Theory of Group for Physics Applications
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Lecture - 21
Orthogonality for Characters - I

So last time we proved this very interesting theorem, the great orthogonality theorem or the wonderful orthogonality theorem.

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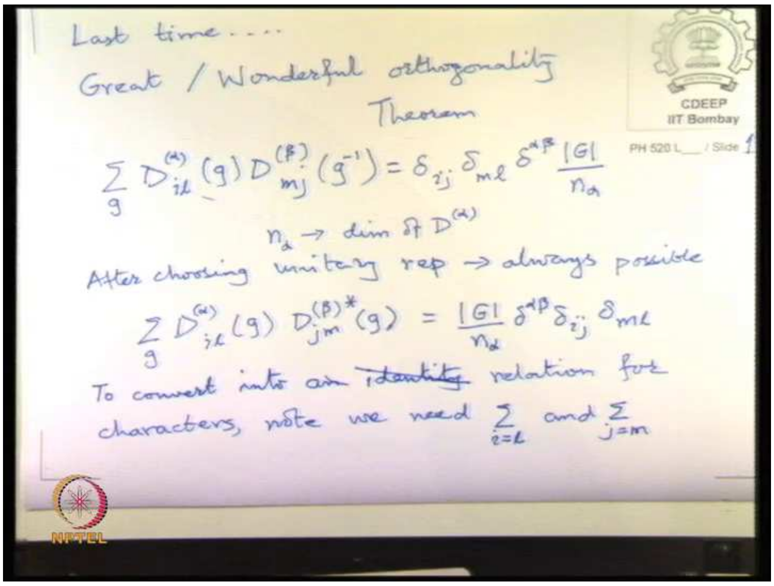

And the theorem basically was about representations basically expressing some kind of linear independence among the representations.

$$\sum_g D^{(\alpha)}_{il}(g) D^{(\beta)}_{mj}(g^{-1}) = \delta_{ij}\delta_{ml}\delta^{\alpha\beta}\frac{|G|}{n_\alpha}$$

So, this is for irreducible representations. So, where n_α is the dimension of the representation of α and by that actually we mean that D_α s are n×n matrices.

And then for since most representation since if you since you representation can always be chosen unitary. So, after choosing the rep unitary which is always possible this we proved as separate theorem, basically the representative matrix are of g^{-1} would have to be the inverse matrix because it represents the it is a homomorphism it realizes the what are the group does. So, the D(g) inverse has to be same as $D^{-1}(g)$, but since it is a unitary representation D^{-1} is simply equal to complex transpose conjugate. So,

$$\sum_{g} D_{il}^{(\alpha)}(g) D_{mj}^{(\beta)*}(g) = \delta_{ij}\,\delta_{ml}\,\delta^{\alpha\beta}\frac{|G|}{n_\alpha}$$

Now, the next thing we will do is to derive convert this into a relationship for the characters, ok. So, this is for the matrices themselves all the representation matrices themselves, but we can now we can convert it into identity for the characters. So, to convert into an identity rather than formula for a relation for characters note that we need to we need to trace this. So, we need sum over i=l and separately sum over j=m; that is what tracing would require that will produce traces on the left hand side. So, if we do this summation so on the right hand side by the way this RHS, LHS is very English British and India it is extremely common.

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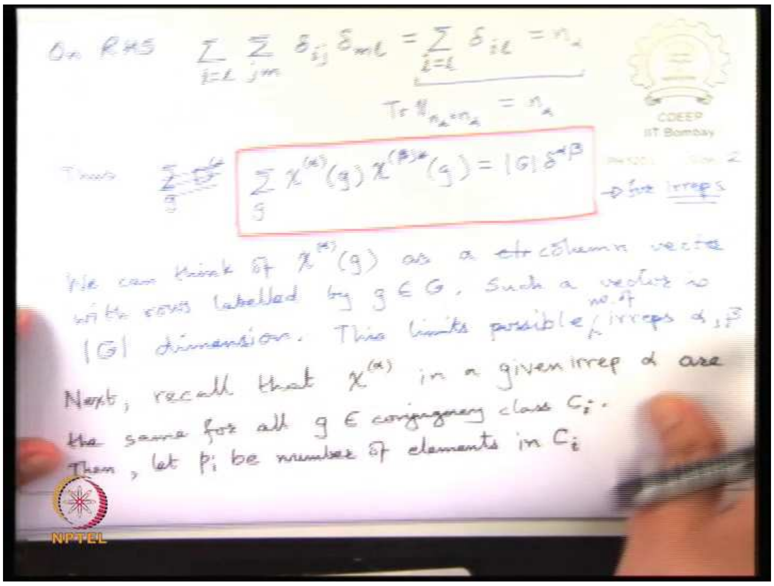

