Theory of Group for Physics Applications
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Lecture - 19
Schur's Lemma & Orthogonality Theorem – I

Today, we want to finish of the main theorem that concerns this, a discrete group theory and it is called the Great Orthogonality Theorem. In another book that I found it is called Wonderful Orthogonality Theorem. So, it depend on whether you like GOT or WOT you can call it what you like.

So, let us put down the various concepts we are dealing with; one is of irreducible representation.

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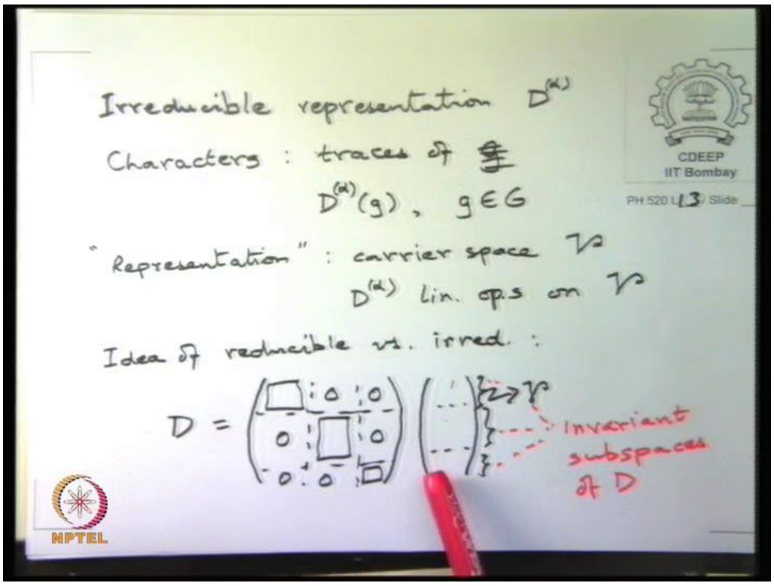

And we want to move towards and we are different characters which are essentially traces of $D^{(\alpha)}(g)$. So, the irreducible representation we will denote by D and α is the dimension of the representation or something else that characterizes it and the character that traces of the matrix is $D^{(\alpha)}(g)$ which represent elements g for $g \in G$ ok.

The representation is meant to be as we have already said with carrier space some linear vector space and $D^{(\alpha)}(g)$ are matrices. So, in our representation it almost always means carrier space, some vector space $\mathcal{V}$ and $D^{(\alpha)}(g)$ as linear operators. Now, some people had a question last time at the end because the idea what is irreducible was not completely clear. And we had said something like if there is invariance subspace; you know if space breaks up into two subspaces which remain invariant.

But the idea is of reducible verses irreducible is simply that if I have a matrix D which breaks up into block diagonal form such that only these are non-zero and all these are 0 ok. What this means is that in any basis we have constructed or so, any vector we take belonging to $\mathcal{V}$ and if this matrix acts on that $\mathcal{V}$ it is going to take the first few only into themselves. Those components will only be mapped into those components; the next few will only be mapped into themselves and the last segment will be only mapped into itself.

So, those basis vectors will never be transformed into these basis vectors or these will never be transformed into those ok. So, each of these is called an irreducible subspace or it is called invariant subspace. So, these become all invariant subspaces of D. But the point is that if for the entire representation, for all g belonging to the group if every single matrix D that represents them breaks up like this; then we are some redundancy because we could have just looked at this last few components and we would have got a representation.

We could have looked at this subspace and we would have got a representation. So, the idea this is called a reducible representation and when you restrict yourself in such a way that there is no invariant subspace to it then it is called irreducible. So, if it can very well happened that by accident one element happens to have some form like this, but if it happens for all the elements of G then we call such a representation reducible.

