Theory of Group for Physics Applications
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Lecture – 14
Point Group Notation & Factor Group - II

And the other part of this PDF page is the interesting way that people are taught to decide very quickly on what symmetry group a molecular obeys. If you take a group then you may start wondering I have to identify $_v$ plane, I have to do this. So here they were given a flowchart. This is the sequence in which you will go, and you will you can quickly arrive at what is the symmetry group of the object you are looking at, start with does the object have a rotation axis, well it does and then it falls in to those trivial groups.

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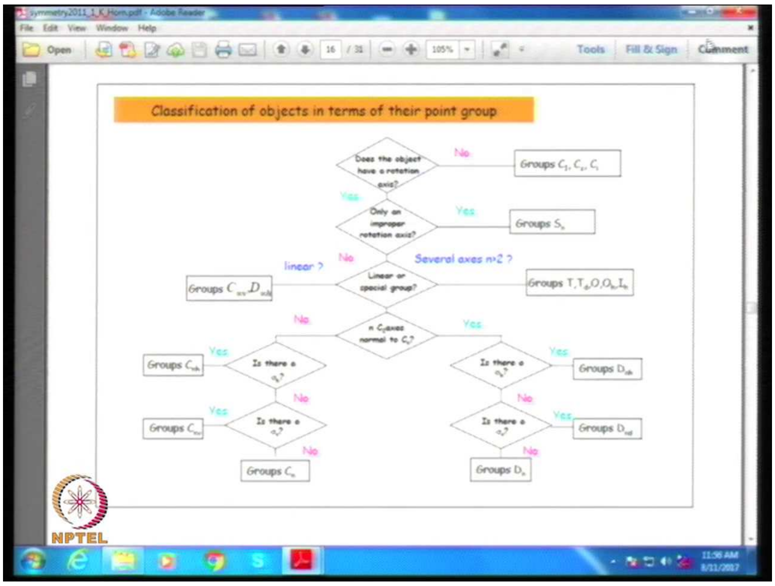

If it does well then look does it only have an improper rotation axis, so it may not have proper rotation you may have to have reflections then it becomes S_n group, but it does have regular rotation axis then you come to this and so on. So you go down the flowchart you get all the groups. So for example if you just went down this saying it has proper rotation axis is it a linear or special no let us say you have ammonia so it is not.

So you come down. Does it have n C_2 axis normal to C_n; well ammonia may not work, but then suppose if the answer is no then, are there any vertical planes containing axis; no, then you would get C_n. If you do not have any other symmetry elements you would have a plane C_n, if you have other ones then you will get shrunken into C_{nh}, C_n, D_n , so on. Similarly if it has perpendicular plane which is bisecting the which is perpendicular to the z axis then you will get the D_{nh} and D_{nd} etcetera and if there are no dihedral planes then you will come to plane D_n. So, this is an interesting way of thinking about how to quickly go down option chart tree to decide the symmetry of a molecule ok. So much for this you can continue to read this transparency, if you like it and second half so I think this was a D_3 example.

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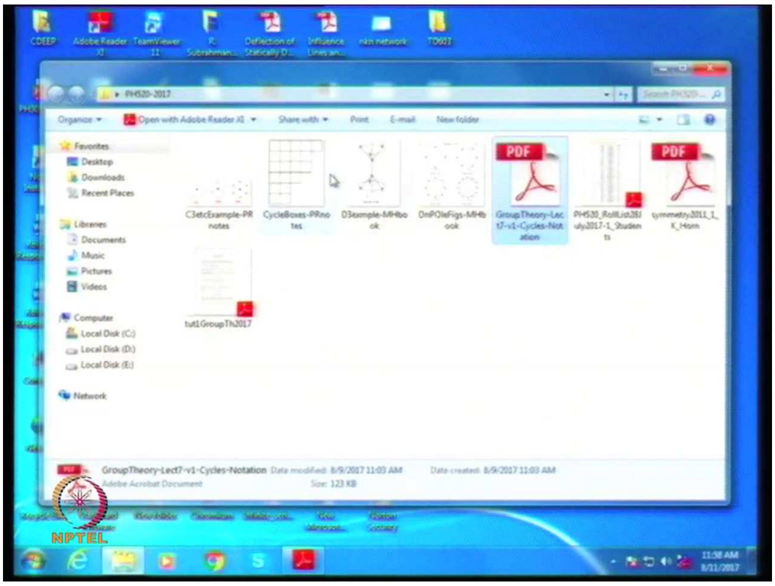

