Theory of Group for Physics Applications
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Lecture - 02
Algebraic Preliminaries

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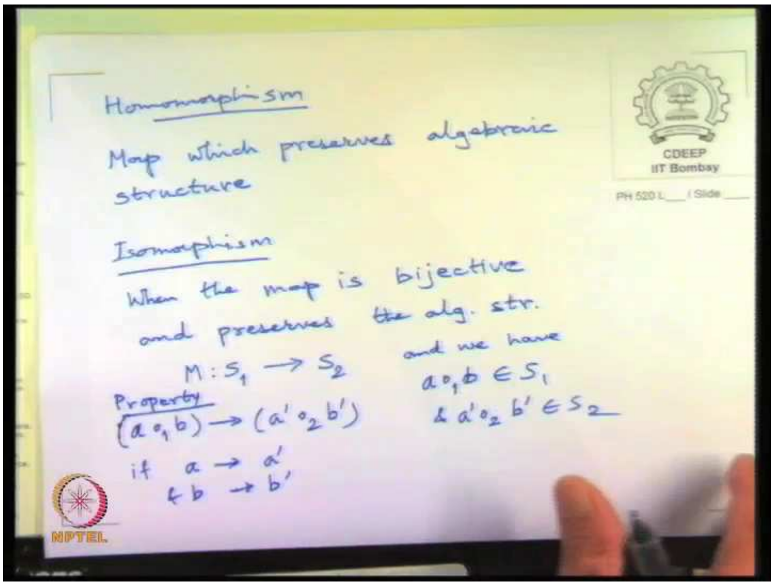

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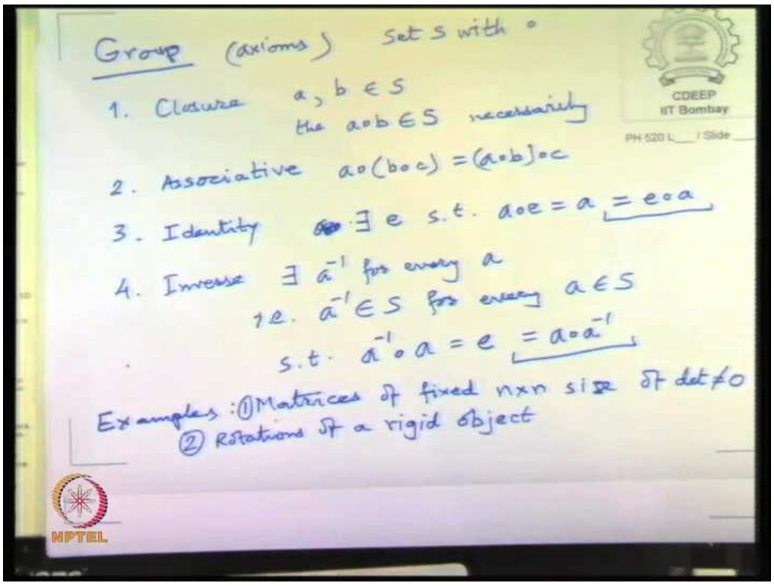

The next thing we take up is the idea of specific algebraic structures and we can begin by saying group. So, group as an algebraic structures since, we already discussed those generic properties, we can write down quickly now but the axioms are usually stated as follows, closure that is the set S is such that given any two elements in it and if you combine them using this binary operation whatever you have you get back something within the set ok.

So if a, b $\in$ S then a $\bullet$ b $\in$ S, so we have set S with the binary operation o, we can say then the first requirement is closure. Sometimes it looks a little trivial, but it turns out that it is important sometimes you can fall out of you know the original set that in your mind, so then the closure itself would be violated; number 2 is associativity.

So, that we have already talked about; then there is an there should be an identity for it to be a group, there has to be an identity operation meaning some operation which actually does not change anything ok. So the statement is that there exist e such that a $\bullet$ e = a = e $\bullet$ a, but as I said purist say that it is not necessary to add both side you can get one from the other.

