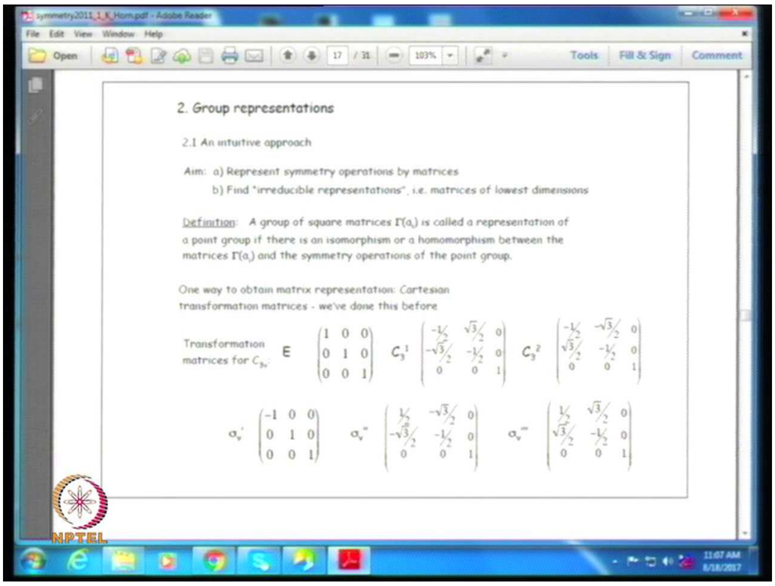
Theory of Group for Physics Applications

Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

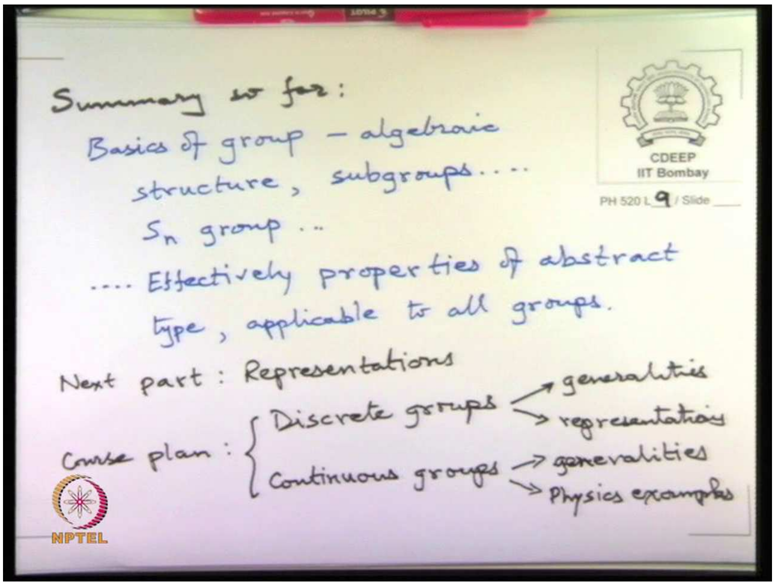
Lecture – 15
Representation Theory – I

So, far what we have done is that we covered basics of group structure, the algebraic structure subgroups etcetera.

(Refer Slide Time: 00:18)



(Refer Slide Time: 00:25)



And cycles, cosets you know and then the S_n group and Cayley's theorem, important theorems, normal subgroup. So, all this we have done for a group at a general level what we are going to do next is to study representations, because those are the things that are actually used. So, so far we can say that effectively abstract.

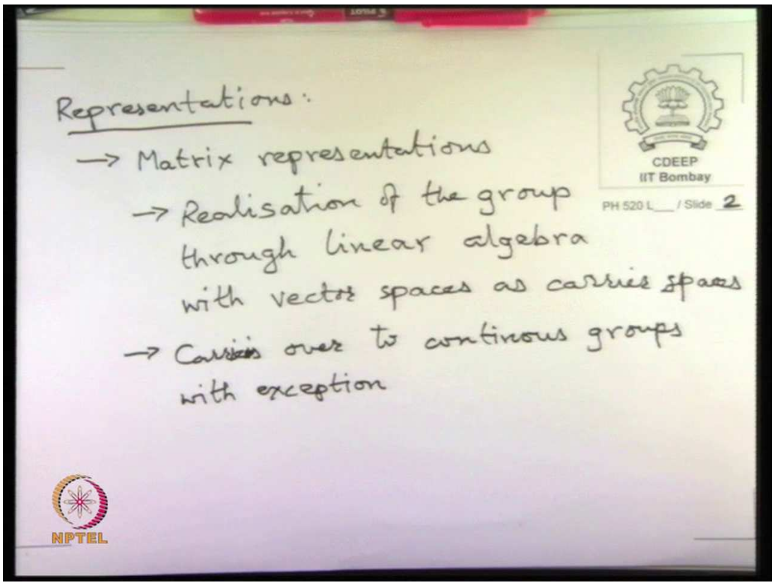
So, properties of abstract type applicable to all groups. So, now, we begin next with the idea of representations, and just to remind you we have divided this course into two roughly equal halves of discrete groups and continuous groups. So, far we are doing discrete groups. So, all the lot of the general things said about groups are will carry over for the continuous groups, but there are some essential differences.