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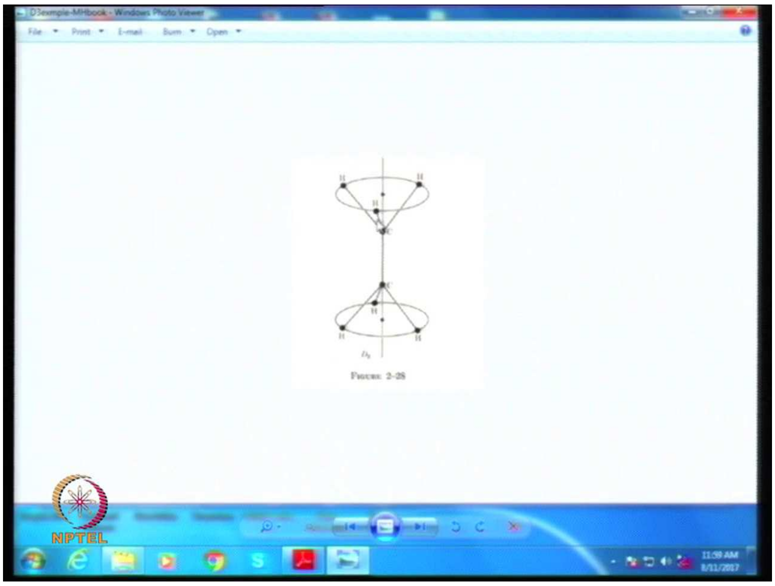

So this I was waving my hands about that this is C_2H_6, and this is the so called obscured ethanol because the 2 H's are just about each other. This is an example of a pure D_3 because there is a plane perpendicular to this which maps this in to this. So, there is this hydrogen which is let us say towards us in this ring, but then in the bottom one that the hydrogen is in the behind the, so this is actually the case of having 2 rotate and then reflect.

Let us go to the Otterbein and check quickly a few of the things then I will come what I want to do for completion is the proof of normal subgroup giving rise to a factor group. Ok so maybe we will do that first and then so actually we could as well since them you can always look up those molecules under animations yourself it is not so difficult.

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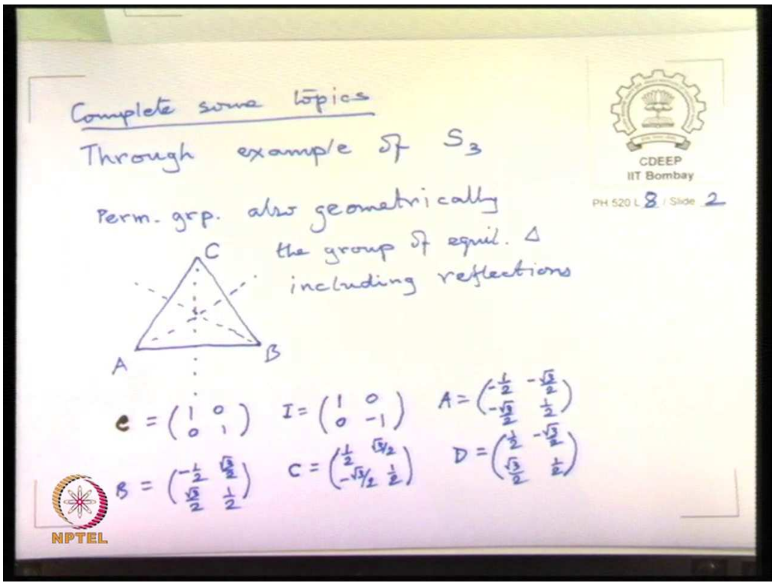

Let us go through one example in some detail which will illustrate what we have been doing. So S_3 is the smallest non trivial permutation group, permutation of 3 elements, but it is permutation group which is also the group of the triangle equilateral triangle including reflections including. So we have to have its geometric centre around and so if it designate this A, B and C then of course we have 2 /3 rotations and so there are 3 elements, but additionally we reflecting these planes.