And fourth is inverse there always exist a^{-1} for every a and of course, $a^{-1} \in$ S for every a $\in$ S, such that $a^{-1} \bullet a = e$, and it turns out that left identity and right identity are usually the same. So, these 4 axioms define what is the group ok, and the two very common examples we can immediately pick up are matrix multiplication of square matrices; square matrices of a fixed size say 3 × 3, 5 × 5, any n × n fix size square matrices.

We can see that matrix multiplication you will get another matrix of the same size. So, there is closure a matrix multiplication is associative as you know. So, you get associativity the identity element is the diagonal 1 1 matrix, which as you know leaves the matrix any other matrix unchange if you multiply it on left or on right, and the inverse exist only for determinant not equal to 0.

So, if you do not if the say example is of determinant not equal to 0. Then you can always invert such a matrix, so the inverse then exist. So, this is a very standard example. A geometric example is we can say examples, and put this is as number 1, and number 2 is rotations of a rigid body of a rigid object. So, you can think of any you take a pen and then say you rotate it, you combine it with any one rotation is one operation. So, the set S in that case will be all possible configurations of the pen all the possible ways it could be lying, and you are start with any one way it is lying you can always bring it to any other and that is one operation, but if you combine such two such operations you rotate it from one configuration.

So, let us say with a one point fixed to make things a bit simpler, and then you can always rotate it to one configuration, you can then rotate it to another configuration, that combination of two rotation says yet another which could have been done directly as well ok.

It could be rotation about some third axis. Associativity is there for rotations because you can do them in any this sequence it does not matter that has to be checked of course, the identity is you do nothing. So, that operation you do not rotate it at all, so the identity operation exists and the inverse is you undo the rotation. So, Euler proved an important theorem in early days of dynamics that said any general motion of a rigid object is translation plus some 3 D rotation about a fixed point in the object.

So, that is an important way of characterizing actually, now you can use it in reverse to characterize a rigid body, but rotation first studied in great detail by Euler. So, in that sense he had done group theory in a in it is preliminary form right then although he did not emphasize the group structure at that time, but he is specified all the machinery required to grapple with rotations they are called Euler angles which in mechanics you will probably learn alright.

So, that is that lays out the very basic requirements of a group these are the most essential requirements of a group in addition there is a fifth property which is commutativity, which simplifies it make it is a it is an additional requirements.

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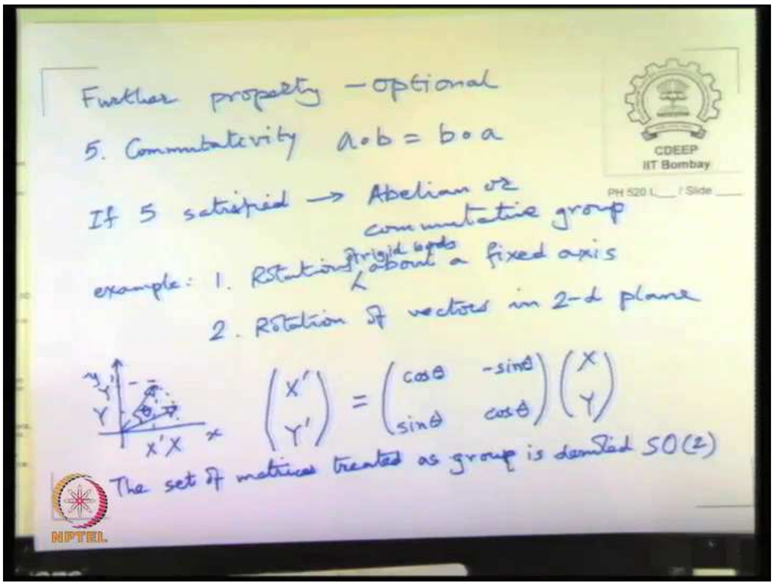

So, it is more restriction which is optional and that is commutativity, and if this additional axiom is satisfied then it is called an Abelian group, then it is called Abelian or commutative group.

So, rotations about a fixed axis rotations of a rigid body or you can just think in terms of rotation in the plane rotation of vectors let us say in 2 D plane. this is the most is a simplest example of an Abelian group and we deal with it all the time. So, you may be let me just write out how it works I have X and Y axis, and let us say I have position vector in 2 dimensions.

