Theory of Group for Physics Applications
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Lecture - 18
Representation Theory - IV

So, let us now continue, and let me finally put down since we did define all the notation put down the definition of reducible and irreducible. For the next one or two lectures we will also be using a slightly abstract notation.

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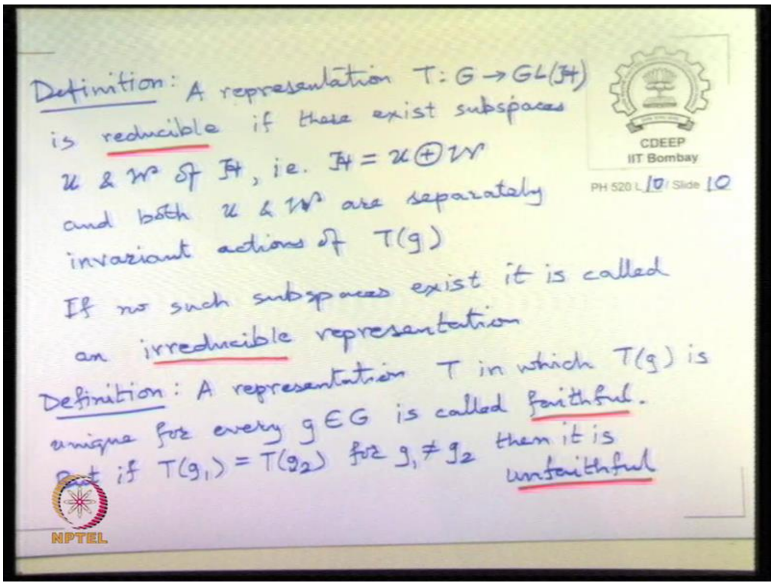

So, we say that a representation is the map after all T : G → GL ($\mathcal{H}$) is called reducible, if these exist subspaces $\mathcal{U}$ and $\mathcal{W}$ of $\mathcal{H}$, i e. $\mathcal{H}=\mathcal{U}\oplus\mathcal{W}$. And both $\mathcal{U}$ and $\mathcal{W}$ are separately invariant under actions of T(g).

So, invariant means that subspace never gets mapped into T(g) acting on the subspace $\mathcal{U}$ will never map anything of $\mathcal{U}$ into $\mathcal{W}$ and vice versa. So, if this happens then this particular representation given by this map T is said to be reducible and if there is no such subspace exist it is called a irreducible representation.

The other definitions of faithful and unfaithful; A representation T in which T(g) is unique for every g is called faithful, but if $T(g_1)=T(g_2)$ for $g_1 \neq g_2$ then it is unfaithful and we already saw the example first the worst kind of unfaithfulness is where everything is represented by just 1.

Nevertheless that severely unfaithful representation is irreducible because you cannot reduce 1 to anything else ok. So, it is an irrep and as we will see it very importantly anchors the whole calculation of Schur; the whole calculation of the great orthogonality theorem ok.

So, the only next thing I want to remind you is that we also checked last time that you can always make any representation unitary. So, all this matrix representations are unitary can be chosen to be unitary.

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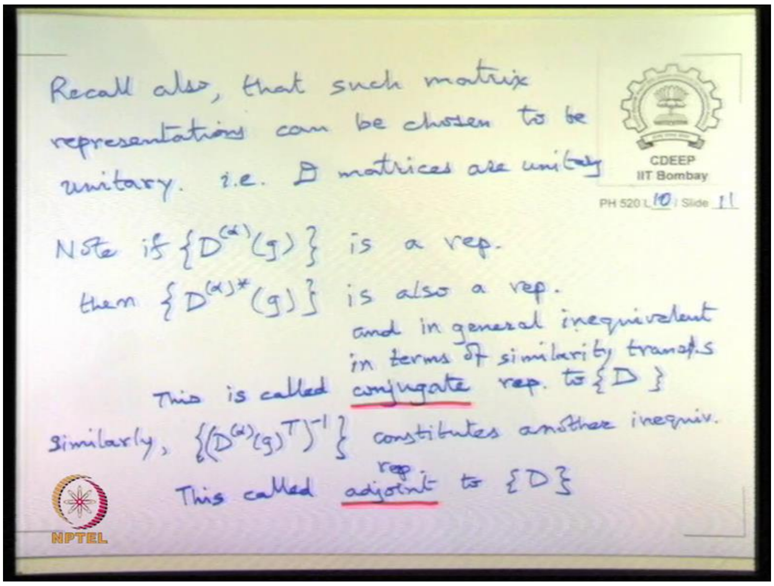

So, in the other words the matrices D are unitary. If there are representations called adjoint if so, D is the representation and D^* is called the conjugate representation.

