Theory of Group for Physics Applications
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Lecture - 12
Cycle Structures & Classification - II

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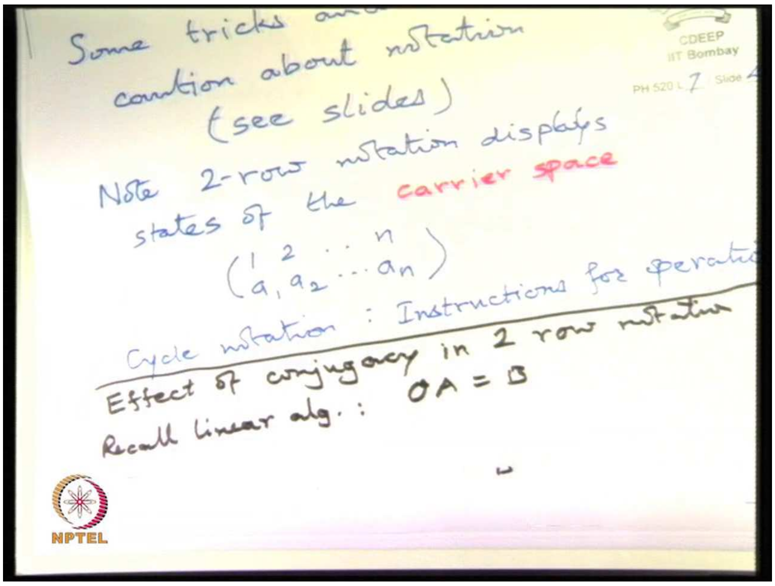

The next thing is what we started last time towards the end of the class. Now, we are moving towards getting sort of control of everything. One thing is about the possible conjugacy classes in S_n and as you know S_n is the big daddy group you know if you know everything about S_n and then you know a lot about all the subgroups as well.

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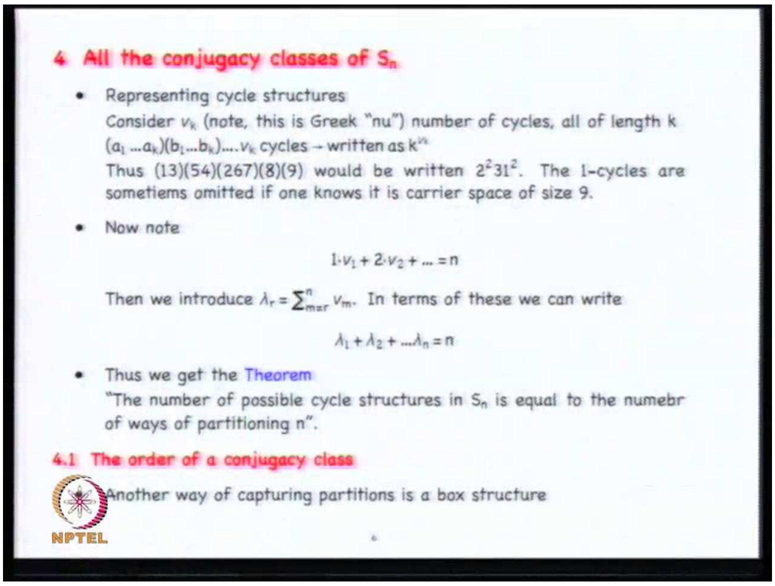