So, though those reflections exchange A and C or exchange A and B at a time, by allowing all of these we have effectively allowed all the permutations of 3 objects ok, so the group is isomorphic to S_3. So let us write down its elements, we can write them as a matrix which sometimes comes out useful, but we can also remember them. So I will use a particular notation for this I mean particular way of listing these suppose we call e; e the identity. So let us call I to be the improper rotation or the flip so $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

And then write down this A and B a little bit later, and we write C and D. So this C and D we take to be our usual 2 /3 rotations, so they have cos(2 /3) and

$$C=\begin{bmatrix} 1/2 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1/2 \end{bmatrix} D=\begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix}$$

So this is 2 /3 rotation and this is 4 /3 rotation.

Now, this basically will show these matrices act on x-y axis and we are claiming that these are symmetry elements, but they have a correspondence in this. So I is the one where only one of them something gets flipped, it is an improper rotation. So now we combine these proper rotations, so it basically one way of thinking about it is we change the signs of these in 1 row, but not both.

This is what will happen if you say multiply by inverse of this which is (-1) here ok, and so there we change and here we will change the signs of this, I think you can use this. So here this C has a -1 in the upper corner and 1 lower corner so that we will do, and we can compare the proper rotations. So this is the proper rotation with minus sign, this is very strange we have writing. See for an orthogonal matrix you must have sin and-sin , there must be opposite signs here, but then I would have cos and cos under diagonal. So this is also not what I like, we will live with this you can check that this is correct. So we have this cos , cos on the diagonal and off diagonal we have minus signs and then we should actually have multiplied by this and that is the only thing I am not sure of probably I have multiplied by -1, you can check it later.

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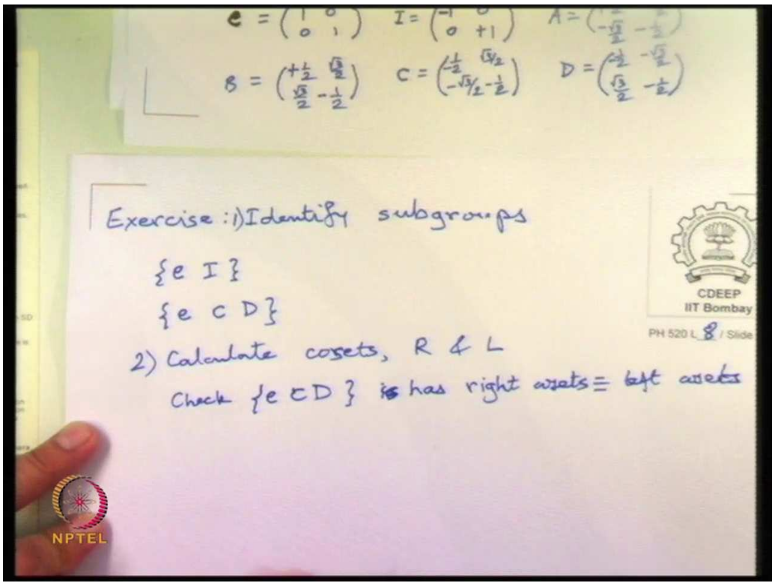

Now let me just tell you that if you start with this we can prove a few things quickly. So Exercise: Identify subgroups, so I think the only change I have to make here is put minus up here and make this plus then this is correct representation of the group. The other ones are equivalent sign changes, but this is this is fine, the one in the slides is fine, but it is not what physicist will write.