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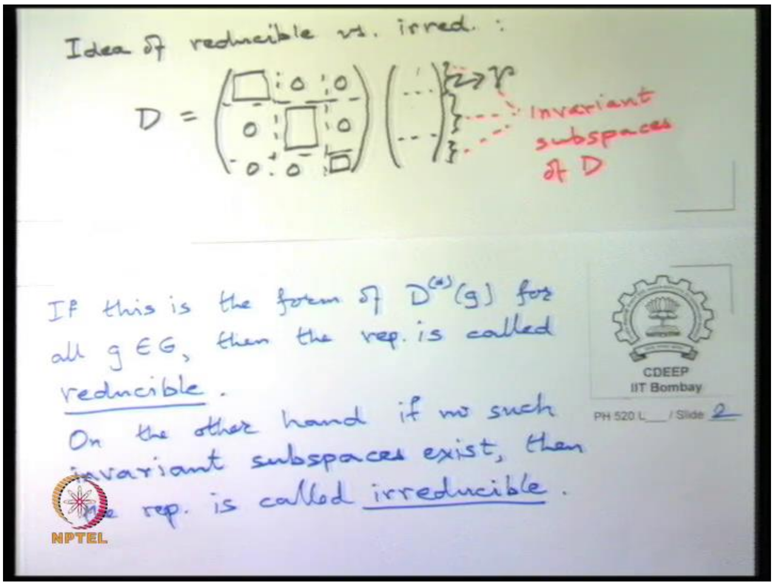

We will find that for a given group the number of irreducible representations is also limited; you cannot indefinitely keep finding newer and newer irreducible representation. If a newer representation you find larger one you find it will be possible to break it up into this smaller ones which are irreps.

So, we will see that there is a limit there are only finite number of irreps. Any larger rep will be reducible, reducible into this and we will find that the irreps actually had somewhat like a basis.

You find any general large reducible representation you can almost represent it like a linear combination of the smaller ones. The whole problem has been transferred to linear spaces now into linear algebra and lot of the powerful results that follow from linear algebra apply. And so, we will find that there is almost like a the irreps almost form a basis.

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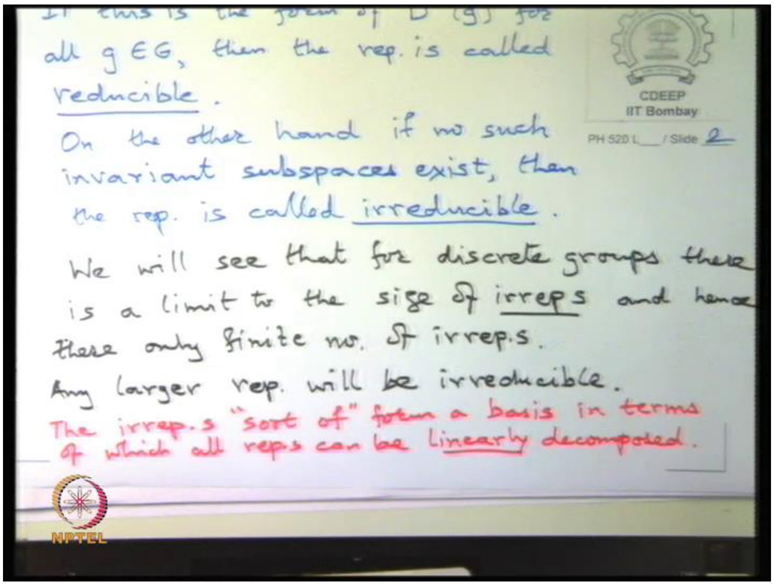

I am just writing sort of because I do not see any theorem stated of that type, but the statements we will prove will convince you that this is what it means; in terms of which a general rep can be linearly formed. So, linearities always a great thing because it keeps thing simple can be linearly decomposed.

So, that is the general idea by the way I finally, discovered on to a good book it is rather comprehensive it is a big fat book, but it is also free put up by MIT and I have put the link the model. So, the book by it is actually written for condensed matter physics, but its first few chapters will be useful for us and it goes by example. So, that is a very nice thing it. So, any larger rep will be reducible and of course, when I said larger I meant larger than the largest irrep that exists.