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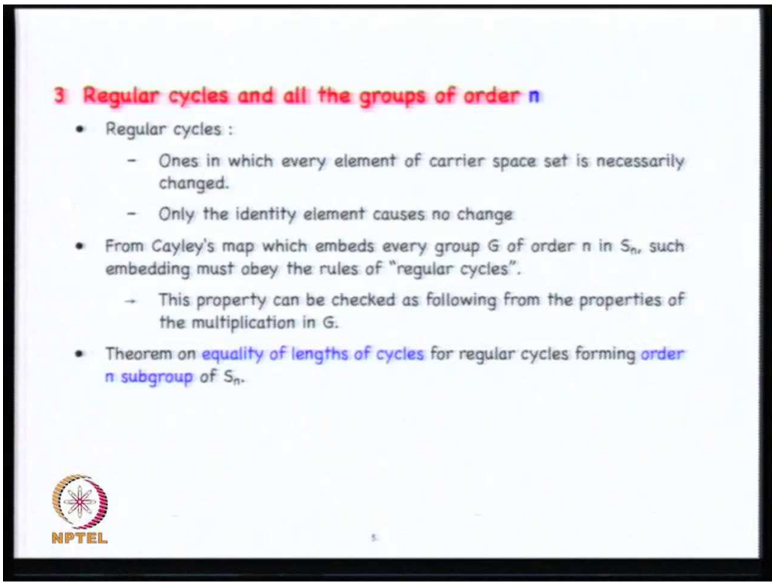

So, you did not see this transparency.

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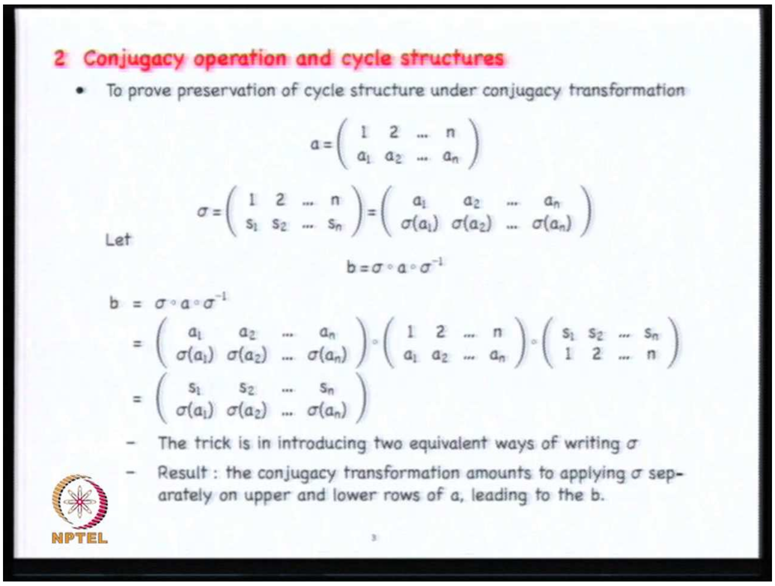

And we have the theorem about equality of lengths of cycles for regular cycles forming order n subgroup of S_n, ok.

Now, comes this statement we were trying to make last time. So, we want to study the conjugacy classes of S_n written as cycles generically we will have something like this $\begin{bmatrix} a_1 & a_2 & .. & .. & a_k \\ b_1 & b_2 & .. & .. & b_k \end{bmatrix}$. So, suppose there is a cycle of length k and suppose it occurs several times; there is one cycle of length k, then another cycle of length k, there are coming several times. Suppose this number is new subscript k these all going to get a little confusing, but not later once you read it again it will not be so bad. So, consider $_k$ which is the number of cycles all of length k, ok. This we note as that notation for it is k^{v_k}, and that is in a sense the total number there as well because its product of k $_k$ times. So, it is k^{v_k}.

Now, suppose you have a cycle like this in S_9, let us check that every element occurs in it 1 2 3 4 5 6 7 8 and 9, ok, but there is a 2 cycle, there is a 2 cycle, then there is a 3 cycle, then there is a 1 cycle and 1 cycle actually this is not how the educated books will write. They probably order it by putting 3 cycle first then 2 cycle and so on, but this how I have written it if you do it then in this notation that was suggested here what it means I have 2 cycle 2 of them. So, k is 2 and there is 2 of them. Then here k is 3 the cycle size is 3, but there is only one of it and then there are two 1 cycles, so 1^2.

So, the cycle structure of this is written out actually it should be written 3 2^2 1^2 and sometimes the 1 cycles are not written, that is when you know from context that the carrier space is already size 9 and if you then see it 2^2 and 3 and stops there and then you guess that, yes there are 2 cycles of size 1 which are not written. But, so this is writing one notation for writing out cycle structures in compact form. So, that is when it does not matter what numbers and what details these are, ultimately it only matters how many cycles of a given lengths there are in a particular element in the pi n the in a S_n.