So, note that if $\{D^{(\alpha)}(g)\}$ is a rep then the matrix is $\{D^{(\alpha)*}(g)\}$ is also a representation and it may not be transformable into D under similarity transformation. So, they are inequivalent. So, I should say in general that it cannot be brought to D by a similarity transformation and is called conjugate representation.

This we are not going to need in near future, but it is words keeping in mind to the one with D and there is another one where we take $\{(D^{(\alpha)}(g)^T)^{-1}\}$. Similarly, transpose and then inverse constitute another inequivalent. I have written constitutes because now I am referring to the whole sets. So, there is one entity. So, this set constitutes another inequivalent representation and it is called the adjoint representation.

We will see that so, there are some accidental cases for very small groups it turns out that sometimes can actually be put in. So, that is why I said in general it is inequivalent ok, but there are some special cases where coincide or they are equivalent under a similarity transformation. The classic example is the group of SU(2), the spin group generated by Pauli matrix is turns out that Pauli matrix can be complex conjugated by similarity their own transformations. You can form that complex conjugates.

So, but aside from those special cases you do not have and I just said it without too much details; we will come to it in the second half of the course where we go to continues groups. Now, let us prove the so, called Schur's lemma.

Let me state Schur's lemma in the formal sense. I am probably mentioning Schur's lemma the fourth time, but it is important enough that one can say it several times. Suppose we have one representation $T : G \rightarrow GL(\mathcal{V})$ and $T' : G \rightarrow GL(\mathcal{V}')$.

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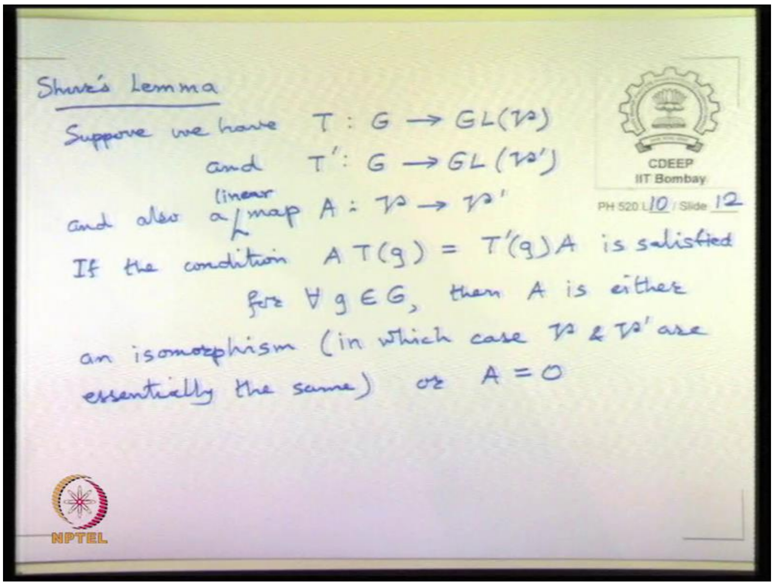

And also a map $A : \mathcal{V} \rightarrow \mathcal{V}'$ which is a linear map or a linear operator. So, these are the m×n matrix is that convert an size n space into a size m space. Now, if such if it happens that if the condition that $AT(g) = T'(g)A$ is satisfied for all g, then there are only two options A is an isomorphism or A is zero and let me add here in which case $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic are essentially the same or A is zero; the matrix A is zero ok.

This particular lemma within some sense even looks a little too simple; it does not seem to say anything very profound turns out to be key to proving the great orthogonality theorem and the way one proves it to prove it. And so, we will only setup the beginning of the proof and we will go to the proof next time. So, to prove this we need to introduce a concept called kernel of a map and this proof I am taking from the book by Hassani.