So, now we go back to what we were trying to do last time which was first to prove Schur's lemma. So, we derive into a bit of mathematics for a while and then we will come out and then prove this nice result that every irrep is linearly decomposable and just to make the larger connection; the point is that if you have any vibrational modes of a molecule then you can analyze.

Then by their the vibrations essentially fall into some representation of the symmetry group of the molecule and if you have arbitrary vibration then you can decompose it linearly into so called normal modes of vibrations which form basis in terms of which irreps are represented.

So, if you have to calculate the intensity of the particular transition it will amount to finding out the weightage of a particular representation that is contained in the particular vibration that you have ok.

So, if it makes it will sound way right now I am just going over it quickly, but that it what it boils down to so, you can decompose motions. Ultimately there is some intrinsic linearity the vibrations are small. If there are large vibration then there can be non-linearity, but usually the effects that human beings can cause are small until laser science. Laser can cause non-linear effects at atomic level, but the otherwise all the electric or magnetic fields you can produce arsotinic compared to the intrinsic forces electrical and forces binding the atom that you can only make very small disturbances.

So, they can be analyzed as just linear vibrations you can treat it like a harmonic oscillator general sense. So, linearity applies therefore, most of chemistry use this kind of linear representation theory ok. So, we come to this Schur's lemma and I remember that we were doing these statements about so, called kernel of map and I think this proof is probably the most abstract one that we are going to do.

Probably this Dresselhaus's book has a better proof, but since I have not prepared for it; I will just tell you what is from Hassani's book which is easier to which is at least need it is clear. So, we had defined.

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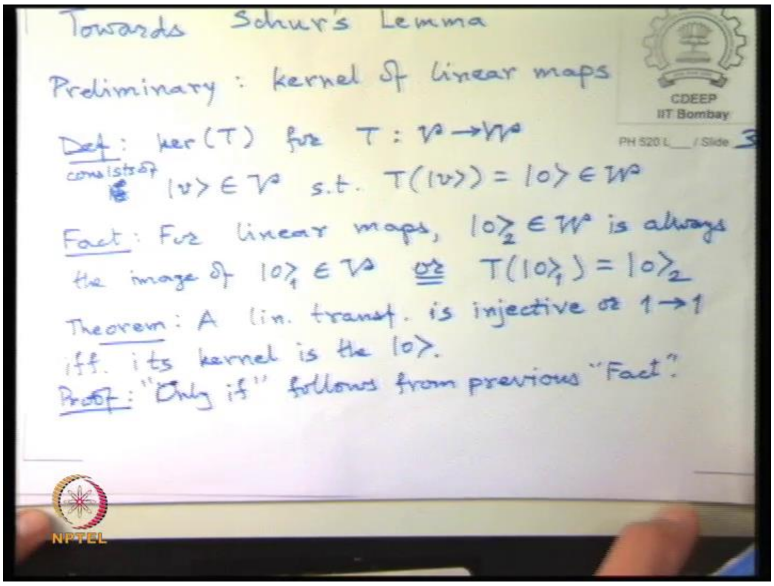

So, towards Schur's lemma the first thing was to say something about kernel of linear maps. So, first of all the definition was kernel of a map T for T : $\mathcal{V} \rightarrow \mathcal{W}$ lets say is all elements $\mathcal{V}$. So, kernel $|v\rangle$ consists of $|v\rangle \in \mathcal{V}$ such that T ($|v\rangle$) = $|0\rangle \in \mathcal{W}$.

So, whatever is mapped into the $|0\rangle$ is that set is the in the domain set is called the kernel and one of the immediate results is that for a linear space the $|0\rangle$ on the other side is always in the map; so for a linear map is always in the range.

So, right always the image of ok; otherwise the linear transformation properties will not work you have to add. So, you know that adding a $|0\rangle$ does not 0 is like the identity element and so, if on this side if you add $|0\rangle$ it should not change the vector then the image also should not change. So, the on the other side also you have to have this.