So, we did the generalities and then we are going to do representations, and the next part after the mid sem approximately will be continuous groups, where again we will need generalities. So, I will say more general I should not say more generalities, but generality special to them.

So, this very basic properties will mostly apply except for a few exceptions, but there is the structure of the group as a continuous groups leads to some very interesting property, then again representations, but more through some physics examples. Here also in discrete group representations, there are physical examples, but it is difficult to actually cover any of them very seriously except for molecular vibrations.

So, we will take molecular vibrations, but the applications to lattice groups is rather involved. So, you really have to delve into some condensed matter systems to really see their applications ok.

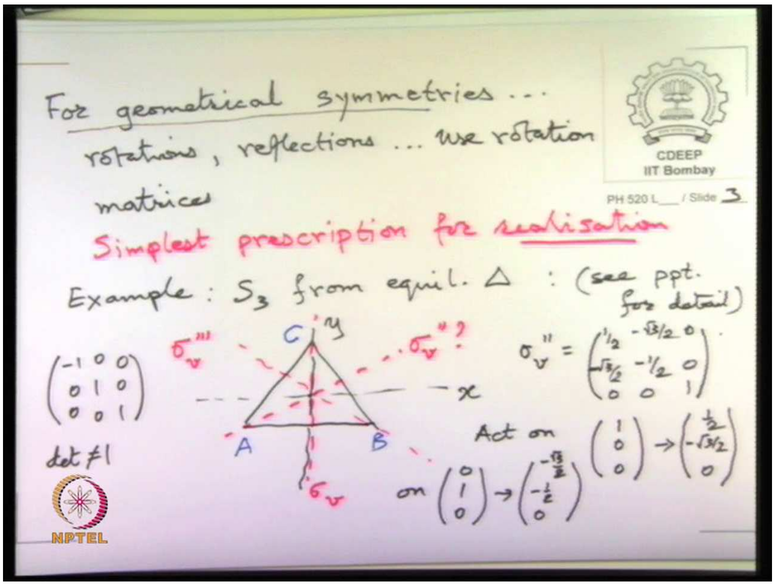
(Refer Slide Time: 04:21)



So, let us begin with this idea of representations and by this we mostly mean matrix representations and what; that means, intern is, in the terminology we have defined earlier; The realization of the group through linear algebra, with vector spaces as carrier spaces. So, the permutation groups were very powerful overarching groups for discrete groups, but that does not carry over to the continuous groups whereas, the this particular matrix representation idea carries over to continuous groups more or less with only the special things being different carries over to continuous groups case.

Again with some changes of due to continuity conditions ok, but the use of the cycle representation etcetera is going to go, because we no longer have S_n groups in the case of continuous groups ok. So, what is it about matrix representations? One is whenever we have a geometric quantity, we have a geometric system or a geometric symmetry, we can always represent rotations as matrices this is well known.

(Refer Slide Time: 07:05)



So where we have things like rotations and reflections, we use the rotation matrices.

So, this is the simplest way of and we will see that all the discrete groups can be described in this way as linear algebra operations, but when it is geometric it is the most obvious. So, the as a first example and since I have the slides from this I will put up the slides, all his affiliation is there so.

So here is the set of matrices remember the triangle, you can an equilateral triangle you can rotate it and you can also flip it along the its altitudes, you can represent this as matrices as 3D matrices. So, suppose you put it in the x y plane, put the triangle in x y plane all the operations you now do are not going to affect the z axis ok.

So, in this 3×3 representation of rotation matrices, the lowest corner is always 1 and lowest row and last column are just 001 ok. So, in all the 6 operations written out it is like this and this transparency of course, goes into some detail of how this has to be got. But the point is that we have firstly, of course, identity element which will be represented as identity matrix.

So, you can either make a geometric association I think in this case just think of geometric association. So, if you do not rotate the triangle at all then you should represent it by identity matrix, then we have rotations by 120 degrees and 240 degrees, for which I in a hurry drop thus minus signs last time, but this is the rotation matrix for 2 π/3 and 4π/3.

So, you can see there is -(1/2), -(1/2) and the whenever there is a real rotation, there is a $\sin\theta$ and $-\sin\theta$ it is always antisymmetric in terms of the signs and. So, clearly this it is in what a second quadrant, then it is in third quadrant. So, you get this minus sign.