So, it has components X and Y and then I can rotate the vector to a new position, where it may have components X' and Y' , and the angle through which it rotates is theta then we know that

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

So, if you think of the same vector as having rotated it is X component is reducing. So,

$X' = X\cos\theta - Y\sin\theta$ and $Y' = X\sin\theta + Y\cos\theta$

and then for all other angles theta. So, the group here you can think of it in different ways we already have abstracted in the sense of we are talking about vectors, we are not talking of much of a physical quantity the first 1 is actually you can think of a pen and think of rotations in a plane of about a fixed axis of the pen, but when we talked of vector we already became a little abstract.

But furthermore you can now think simply in terms of these 2×2 matrices and then say my set consist of 2×2 matrices this obviously, as determinant 1 because $\cos^2\theta+\sin^2\theta=1$.

So, 2×2 matrices of determinant 1 we have we can easily check that the form a group because the product of any two of them will just be another rotation, and inverse will be just rotating backward identity is when it is identity associativity is inherited simply because it is a matrix, but furthermore it will be commutative because you rotate by θ_1 and then by θ_2 it is same as rotating by θ_2 first and then θ_1.

So, the set of matrices 2×2 matrices of size determinant 1 and determinant 1 is an Abelian group. So, that group would be called SO(2). So, this is like giving a trailer to what will come later, so thus set of matrices treated as group is denoted SO(2), O means orthogonal because these matrices also have the property that the transpose is the inverse.

So, they are orthogonal matrices the size is 2 and S is for special orthogonal group means that they have determinant 1 ok. So, this is these are the essentials of a group the next thing I want to cover is vector space, which you are all familiar with you have been using Newtonian vectors for quite a while.

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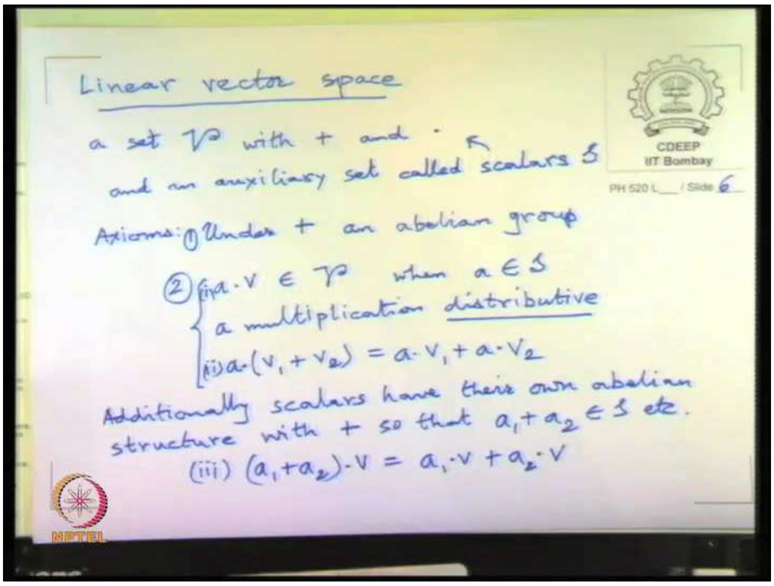

The word linear is often not even mention vector space is generically linear, so what is a vector space basically you have a set of vectors, set $\mathcal{V}$ which two kinds of operations + and $\cdot$. So, the plus is the linearity part that you can add anything, and the $\cdot$ is multiplication by scalars. So, there is v which + and $\cdot$ and an auxiliary set auxiliary set of scalars and of course, the dot enters only if you have those scalars.

So the axioms can be stated in two parts under + the set forms an Abelian group. Ok, because there is closure there is associativity of addition identity element is because there is a 0 vector, and inverse is negative of a vector. So, under plus an Abelian group and it is Abelian right , in any n dimensions is nothing to do with 2 dimensions. So, under plus an Abelian group, and under scalar multiplication it is a linear and it is distributive. So, we can say $a \cdot v \in \mathcal{V}$.

So, let us set the say the set of scalars is S, and a is distributive so that is the word use distributive in other words $a\cdot(v_1+v_2)=a.v_1+a.v_2$. Additionally the scalars also have a algebraic structure of their own. So, that their own Abelian structure which a sign which without much first is called the same sign + although; that + is technically different from the plus of the vector space. So that $a_1+a_2 \in S$ etc etc and then we have the third property. So, here I will put this 1 2 and 3.