So, that is a very simple and obvious fact then the next thing we were trying to proof was that a linear transformation is one to one if and only if its kernel happens to be the $|0\rangle$. So, in other words after all it is one to one then it should take only one elements into a specific element and we already have this thing here this statement here which is little obvious.

So, this have in this case it is we are trying to state a I mean if and only if statement. So, not only does it map $|0\rangle$ into $|0\rangle$, but conversely if the kernel is $|0\rangle$ then the linear transformation also has to be one to one.

So, the if part is sort of clear the only if that is this we have already proved and then to prove the forward statement that the map has to be one to one if the kernel is only $|0\rangle$; then note that if we have two different elements not if.

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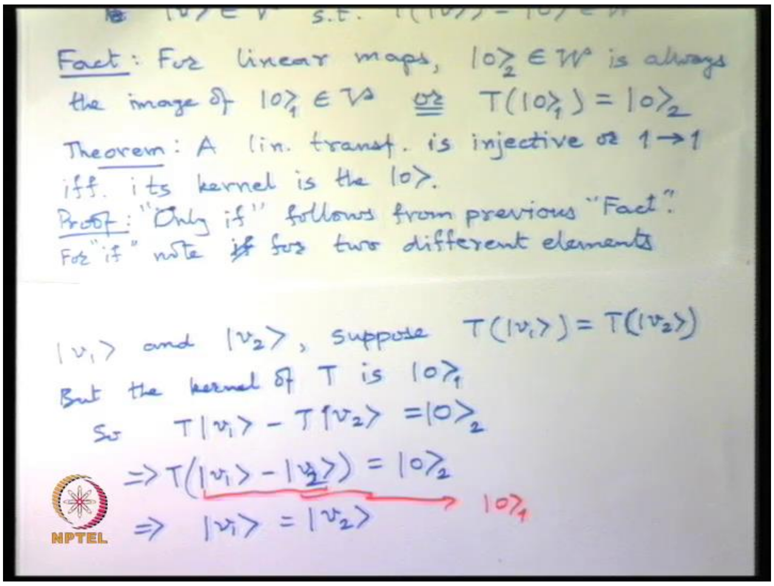

So, for two different elements $|v_1\rangle$ and $|v_2\rangle$ so, suppose that $|v_1\rangle$ and $|v_2\rangle$ are mapped into the same element on the other side; what we want to prove is that that implies that $|v_1\rangle$ must be actually equal to $|v_2\rangle$ because it is a one to one. But then we know that, but the kernel of T is $|0\rangle_1$. So, T $|v_1\rangle$-T $|v_2\rangle$ = $|0\rangle_2$ on the other side implies that $T(|v_1\rangle-|v_2\rangle)$ is also equal to $|0\rangle_2$ because the kernel is $|v_1\rangle-|v_2\rangle$ right.

So, implies the T of by linearity in other words this has to be essentially $|0\rangle_1$ because that is the kernel. So, if it maps into 0 then this vector has to be $|0\rangle$; that means, that $|v_1\rangle=|v_2\rangle$. So, if images are the same, the domain elements are also the same and so that proves one to one.

Now, we are going to use for Schur's lemma and as I was trying to tell you Schur's lemma should actually you have been called a theorem. But I suspect that he proved it along the lines of proving the great orthogonality theorem so, since the wonderful orthogonality theorem. So, because that is the bigger theorem people call this previous result a lemma.

So, what is Schur's lemma? So, the statement of Schur's lemma that I may have written last time; we have to emphasize that it deals with irreducible representations such that that define irreducible representations, and if there is a operator A which so, here the T' is on $\mathcal{V}'$.