So, this the top 3 represent just the usual rotations and it constitutes group C_3 you know the cyclic group of order 3, then of course, there are three possible reflections and the way they are captured are here. So, you can think of this as. So, this flips the x axis basically without touching the y axis. So, this is when the triangle is coming back here.

So, we have x and y and z is coming out, if I do not touch the. So, if I flip about this axis then basically x → - x. So, this one matrix would just send x → - x, but then there are two other matrices, which are flips about this and the other two altitude. So, one is this and then flip about this and then flip about this.

So, this become and let us try to identify by going back and forth a little bit. So, you can see that σ_v' is the x axis flip, now let us try to identify what this does. It is one way to think about it is, you compare it with say one of the these are the proper rotations. So, if you compare this and this, then basically the top row is its signs are changed whereas, the bottom row signs are not changed ok.

Of course if you will compare with this then you will find that the first row the first column signs are changed and second column signs are not change, but let us think in terms of the row; So, this is called σ_v'', and where this sign changes. So, can you think of what σ_v'' is, in our diagram yeah. So, this is σ_v and let me write down the σ_v'' matrix, which is

$$\sigma_v'' = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, one thing is to put here and labeling for convenience call this A, B and C. So, A B being exchanged is this, then if you exchange C and A is that what is σ_v'' one thing is to let it act on $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

So, if I have a vector in this direction, let us see where it goes under this operation so, acting on. So, what does it do? We will have $\begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$. So, it will send it into this direction, that does not tell us. So, we have to study both x and y and if we take y this is taken to ok.

So, in generic vector here the x coordinates remain within the first quadrant, but for the y coordinate it goes into this quadrant. So, that is yeah. So, we are trying to decide which flip it is, which of the flips. It is not a rotation matrix, but it is a flip it is an improper rotation right it is not orthogonal, but it is a transformation of the 3 points A B C into themselves and we are trying to identify, which it is it is a. So, we note that the matrix diag (-1,1,1) is also not orthogonal it is orthogonal, but determinant is $\neq 1$.

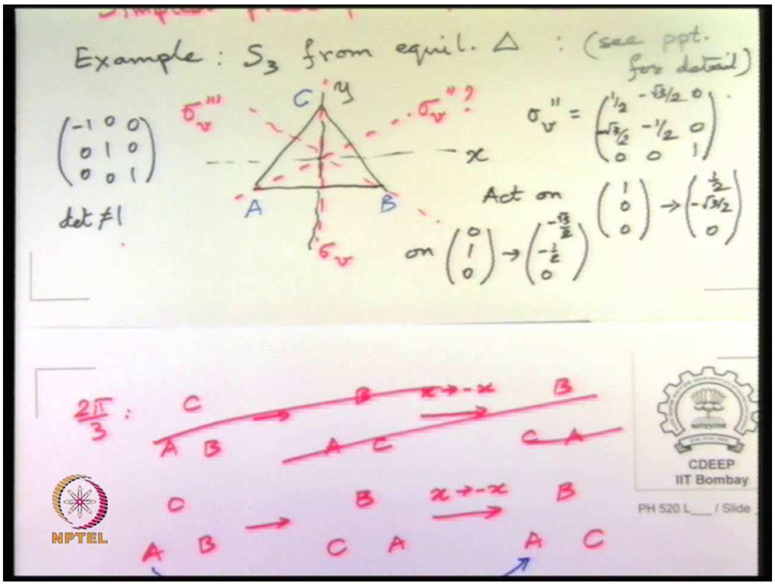
So, this kind of a transformation will be a rotation plus a flip, combined with it is an improper rotation let us see. So, σ_v'' has 1/2 and (-) this -2/3 right that there is a -1/3 is there. So, in this writing there is a minus sign here, this acting on this produces this and minus and acting on this it produces. So, this is correct in. So, in other words the question is what is the eigen vector of this, which is the vector this matrix does not a flip, it does not change. And so, try to figure out whether this is σ_v'' or this probably shown in this, but I just thought that we would think about it before going to it.

So, basically those are the operations yeah. So, is that now with this (-) sign. So, this is what we are saying right. I understand that you are saying this was not orthogonal, but with this sign or it will be orthogonal, but the determinant is not (-1). So, coming back to this basically the top row contains the proper rotations, and then the bottom row contains one key reflection which is a (-1) and actually the other two you will get out of this proper rotations by just a multiplication of this to them ok.

So, that is one way of thinking. So, this (-) sign multiplying yeah. So, first we perform the rotation and then if we multiply by this, we will change the signs of the top row that will give us this.