Now, we note the following if I take 1 $_1$, $_1$ are all cycles of size 1, I multiply it by 1s. So, that counts all the elements that have fallen in cycles of size 1 when I take 2 $_2$, $_2$ is the total number of cycles of size 2 I multiply it by 2 I get all the elements of the carrier space that went into 2 cycles. If I do this sum up to k times in $_k$ I should recover all my balls in the carrier space, all the n elements of the carrier space. So, this summation equals n the total number of elements in the carrier space. Here you can see (2 2 + 3), so (4 + 3 + 2) is going to give back 9, ok. So, this is the fact.

Now, the way it is written it does not look very powerful, but it so clever people of course, set around and said instead introduce $_r$ it also trade with . So, here I should have added something in the notes, but I will add it later, but let me switch here.

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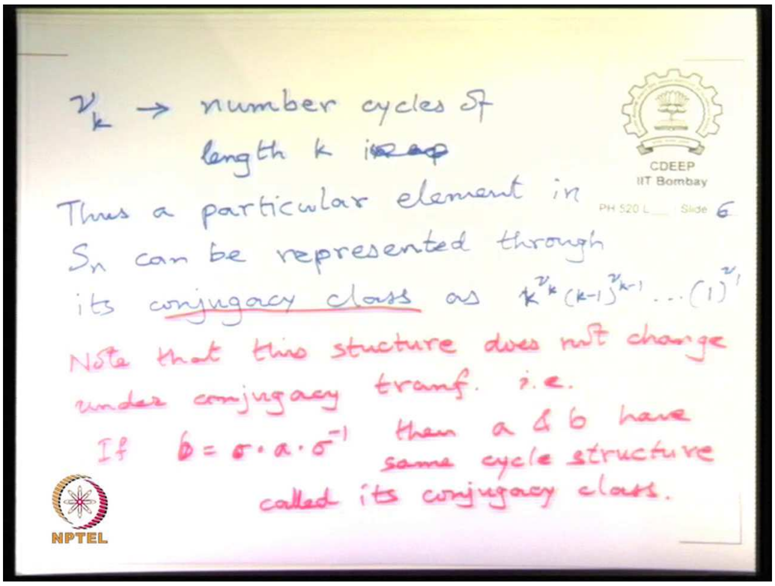

And sometimes this is called length of the cycle of length k in a particular length k, ok. Thus a particular element in S_n can be represented. This does not represent the does the plus of a particular, the conjugacy class, can be represented through its conjugacy class as $k^{\nu_k}.(k-1)^{\nu_{k-1}}$ etcetera, and times 1^{ν_1} which is some time omitted.

Alternatively, one statement that I do not know whether I have written earlier in the notes and did not say it; so the size of the conjugacy this particular, let me write over here. This structure does not change under conjugacy transformations i.e if $b = \sigma a \sigma^{-1}$ then a and b has same cycle structure and this is called its conjugacy class. So, that is why it is interesting to focus on this because this way of writing out the designation of the class is not going to change under conjugacy transformation conjugate transformation. So, conjugacy class remains the same under those under conjugacy transformation.

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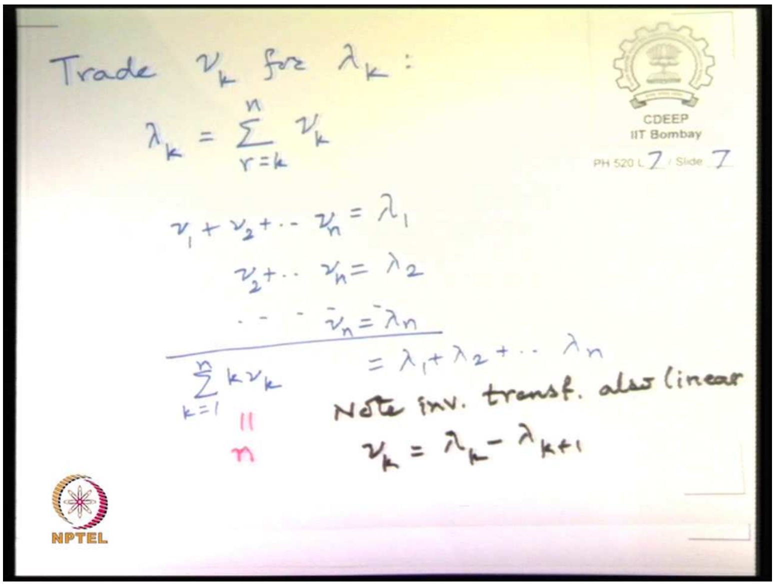