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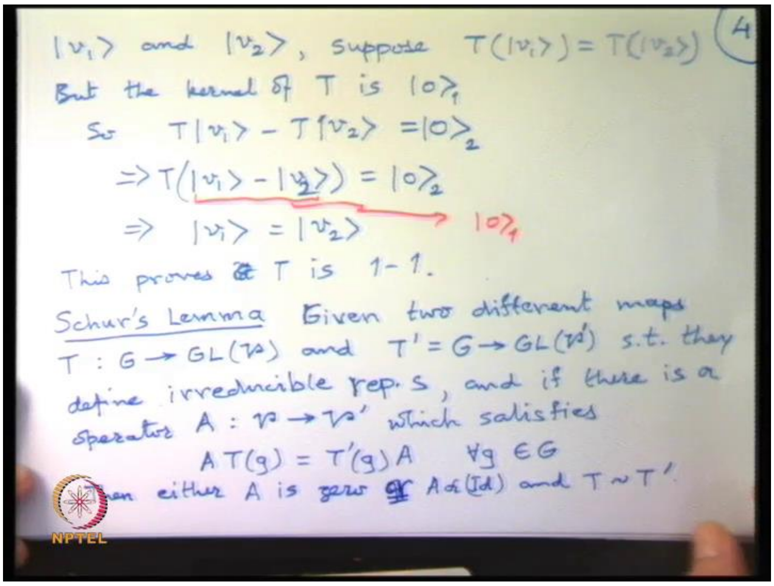

So, the carrier spaces are different ok; GL just means the general linear group it means matrix some matrix representation and $\mathcal{V}$ and $\mathcal{V}'$ are vector spaces of potentially different sizes. They are not the same vector spaces.

So, we are trying to realize the group in two different ways we have one carrier space $\mathcal{V}$ on which some the same size may. So, if it is size n then $n \times n$ matrix is can act on it or there is a $\mathcal{V}'$ whose size may be n' and there will be $n' \times n'$ size matrices is acting on it; that would be called a map T' of all the elements of G into those matrices.

So, and if there is now an operator $A : \mathcal{V} \to \mathcal{V}'$ which satisfies $AT(g) = T'(g)A$ for all $g \in G$. So, in simple language what we are saying is ultimate although we are writing this as maths they will become some representations $D^{(\alpha)}$, $D^{(\beta)}$.

If I have two sets of representation matrices for the same group and then I find some $n \times m$ matrix which code commutes with. So, if it is applied to representation α; it produces $D^{(\beta)}A$ on the other side such a has to be either trivial I mean it as to be 0 or identity that if what it shows. So, the in other you can you can think of it another way, you can try to apply A^{-1} on this side you know flipped.

There is no way of converting an α representation into β, if they are both independent irreducible representations what it means. Some kind of similarity transfer; if you seek a similarity transformation like A that does not I mean its trivial, if there is just identity element or it is actually 0. There is no way of doing it and that is actually; what is the significance of an irreducible representation that you cannot convert one into the other by multiplying it by some.

So, given an $n \times n$ representation you will not be able to find an $n \times n$ operator acting on it which will convert it all magically into $m \times m$ representation that will become a representation that is the meaning of this statement. So, let me just very briefly say loosely speaking . So, I did not complete the statement. So, if such A exist then such A is or A is identity and T and T' are the essentially the same or 0 or.

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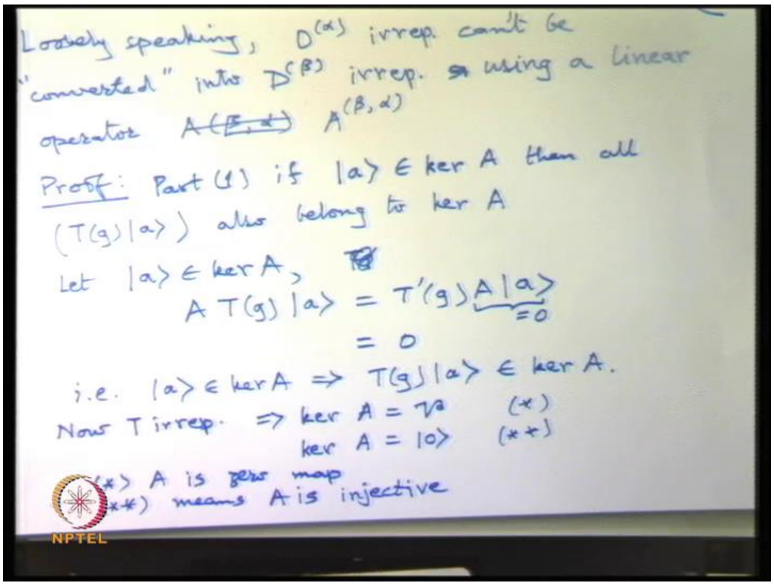