You go America people will say what is LHS. So, you have to every time tell them that LHS means left hand side ok. So, it is not a standard convention everywhere in this world they will really ask you every single time that I that I try to say this I get asked. So, on a right hand side if we do this, sum over i = l, sum over j = m of now what is there on the right hand side well these are things independent of i and j, but if I put the things that are dependent on i and j inside then I have this. But if I do a summation over mj then i summing this and this, but that is like matrix product of this δ with this δ, so it actually results into i equal to l the outer summation is still there.

The inner summation as simply produced δ_{il}, but the sum over δ_{il} only gives me n_α because that is the trace of identity element which is size of the representation. So, this is the identities $n_\alpha \times n_\alpha$ matrix. So, it is equal to n_α. So, if we now carried here then this summation as given as n_α. So, it will just basically cancel this n_α and we can get a much prettier looking relationship which simply says,

$$\sum_{g} \chi^{(\alpha)}(g)\chi^{(\beta)*}(g) = |G|\,\delta^{\alpha\beta}$$

This is much more like a orthogonality relation.

So, this is as if we have vector χ which living dimension equal to the order of the group g because if you think g as a component of entry χ, is column vector with entries in it there as many entries in a column vector as there are column vector g. So, χ can be taught of as a |G| dimensional vector, as a column vector with rows labeled by elements g such a vector is same dimension as the order of G. So, one thing becomes clear it is difficult to have too many representations because suppose alpha and beta there were large number of them very large number of them, but then the dimensionality restricted to |G| so obviously, you cannot have a mutually orthogonal vector at a larger then the order of group, right.

In fact, the relationship is tighter, so this limits possible irreps. Remember the whole thing is for irreps it is the relationship is for irreducible representations. So, this limits possible irreps, possible number of irreps $\alpha\beta$ because you cannot have in definitely large number of them because you will not be able to produce so many independent vectors. In the relationship is also some time written, by now we remember that the characters in a particular conjugacy class are the same because under conjugate transformation the trace does not change. So, characters in a conjugacy class are all the same.

So, all g $\in$ conjugacy class C_i , then let p_i be number elements in C_i and and how many conjugacy classes that also we can introduce a symbol for. So, we are just going to rewrite this summation for directly for conjugacy classes. So, we can now look at this. This summation will breakup into summation over classes and when you are summing over particular class the value of χ remain the same. So, if there a elements p in that class then all we need to put p_i as the weightage factor. So, this relation becomes r independent classes. They are of course, independent as you know because the classes are found by an equivalence relation, an equivalence relation produces the join subsets.

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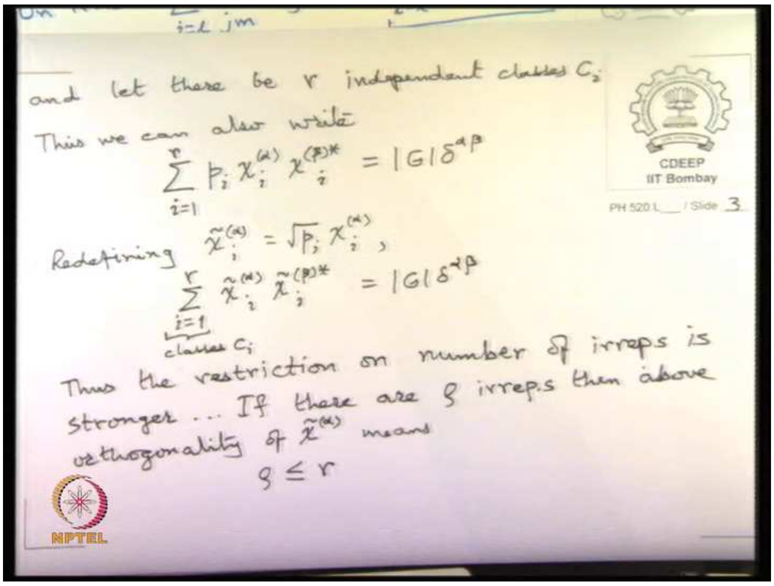

So, therefore, we can or thus we can also write,

$$\sum_{i=1}^{r} p_i \chi^{(\alpha)}(g)\chi^{(\beta)*}(g) = |G|\, \delta^{\alpha\beta}$$

This again is like a inner product, but with some weightage factor sometimes you can, so can have any inner product where you just multiply the components. But some time you can have a inner product where there is a weightage associated with each of the components it is sometimes called a metric.