Next way instead of $_k$ we change notations to by defining $_k$ to be equal to sum from r equal to k this whatever k here is up to n of $_k$. Now, this may look a little silly why are we doing this is. So, what happens as a result of this is that $_1 = (_1 + _2 + \ldots\ldots.. + _n)$ right then $_2$ that from this instruction starts with r = 2 and goes upward.

So, $_2 = (_2 + \ldots\ldots.. + _n)$ and so on. So, that if I add all these then here I get $(_1 + _2 + \ldots\ldots.. + _n)$. So, last entry will be $_2 = _n$ right because if you ask for what is $_n$ the summation has to start with n. So, it just gives $_n$. So, this set of equations sums to giving sum of s than this side, but on this side it exactly produces k $_k$, right. Because you can visually see it has $_1$ once it has $_2$ twice it will have $_r$ r times and it will have

$_n$ n times. So, it exactly produces k $_k$ summation, but this summation we saw is equal to n.

Therefore coming back to the ($_1$+ $_2$ +……..+ $_n$) = n, now you will say what is the great achievement but one thing that has to be pointed out is that the transformation is reversible the s are defined in terms of you can always recover the s out of the s because this set of equations is linear and you can extract any particular $_r$ by just subtracting 2 successive s. So, if I subtract $_2$ from $_1$ I get $_1$ out. So, also note, so note the inverse transformation also linear which is simply that

$$\nu_k = \lambda_k - \lambda_{k-1}$$

because (k+1) is going to start with ($_k$+1) whereas, $_k$ would have started with $_k$ and then also have k+1 and all that. So, if you subtract you recover $_k$. So, it is a unique mapping back and forth without any loss of information.

So, coming back to this, so we can now write this as ($_1$+ $_2$ +……..+ $_n$), so there I was saying why would you change from one set of a symbols to another after all its a one to one mapping back and forth. But the point is now we can read this of as the Ramanujan would read and say these are partitions of n, given an integer these are its partitions. Therefore, the number of possible structures you can create in S_n is equal to the number of ways of partitioning n.

In number theory there is a now you learn number theory here, given a number n what is the all what are all the ways of partitioning it into smaller integers this is called number of partitions. So, let us see an example.

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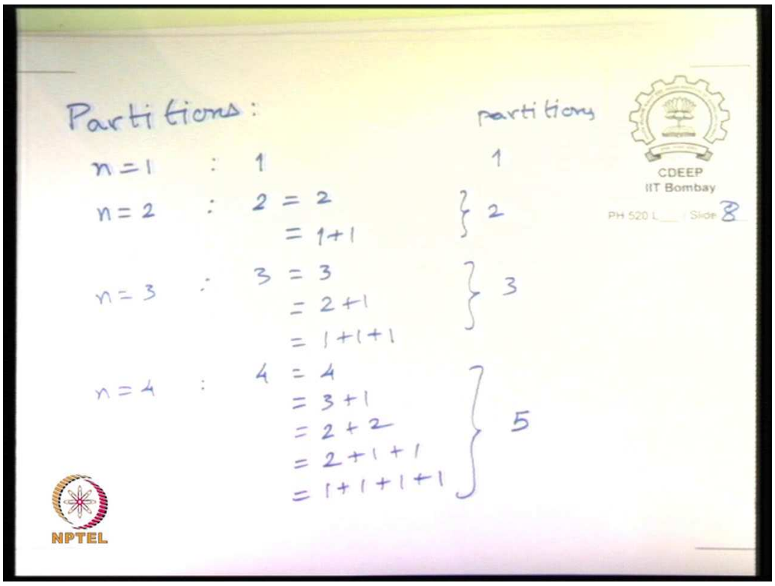

So, start with one if n = 1 then there is only one partition we can write it as 1. If n = 2 then 2 can be written as equal to 2 or can be written as 1 + 1. So, there are so number of partitions is one here, here it is 2.