Third being that $(a_1+a_2)v=a_1.v+a_2.v$ so there are 2 pluses, but you can see that they merge because here it is sum of only the scalars, but here is sum of the 2 vectors.

So assume it is very easily a transferred property and people do not distinguish between the two plus signs this vector space structure is very important in worths keeping in minds because we will be using it a lot, and it will be used especially when we get to representation theory groups are represented usually as acting on vectors like we already did right this example we saw was algebraically you can think of this as this set of matrices, but geometrically you can think of this as operations onto by onto dimensional vectors.

So, we already have sort of a trailer of what is going to come later there is an abstract group where you may just specify a list you can just specify table you have some elements $g_1, g_2, ,...., g_n$ you just specify a 2×2 table which specifies what is $g_1\times g_2$, $g_1\times g_5$

etc. That is sufficient to make it a group provided it the axioms are satisfied associativity and so on, but then in practice that is not have you actually use the group you use it by using those g elements as operations on something else So, the other classic example is permutations, so we can actually move to that.

But just to finish what I was saying here we will always have these two versions of thinking, either you think only of the set of matrices and their properties and think it as a group, or you think of it in terms of how it affects 2 dimensional vectors. The advantage of this second way of thinking is that you could have also created matrices you can have higher dimensional representations, which we will see later ok. So, the representations can be many for the same group the same group may be realized in more base than one. So, you can think of say moment of inertia tensor and so we will come to the examples later more detailed examples later.

But there are groups by themselves and their general properties and then how they affect a particular representation, where they are realized as actions on something. So, one such classic example is permutations as actions ok.

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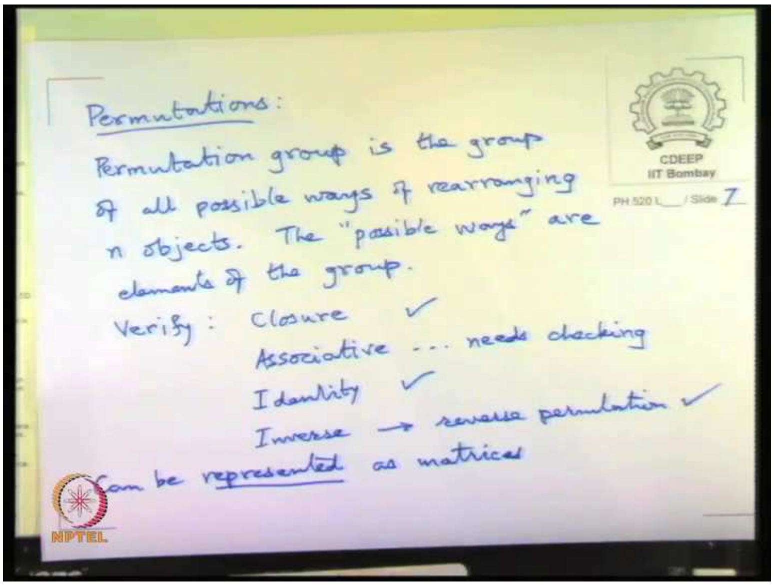

So, the permutation group is the group of all possible ways of, so group consisting of the action all possible ways of rearranging say n objects. The possible ways are group elements.

So, we can see that this particular example permuting objects is actually a group, because you think of one particular permutation as one group element then you think of another permutation as another group element. Now, you know that if you do them in sequence you first do one permutation, and then follow it by another permutation it is again after all permutation of the same set, so you will get some other permutation.

So, there is closure this is the first thing, so we can verify this, so given any example you have to mentally or physically or by through some modeling go through the exercise of checking these things. So, is there closure for permutations values then the question is whether they are associative this requires work, but one can do it needs to be checked, and identity of course exist, because you do not permute you do not anything to that set of objects then you have identity, and then inverse means you undo the permutation you go back to the previous one.