So, you could just put weightage factors and think of this as a more general orthogonality relation or you could just redefine $\tilde{\chi}$. This is a temporary thing we are going to other topic, but just as a end the comment on this to writing redefining $\tilde{\chi}_i^{(\alpha)} = \sqrt{p_i}\,\chi_i^{(\alpha)}$ we can find another type of relation.

So, we actually have a more strict constraint on a how many $\alpha\beta$ we can have because earlier I was saying there are as many components to χ as there a elements g, but to $\tilde{\chi}$ there are even fewer components only as many as there a classes ok. So, in fact, the total number of irreps will be restricted by how many classes there are. You cannot have indefinitely large number because you cannot produce so many independent or orthogonal vectors. You are living in a dimensionality that is only r, you can't produce more than r dimensional vectors. So, thus the restriction on number of irreps is stronger, sometimes people write number of irritable representations to be ρ if there are ρ irreps then above orthogonality means I try to write this out because if you later read the notes if I just jump to write in the equation then the continuity lost.

So, at the risk of spending a little more time I just write this down above orthogonality of $\tilde{\chi}$ means that $\rho \leq$ r, where r is the number of independent classes. So, there is a natural restriction so in fact, it is also possible to prove the opposite namely ρ are not be less than r ; so there is are another orthogonality which we will not prove because it takes a little time and just gets more formal. But it is possible to prove where

$$\sum_{\alpha}^{\rho} \chi^{(\alpha)}(g)\chi^{(\beta)*}(g) = \frac{|G|}{p_i}\delta^{\alpha\beta}$$

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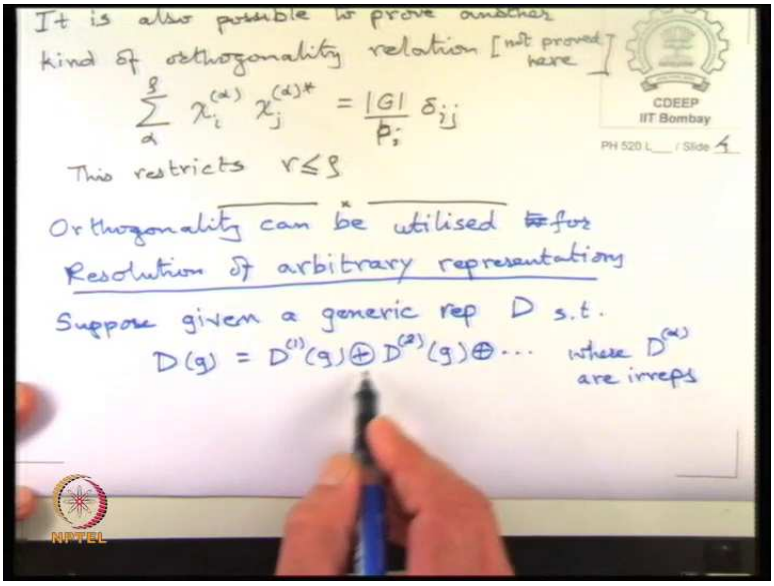

So, we will not prove it here. So, this then restricts the number of classes to remain the less than the number of representations. So, putting together the 2 inequalities r ≤ ρ.

Now, we prove this using orthogonality. We have another interesting fact, so this is sought out end of up to this point. Now we convert this orthogonality into a more powerful tool we can analyze any representation. So, now, somebody hands you a representation and you want to know whether it is a irrep or not, you know somebody just gives you a representation that this is a representation you can check that it is a homorphism by checking that the group table comes out the same.

But how do you know if it is an irreducible representation or not? So, this orthogonality actually even allows you to analyze because of it is a linear kind of relation. So, you can split up a given reducible or arbitrary representation into irreps by using this method utilized for resolution of arbitrary representations. So, suppose we are given a generic representation D such that $D(g) = D^{(1)}(g) \oplus D^{(2)}(g) \oplus \dots\dots$,where $D^{(\alpha)}(g)$ are irreps and this means direct sum.