Now, you have to count how many of these are n = 3, 3 equal to we can write of course, 3 itself then we can write 2 + 1 and then we can write 1 + 1 + 1. So, there are 3 ways of partitioning 3. Now, n = 4. But you can always write it out in terms of saying this is 1^3, this is 2^1, 1^1. So, can also be written as you would have written in cycle notation. So,

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$$

So, here we got 5 partitions.

The point is this enumeration cannot be generalized there is no formula which is why Ramanujan went at it with full force. So, he guessed all kinds of things about where the partitions were easy to calculate and finally, Hardy and Ramanujan proved a very difficult theorem which gave an asymptotic formula in the limit of large N what are the number of partitions you can have there is a hardy-Ramanujan formula. But right now just to get over boredom you can partition 6 in your note book what are the partitions of 6.

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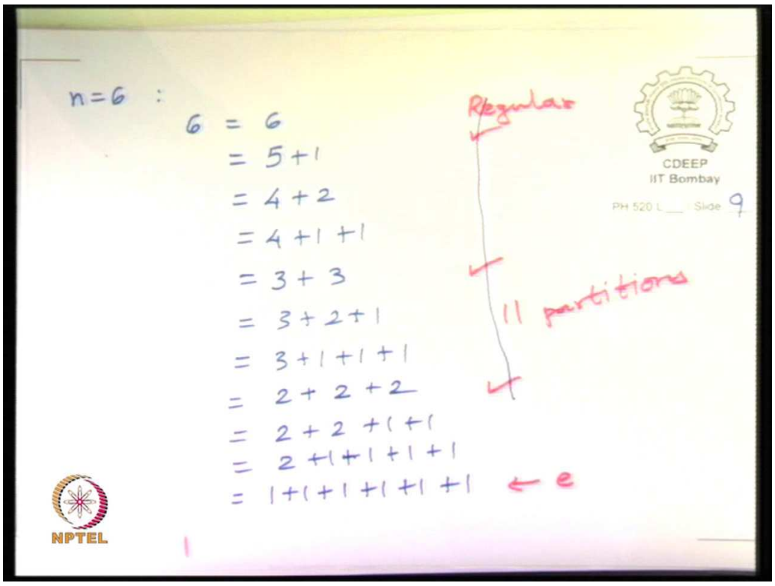

So, we get 11 partitions. So, this formula and now you have control about over S_n because you have all the possible cycle structures that can possibly occur in S_n, ok. And if we go to our 6 example then you can see that all the groups of size 6 that will get embedded in S_6 will necessarily have 6 cycle structure. Remember that they have to have n things listed and they can they are regular cycles. So, they have to have size 6.

They may be split up as they can have size 6 or they can have 3 + 3. So, we can highlight things that are regular here. So, 6 qualifies 3 + 3 qualifies because remember that to have cycles of equal size and 2 + 2 + 2 qualifies and then 1 is of course, identity element where nothing shifts, yes.

So, this is not this does not list the regular cycles here quite right, ok, but the point is you can work backwards to see how many of these partitions will qualify not these partitions, but after recovering s from them which are the partitions which are acceptable as regular cycles because they will be necessary of equal size and the subgroups that you form will only be formed out of those.

The n size groups will only with those. So, you have a lot of control over this and of course, any things that you are what the other thing I wanted to say from this was all those things that are our plus 1s are essentially members of S_{n-1} lower size S_n groups. For example, this 1 + 1 + 1 is essential in S_3 member. So, only 3 cycles are important other 3 elements are not being touched.