So this is fine because then you can see I, this inversion thing times C exactly reproduces A, so this row into this column use (-) signs to the top row and the lower rows do not change; similarly this into this gives (-) signs to these and does not change the lower one, so I think this is fine. So, this will this set of matrices will form a group. So, I think this is now correct, I am happy with that and actually then it tallies with this -1, -1 up yeah, now it tallies completely with horns way of writing.

Now, the thing is we identify the subgroups; one of them is just {e, I} and the other one is {e, C, D}, in fact I should have written capital e, but now you ask so, the as exercise I would say calculate the cosets, right and left ok. So, what would we get? So we need the multiplication table for that which you can put in the corner. So, this I meant to do as class exercise, but let me go and prove the theorem that we wanted to prove about normal subgroups.

So here it turns out that if you do this then check that {e, C, D} is a normal subgroup, so has right coset equal to left cosets. So, we will come back to this in a minute we will let us do the abstract part first which is a little easier to do then working out the example.

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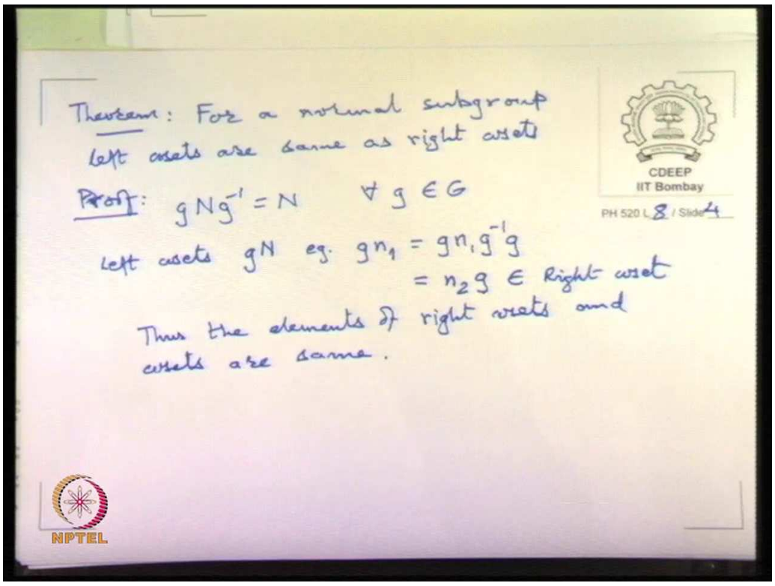

So, we want to check two things for a normal subgroup left cosets are same as right cosets ok. Now by definition by definition we mean by normal subgroup is meant that $gNg^{-1} = N$ for all N for all $g \in G$. So a left coset cosets are formed by gN. So consider an example where you say that this is gn_1, then,

$$gn_1 = gn_1 g^{-1} g = n_2 g$$

but this belongs to the right coset.

Right, if I start with some element gn_1, then it is also a member of the right coset formed by the same element gn. So, this left coset element is equal to some right coset; I mean the same element of n is obtained through a right coseting operation and left coseting operation with the same g. So, they basically make sure that all the left cosets will be right cosets. Thus the elements of right cosets and left cosets are same.

So, this is one property of normal subgroups this may not always happen. The next theorem we want to prove is that the coset space, G/N formed using a normal subgroup N itself carries a group structure.

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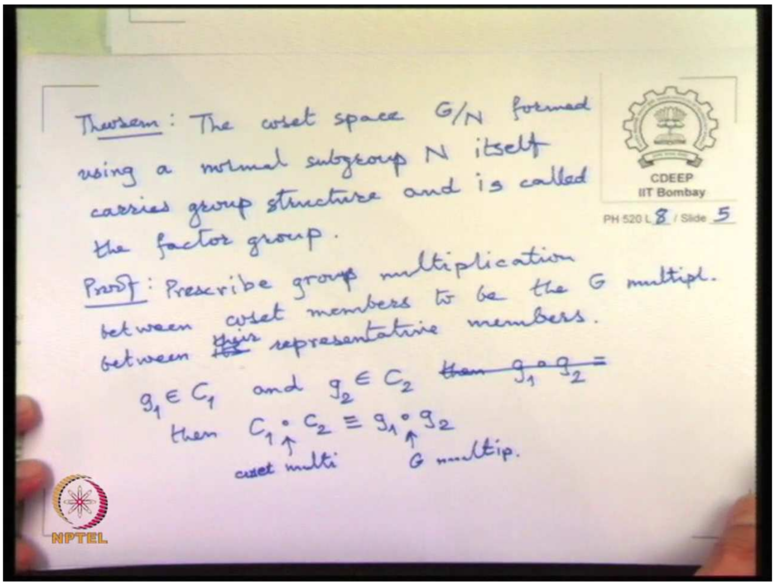