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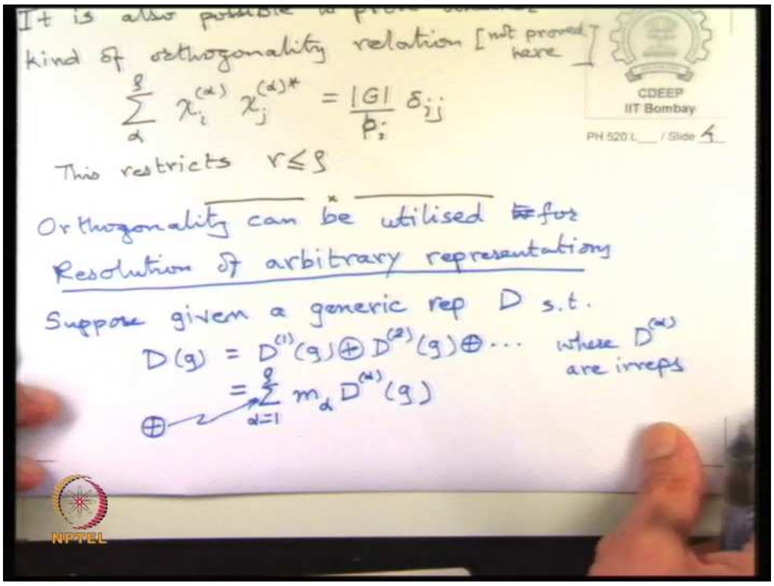

For ρ number of irreps, we will have, $\sum_{\alpha=1}^{\rho} m_\alpha D^{(\alpha)}(g)$. I should have put here not ordinary sum, but actually the group theory sum linear. So, suppose you are given same arbitrary reducible representation, now the trick to play is to simply take trace of both sides, so if you take trace of both side then you basically get the trace of arbitrary rep, but on the right inside you get traces of individual irreps. But those χ_α forms a vector space which are orthogonal so you are essentially resolve a χ a arbitrary representation in that.

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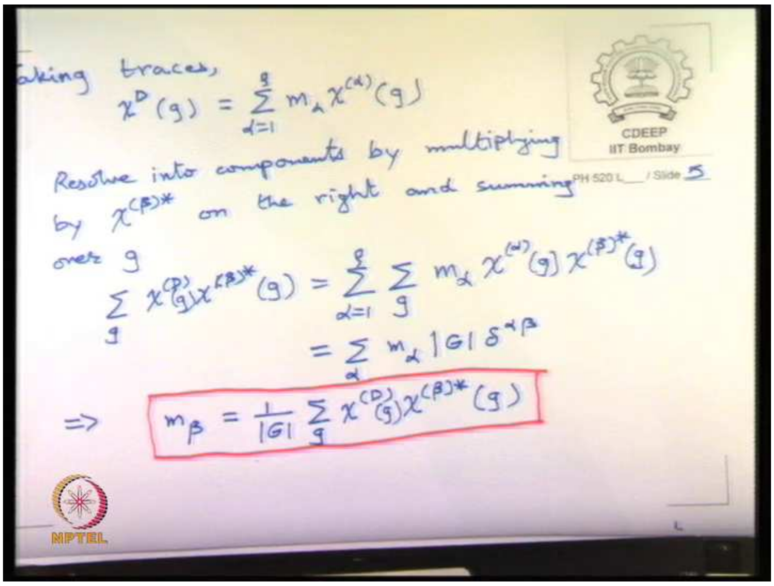

So, taking traces on both side we write this relation and we put a D we put say there it a generic crap it is not a alpha it is not label by the index the index of this. So, we say that this for any element g is going to be

$$\chi^{(D)}(g) = \sum_{\alpha=1}^{\rho} m_\alpha \chi^{(\alpha)}(g)$$

But now you can multiply by $\chi^{(\beta)^*}$ on the right and sum over g. Actually that should have gone inside, but that does not matter m_α. But that produces $\delta^{\alpha\beta}$. So, that produces a $|G|\delta^{\alpha\beta}$. So, we can that basically collapses this summation over α to just been m_β. So, that implies,

$$m_\beta = \frac{1}{|G|} \sum_g \chi^{(\alpha)}(g)\chi^{(\beta)*}(g)$$

So, this is a clever thing that one can do. One can identify the weight age of any rep in the general reducible representation by simply projecting out on to the χ data basis.

So now, we are proved that I can extract the m_α for anything. There is another relationship we can derive an important theorem actually using this.