So, loosely speaking $D^{(\alpha)}$ irrep cannot be converted into $D^{(\beta)}$ irrep using a linear operator A of size $\alpha\beta$, $A^{(\beta,,\alpha)}$.

So, you have to of course, make sure of the domains and ranges, but you cannot and if I flip A it will become an $\alpha\beta$ kind of operator. So, I can always apply $\beta\alpha$ and $D^{(\alpha)}$ and A^{-1} on the on this side and except to get up β size matrix, but such conversions are not possible. So, and that is what leads to that orthogonality. It basically means that the representations contained completely independent vectors with the irreducible representations.

So, now the proof is somewhat technical first it observes that if a $|a\rangle$ belongs to kernel of A then all $T(g)|a\rangle$ also belong to kernel of A. So, we are trying to so, if everything belongs to kernel of A it means that A is mapping everything into $|0\rangle$. Now, belonging to the kernel of A means it is being mapped into $|0\rangle$; such things in linear transformation theory are done only by $|0\rangle$ maps taking a whole space and converting it on $|0\rangle$ on the others in the range from domain means that that is essentially a $|0\rangle$ transformation.

So, we are moving towards that. So, this can be seen because so, let $|a\rangle \in$ kernel of A and we take $AT(g)|a\rangle$, but that according to the hypothesis should be same as $T'(g)A|a\rangle$ on the other side on $|a\rangle$. But that means, that since this is A of this is $|0\rangle$ because it is kernel of A. So, this is equal to $|0\rangle$; that means, that a belonging to ker A implied that T after applying T(g) also it continues to belong to ker A ok.

Now, if T is an irreducible representation then either it is a trivial map which means that either it is going to send all of $\mathcal{V}$ into one element the $|0\rangle$ element or it should not do anything to it or the ker A has to be $|0\rangle$. So, I call this results star for time being a statement star and star.

So, these are the conditions for T being in a irrep and star would mean that A is essentially trivial. A is zero map and if it is two then it means it is a one to one map; this is where we are using that lemma. So, the strategy is to prove a map to be both injective and surjective that makes it one to one ok, and in both cases it is the first option that will work out that sorry in both cases this will happen and so, the first is what will happen and so, will essentially see that A is $|0\rangle$.

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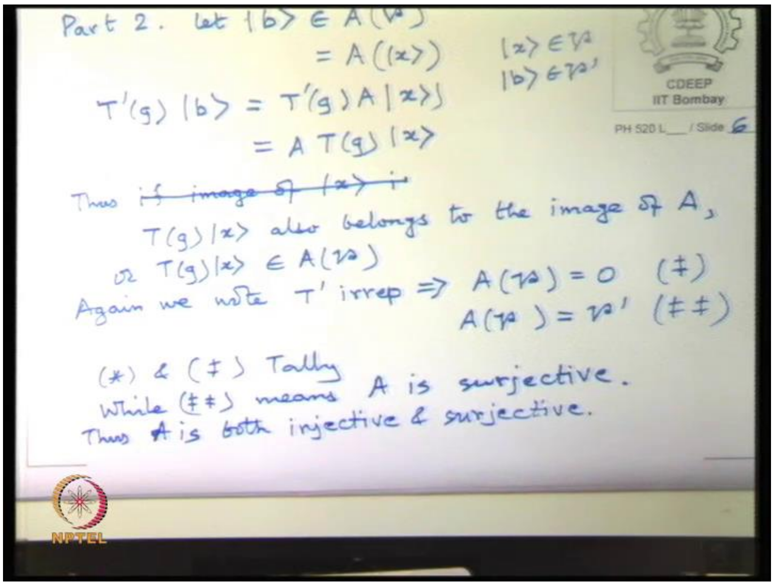

The reverse part basically now, starts from the other side part 2. Let $|b\rangle \in A(\mathcal{V})$ for example, equal to $A(|x\rangle)$ with $x \in v$ and now we start with because this is in the A remember is the map from $\mathcal{V} \to \mathcal{V}'$. So, this $b \in \mathcal{V}'$.

