechanics or dynamics so, all we need is that the rigid body remains the same; no particular symmetry to the symmetry of the shape; and what so, we can of course rotate; we have an interesting theorem of Euler that every motion of a rigid body is translation and a rotation about some axis passing through the rigid body.

So, it is rotation it is translation of the centre of mass by some vector capital R and so, you take any rigid body and this is for any finite motion; you start at some point you wait for 10 minutes and see where wherever it is you can a space station going around or whatever it is effectively the initial configuration and final one are going to be separated by one displacement vector for its centre of mass and a single rotation about one given axis.

So, this is important; however, it is tumbling and rotating. The final answer is between the initial shape and the final shape there is a unique and cap axis about which a single rotation simple rotation by an angle theta, will convert the initial configuration into the final one.

So, this is and that is why the theorem is interesting and we can count the number of degrees of freedom in it because the translation vector capital R has three components and the end cap x which the unit vector which designates the axis as to independent components and there is an angle theta. So, totally there are 6 degrees of freedom as we expect.

So, this set of operations the rotations form a group called as the group of dynamics of rigid body, but that is a more elaborate thing because we know under change of frames Galilean transformations also the motions have some symmetry.

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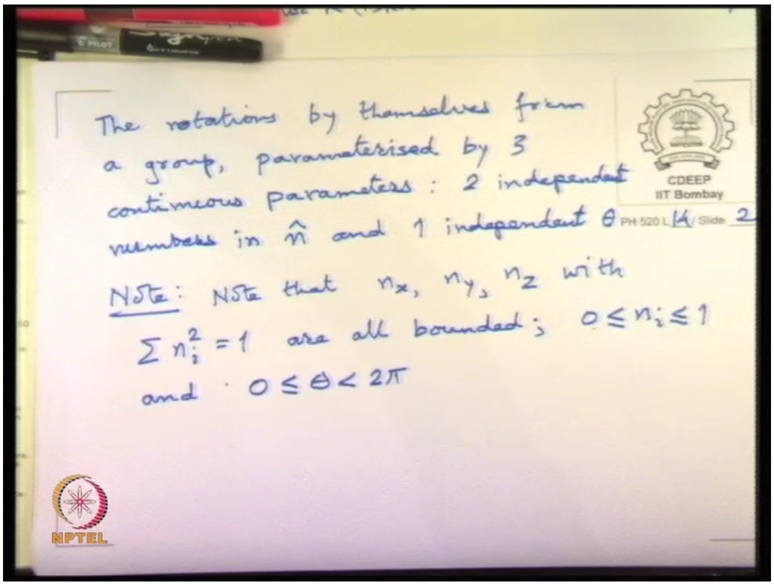

So, for the time being we just note that the rotations by themselves form a group parameterized by 3 continuous parameters 2 independent numbers in n cap and 1 independent theta.

We can also see that these 3 parameters are bounded the values of n cap cannot be infinite because it is bounded to be 1. So, note that n x n y n z with sigma and i squared equal to 1 are all bounded; 0 less than or equal to n i less than or equal to 1 and for theta and 0 less than or equal to theta less than 2 pi. The rotation is maximum by an angle 2 pi. So, all of these parameters are bounded.

This is going to be one characterizing feature of the group, where whether the parameters are bounded or unbounded ok; that is the nature of this particular group.