In other words if the group you are coseting with was just any ordinary subgroup then this will not work, but if it is a normal subgroup then this, so then if it was for a generic subgroup it would just remain coset space it will remain sub some space of equivalence classes, but when N is a proper subgroup then it carries a group structure carries group structure and is called factor group. So, here it is a matter of prescribing what you mean by the group multiplication correctly and then checking that it satisfies all the four requirements.

So, we prescribe group multiplication between coset members to be the G group multiplication between its representative members. Between there, so if I have a $g_1 \in C_1$, which is some coset 1 and $g_2 \in C_2$, then $g_1 \bullet g_2 = C_1 \bullet C_2$. So, this is coset multiplication or factor group multiplication and this is G multiplication. So we can see that the various axioms of group are satisfied.

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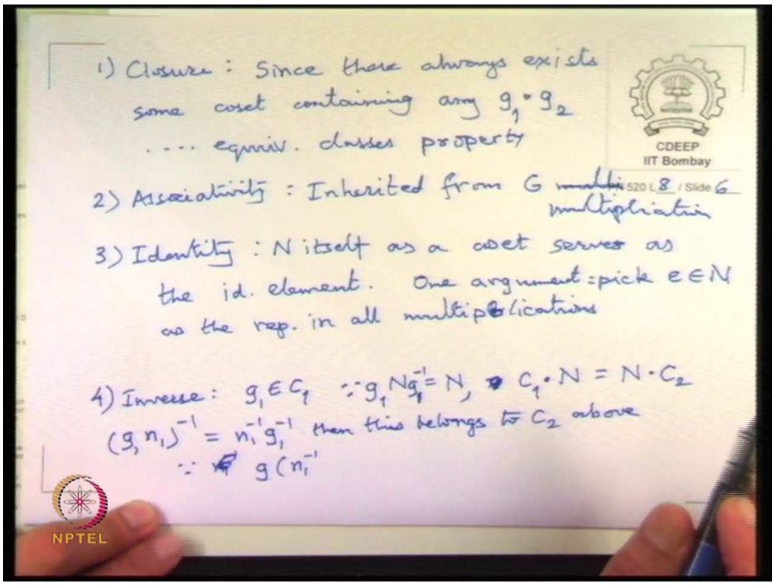

So, firstly (i) for closure clearly if I multiply these 2 I will get some so this has to do with equivalence classes exhaust the whole group. Therefore, if I take 2 representative g_1, g_2 and multiply them, the resulting multiplication should belong to some coset right. So, there always exist some coset containing any g_1, g_2 due to equivalence classes property, then (ii) associativity that is easy because it is directly inherited from G multiplication G multiplication.

Then (iii) we need the existence of an identity, the identity element of this coset space G/N, will turn out to be N itself; the entire normal subgroup N is essentially the identity element, as a coset serves as the identity element. Well one simple argument is to check is that that is the one equivalence class which contains the identity element. So anything to multiply by this particular coset you pick from the representatives, the identity all the time you will get back the same or coset.

But I think one also check that so one argument is to pick $e \in N$ because N is a subgroup so it has to contain e, so pick $e \in N$ as the representative in all multiplications. So I think that is certainly true, but I think you can use the normal we will do it in a minute if we have time and (iv) is inverse, this is where we need to use the normalcy a little seriously because suppose I have $g_1 \in C_1$ I have to now show that there exist some other coset which contains g^{-1}, So, that I get N. So consider the fact that C_1N is going to be N because N is the identity.