Now, we want to use that condition that A commutes with T and T'. So, $T'(g)b = T'(g)A(|x\rangle)$; that is the line about $b = A(|x\rangle)$. But this we can write as equal to $AT(g)|x\rangle$, but that means, that this also belongs to the image of A. So, $T(g)|x\rangle \in A(\mathcal{V})$.

Now, again we draw we remind ourselves of the properties of what is an irrep; that T' is an irreducible representation implies either that $A(\mathcal{V})$ has to be 0 or that $A(\mathcal{V}) = \mathcal{V}'$. And we put some symbols here to tally these all this statement this and this here double dagger this is what I chose one so.

Now, this and this are essentially same that kernel of A is all of $\mathcal{V}$ every all of $\mathcal{V}$ is demolished by A and this is what basically saying; So, whereas, if it is to then it means that A is actually surjective. Thus, we see that A is both surjective and injective and A which satisfies the condition of the lemma that $AT = T'A$. If such A exists then it is both injective and surjective.

So, either it is an isomorphism which means essentially an identity matrix kind of thing or of course it can mean that T and T' are equivalent. But we will see that that is not real

I mean T and T' will actually be the same in this case. And this means that A is either an isomorphism or A is just zero.

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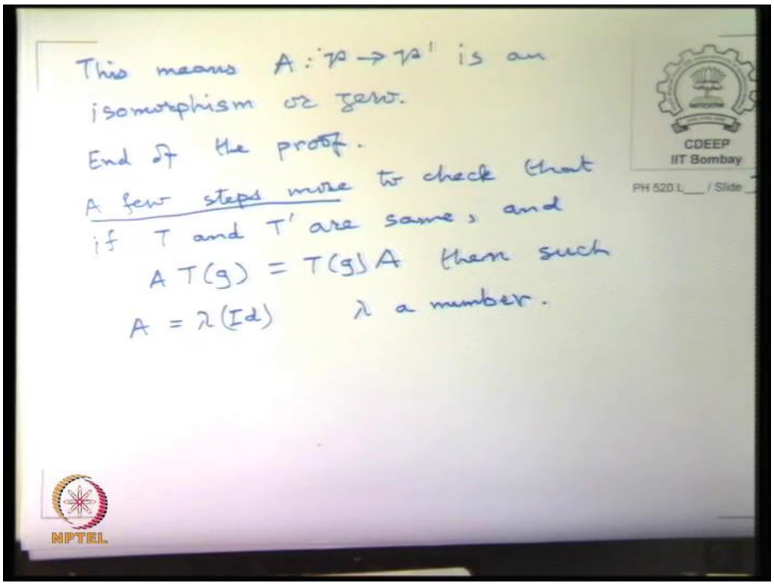

A which was from $\mathcal{V} \to \mathcal{V}'$. So, the role of an A which code commutes with I mean it or code purpose to convert a represent map T into T' has to essentially be either do nothing unusual or as to be just 0 if it is tries to do something unusual so, that is the proof. This is a formal proof and you have to spend a little time thinking about it. So, this is the end of proof.

And actually lot of people so, proving that it is identity can be a few steps more to check that if T and T' are same and AT(g) = T(g)A then such A is proportional to identity. So, we will skip these steps; we are seeing the crux of the proof, most of it is that and it can also be proved by some by an extension of those that reasoning and specializing to the case when T becomes T'.

We can actually check that it is identity because in that case it cannot be all zero and it is bijective and injective. So, it will be essentially equal to identity map so, up to a constant.