Now, permutations are very interesting because it turns out that and this theorem we will prove next time that any discrete group is subset is a sub group of some permutation group ok. So, there always exist some permutation group of whose your interesting the discrete group of your interest is going to be always subset of some permutation group. So, we will prove that next time but this is a very important and interesting reason why permutation groups are studying and we developed special notation for discussing the permutation groups. So, before we go on let us do the last bit of algebraic preliminaries, the reason why we mention permutations is that thought of it is operations there obviously a group, but additionally you can realize them as matrices by if you have n objects then you write the n objects as a vector and then any operation is basically exchanging them.

So, you can write out the so representations can be can be represented as matrices. So this was also meant to give you an example of what is the group and what is the representation. So, the group you can specify simply by listening what permutation followed by which permutation will give you then which new permutation in a list of permutations only, but then you can write out matrices which will actually under matrix multiplication reproduce what would have happen to the permutations. So, that is the idea of our representation.

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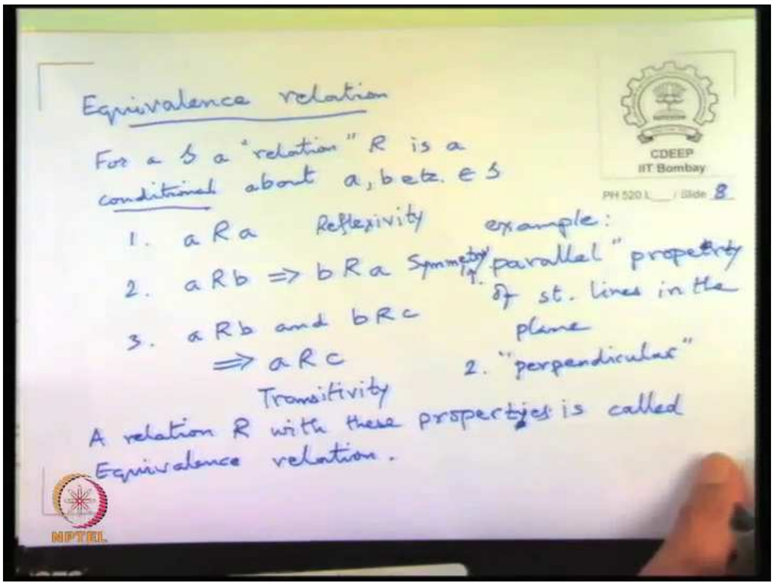

So, the final interesting algebraic structure that we want to discuss or algebraic concept is the concept of equivalence relation, this is a very a cute idea looks very simple, but is very far reaching. So, we say that for a set for a set S a relation R is a conditional about to a, b etc $\in$ S, such that a is always related to a. So, you give some condition. For example, you say straight lines in the plane a straight line is related to another straight line provided it is parallel, a straight line a and straight line straight line l_1 and straight line l_2 are related, if they are parallel and not related if they are not parallel.

So, you can set up some kind of a condition property of straight lines in the plane, and another one; one can think of is perpendicular property, so these two are interesting simple examples. You say that one straight line is related to another if it is perpendicular, and we will see how the axioms we are saying are going to affect going to work in the two cases.

So, we say that a set is an equivalence relation if three conditions are satisfied by that relation R; an object should be related to itself. Now, if you think of familiar relation although mathematics is not going to work there very well certainly you are related to yourself. So, it would work there number 2 a R b ⇒ b R a ok, and if you have if you generically say blood relative or something like that then yes if a is related to b then b is related to a, but if you say something like brother of then it may not work in reverse because the other person maybe sister.

So, this is where these properties are important you have to specify what the relation R is, and the third thing is transitivity if a R b and b R c then a is related to c; a R b and b R c then that implies that a R c. So, these are called reflex reflexivity this is called symmetry sorry about that and this is called transitivity. You it carries on if a is related to b and b is related to c then a is also related to c. So, a relation R satisfying these is called equivalence relation, you can think of many relations and they will not satisfy these.

So, it is a very powerful requirement actually; these properties is called equivalence relation. And we will see a very important consequence that this has very shortly, but now we can look at our 2 examples we thought of parallel and perpendicular. So, if you say that parallel is my relation; then any line is parallel to itself. So, it satisfies reflexivity if a is l_1 is parallel to l_2 then naturally l_2 is parallel to l_1 , so it satisfy symmetry and if l_1 is parallel to l_2 and l_2 is parallel to some line l_3 then it will also follow that l_1 will be parallel to l_3.