So, I will put because the other way round due to so because g_1N is N because the right cosets are itself the left cosets are N itself. So, you get that what we have is $g_1Ng_1^{-1} = N$ ok. So $C_1 \bullet N = N \bullet C_2$. What we have to show is that the so if I take g_1n_1, it is inverse is equal to $n_1^{-1}g_1^{-1}$ and if I have so there will be. So, this if I multiply it by any generic element of C_2 which is of this form because of this relation. So multiplying from the left by g, so what we wanted to show was that there are always exists a C_2 some other coset such that.

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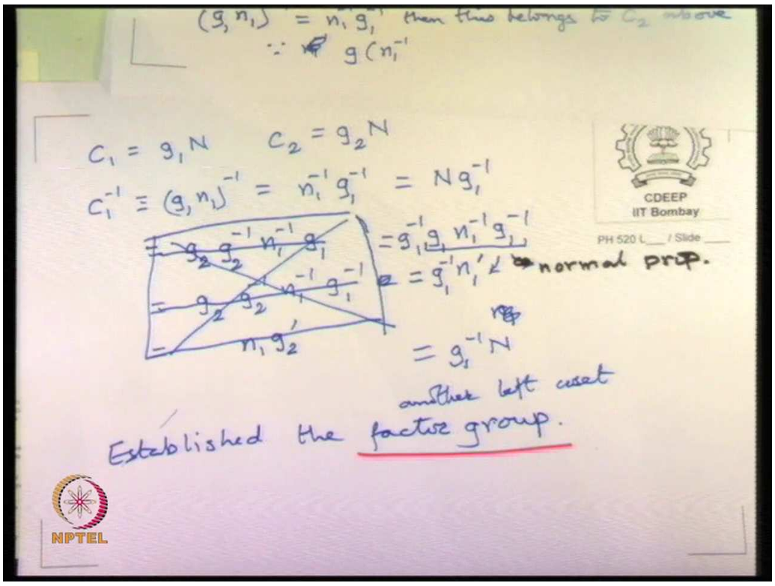

So, let $C_1 = g_1N$ and $C_2 = g_2N$. Then first C_1^{-1} consists of elements that are of the form $(g_1n_1)^{-1} = n_1^{-1}\,g_1^{-1}$, but $n_1^{-1} \in N$. So, this is of the same form as Ng_1^{-1} and so we have to convert this into a left coset, so I can write here g_1. So, we have to show that there is some g_2 that does that right, so we insert $g_2g_2^{-1}n_1^{-1}g_1^{-1}$ that does not help us directly, but we write, so we have to show that this becomes the an element of the N. So, if we are trying to show this if I have $n_1^{-1}g_1^{-1}$. I will now supply on this side $g_1^{-1}\,g_1$, then the $g_1^{-1}g_1n_1^{-1}g_1^{-1}$ is nothing but some other member of the set N.

And therefore, the inverse of C 1 consists of elements of the form g 1 inverse times N itself. So, the inverse of the I am sorry because of the things we are trying to balance concepts we are trying to balance you get it sort of repetitive what you are trying to do, but I think I can repeat it now, it has taken a few minutes more but so, given C 1 that this is of the form g 1 times N. The C 1 inverse will essentially be all the elements of the form g 1 inverse N.

Because the inverse of C_1 will as computed here, will lead to this which is another left coset. So, it shows that given any coset there exists another left coset such that it is the inverse in the usual sense where n acts as the identity. And we use the normal group property here from here to here is due to normal group. So, you get back to N itself there is some other n_1' which multiplying this will produce the coset which is the C^{-1} ok.

So, we have checked all the 4 properties of group and so this establishes what we call the factor group and this completes all the things that we were trying to do up to now which I have to do with cosets, equivalence classes and normal subgroups and we will now go on to representations and characters which are used for classifying the spectroscopy ok.

Thank you for bearing with it.