So, I claim that let us decide which is this rotation, because this rotation gives has a positive y, the y component remains positive right. If I have $\begin{bmatrix}1\\1\\0\end{bmatrix}$ let say 45 degree vector, it will make the x coordinate negative, but it will leave the y coordinate positive. So, it is still in the second quadrant. So, it is a rotation by 120 and this is a rotation by 240. So, if we perform a 120 rotation and then make the x axis flip what do we expect? So, you take this to this and then make flip about the x axis, that will have first sent A C on next page.

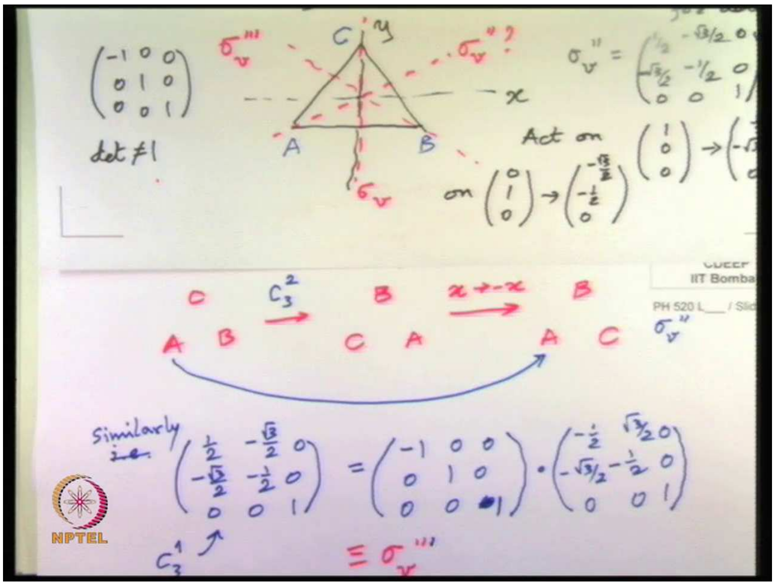
(Refer Slide Time: 19:25)



So, if we rotate this we send B over here, we send A over here A B C yeah. So, B goes there, A goes there and C goes there and then I do on x → - x flip, it will put this B here, A here and C here right.

So, the net effect is to leave A unchanged, A remain where it is and it flip B and C. So, this operation is σ_v'' because we did a sequence of.; so, we have identified some element like this and what we have just checked is that this is equal to; so, we are reading from right to left.

(Refer Slide Time: 20:14)



So, if we first do $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ a proper rotation.

So, our claim is that this matrix with the top row signs reversed, can be got by taking the proper rotation which is this. This is the first step this corresponds to the step rotation by 120 degrees counterclockwise, and then we followed up by x → - x, we can now look at the configuration in compare with the original, we say that the vertex A has not changed where as B and C are being flipped. So, that is what is σ_v''. So, this is equal to. So, in this way we can represent. This one clockwise ok. So, you figure it out.

Then of course, this will get exchanged with this. So, I will not spend more time on this here, because as I said there is input output problem, to look at it here draw it here think about it and tell you is not all easy for me. All I do is take a simple known vector know on a. So, my argument was my argument simply was looking at the signs here, it if I put a vector here $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ which is pointing in 45 degrees direction and apply this, it makes the first component oh it leaves first component positive. Because it is $(\sqrt{3}/2 - 1/2)$ and this is bigger, and then second row and that will make it definitely minus. So, it puts it in the lower in the fourth quadrant. So, it is actually then it is the other one good. So, we actually did find out.

So, this one this is not the one this is a rotation in the counterclockwise direction, which is the way we usually write the matrix C_3^1 written there is not that one. So, this happens with C_3^2 actually is going to produce this. So, I wrote here C_3^2 in that is C_3^2 has given here is going to do this, and this and that is what produces σ_v'' which is flipping C and B. If we start with C_3^1 it should give you σ_v''' and for this the corresponding pictures I do not want to draw too many, ABC was turned like this now it has to go like this and then flipped x → - x. So, we will then get σ_v'''.

So, by taking care of the directions of rotations and the flip, we can get all the matrices. Now in terms of matrices we get we get all the matrices are orthogonal; however, the lower three do not have determinant = +1 that determine their negative determinant this form the usual rotations, now we come back to the idea of representations. The point is that as far as these matrices are concerned, this one which was below goes for a ride nothing is happening to it its representing the z axis.

So, actually from the point of view of representation, we say that this is a reducible representation it has. So, this diag(1, 1, 1) is also a representation of the rotation group of this 3 fold rotations, but the trivial representation where every element is represented by the same element 1. You know 1 is always the realization of any group because when multiplied by itself any number of times gives itself. So, it's a many one man it is an unfaithful representation, but a valid one.