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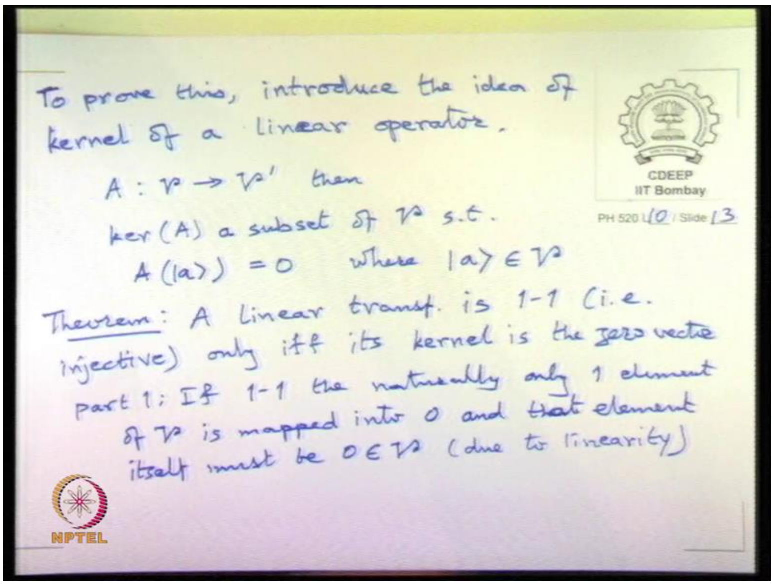

So, to prove this introduce the idea of a kernel not cornel; the idea of kernel of a linear operator kernel of a linear operator is all the elements that it maps into a $|0\rangle$. So, given $A : \mathcal{V} \rightarrow \mathcal{V}'$ let us say then ker $A \subset \mathcal{V}$ such that $|a\rangle \in A$; let me write quantum mechanics notation equal to 0 where $|a\rangle \in \mathcal{V}$.

Now, an important property of this kernel of linear transformations is that it is one to one. So, linear transformation is one to one only if it is kernel is 0. In fact, it is if and only if its kernel is 0. So, the only if part is clear which means that if it is a one to one transformation then clearly its kernel has to be zero.

So, part 1 there is only one element $\mathcal{V}$ is mapped into 0 and that element must itself be 0; $0 \in \mathcal{V}$. This is due to linearity of the transformation. Part 2 is the opposite which is that if it is mapping if the kernel is $|0\rangle$ then it has to be one to one.

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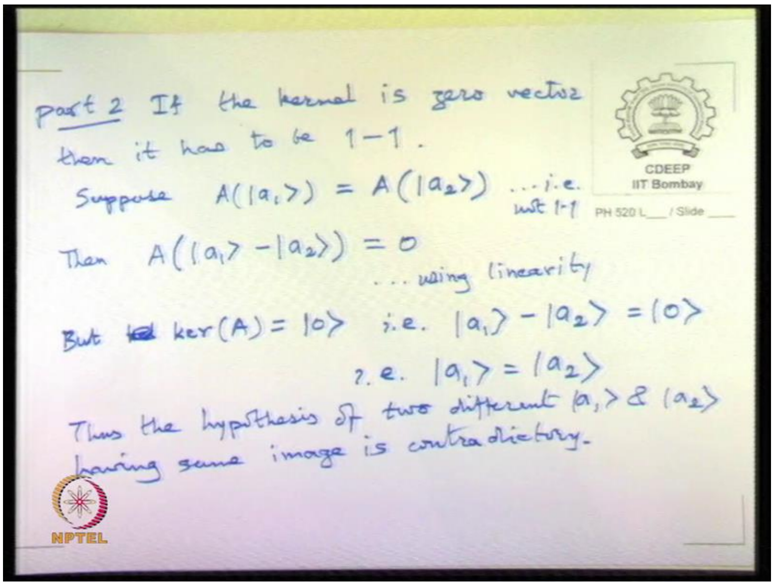

So, suppose it maps two different vectors into $|0\rangle$; suppose it happens that $A(|a_1\rangle)=A(|a_2\rangle)$. So, here we are saying it is not one to one. So, suppose i.e. not one to one, but then because we are using a linear transformation it means that $A(|a_1\rangle-|a_2\rangle)$ is obviously, equal to 0 because I take this to this side and I use linearity of the operation right.

So, linearity is very important in all this, but then $|a_1\rangle=|a_2\rangle$ because the kernel is 0; ker A = $|0\rangle$ i.e. $|a_1\rangle-|a_2\rangle=|0\rangle$ i.e. in usual language $|a_1\rangle=|a_2\rangle$. If the difference between two vectors is the $|0\rangle$ then the two vectors have to be same. So, the proposal that two distinct vectors are taken into 0, two distinct vectors have same image is not possible. Thus, the hypothesis of two different $|a_1\rangle$ and $|a_2\rangle$ having same image is contradictory. So, if the kernel is 0 then the map has to be one to one ok.

So, this we will use in an important way in proving Schur's lemma next time; that the kernel of a transformation is one to one or injective if and only if its kernel is 0.