So, in the plane there is parallel is a property that is an equivalence relation.

But let us look at perpendicular; it fails the very first requirement because a line is not perpendicular to itself ok. So, is perpendicular to is not an equivalence relation. Now, let us state the important theorem. The theorem is that if you have any equivalence relation in a set then that relation divides the group, divides the set into mutually non intersecting subsets ok; divides the set into disjoint subsets.

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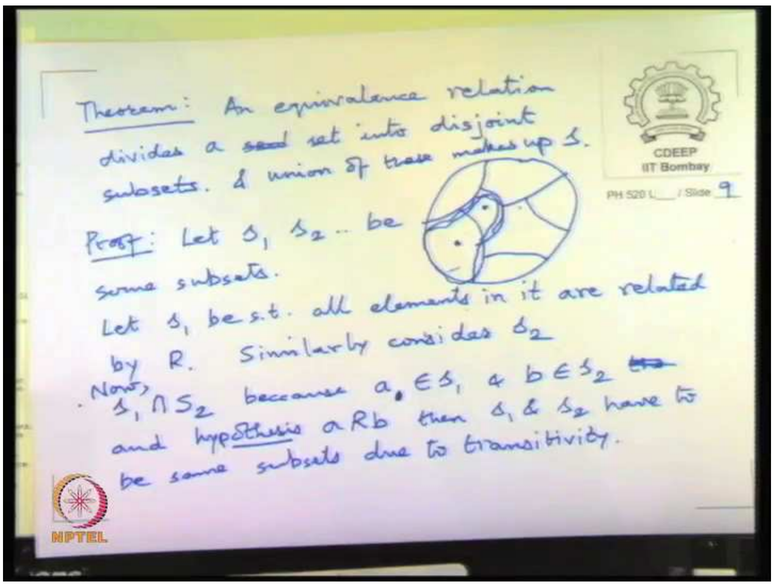

So, pictorially I think of it like this is my big set and I will end up forming subsets such that no two of them will overlap. Now, it is easy to see why I mean we can prove why this happens that, so think of; let s_1 , s_2 ,…. etc be some subsets. Let s_1 be such that all elements in it are related to each other related by R.

Now, first we prove that there is nothing in S which is not already inside s_1, this is so s_1 is a very unique uniquely defined set in S because any other element that belongs to S has to be included in it by because all the elements that are related are included in s_1. And if you have any other set s_2 then no element of s_1 can be related to any element in s_2, because if anyone gets related to s_2 then through transitivity all the elements of s_2 get related to all the elements in s_1 then essentially s_1 and s_2 are the same subset ok.

So similarly, let us say s_2, now s_1 necessarily is disjoint from s_2 because if a $\in s_1$ and b $\in s_2$ and hypothesis a say is related to b, then this hypothesis forces you to conclude that s_1 and s_2 have to be same sets because have to be same sub sets due to transitivity, right because any. So, once you declares that a here is related to b there then because of transitivity this since this a is related to everyone here, but it is now related to b therefore, b by transitivity is also related to all. In other words you get a new set this whole thing has to be really 1 set by itself ok.

So, either the 2 sets are disjoint or they are the same and further more because the equivalence relation actually exist over the whole set; it covers all the possible elements in the set. So, the union of all the disjoint subsets thus, make up the set back as well, so that is also important to know. So, you may have n number of elements in it which are not related to anyone except themselves. The reflexivity holds, but it does not relate to any others. So, I have a special element a which relates to of itself the property specify it is that relates, but it need not be related to anything else.

This point will by itself be a set a subset single point subset and you can have several of them. But any other set which contain any other subset, which contains several elements will be other category of subsets. But these two categories will exhaust all the possible subsets you can have and the union of all such subsets which are all disjoint that union will make up the original set back again.

So, it is a very nice partition of the whole set that is implied by present existence of an equivalence relation. So, but as I said not all relations are equivalence relations and not all sets may admit such nice equivalence relations, but it is going to be a very important concept that we will use improving some of the things ok.

So, I think we can end with this here today.