Again I am sorry I have to say if it was the new picture then that would be correct. So, lot of them will essentially be reproduction of smaller permutation groups and then the essential ones will come at where all the elements occurred the regular cycles occur, ok. So, the theorem as we said is the number of possible cycle structures in S_n is equal to the number of ways of partitioning n.

Now, there is a concept of order of a conjugacy class. So, you might ask given a particular partition let say this is this is the particular cycle structure I have which is actually abstractly written like this because the details do not matter, how many of such elements there are, how many elements in S_n as cycle structure 2^3 , 3 1^2. So, that is called order of the conjugacy. So, all of those will be in the same conjugacy class because they will have exactly same cycle structure and the number of elements in the conjugacy class is called order of the conjugacy class. So, I will now write a formula, but not really prove everything but two things one instead of writing this and instead of writing the summation that we wrote.

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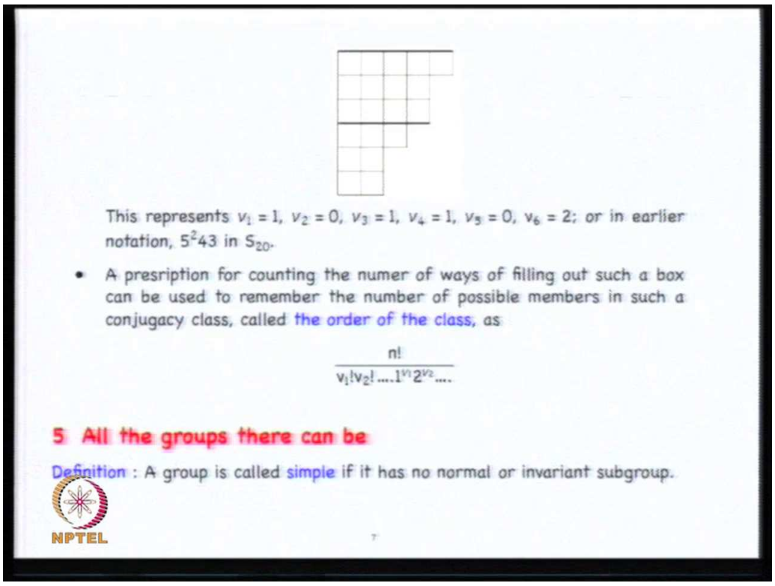

There is another way of representing the possible cycle structures. So, that is what the last line here says. Another way of capturing possible partitions is a box structure. So, suppose I have S_{20} , this 20 is chosen so that you can make drop large enough a figure what you do is you take 20 boxes there are 20 boxes here and arrange them in various possible orders. But starting with 1 over here smallest on this size and growing larger to the left.

This particular pattern that has been drawn reads as follows $_{1}$ the number of cycles of size 1 is 1, $_{2}$ the number of cycles of size 2 are none here 0, $_{3}$ is equal to 1. So, I can write 3 vertically like this, $_{4}$ again is 1. There are no cycles of size 5, but then there are finally, 2 cycles of size 6. So, the thing that would have been written as 6^2, $4^1 1$, 3^1 can be also written out as a set of boxes.

Now, what is the advantage of writing like these boxes is that actually by assigning numbers to them according to some rules which I do not want to enter right now these are called Young tableaux. But you can then calculate the order of a cycle the order of a conjugacy class how many elements of this type can occur in a conjugacy class this can be shown to be found on this, but you can also manual account what is the order of a class. So, a prescription for counting can be used to remember, but I can tell you directly what the order of the classes and you will I think believe me.

Well, what is the order of the class? First of all there are n elements and therefore, there are n! ways of arranging them. But when they have broken up into cycles of size length k and $_{k}$ of each type essentially in each of them you can permute the k elements that are in a k cycle and that does not change the cycle.

So, you divide out by $_{1}!$ $_{2}!$….. $_{k}!$ because within a things within a cycle do not matter. So, those are repetition, so we factor that out and finally, we also factor out by k^{ν_k} because that is the number of the; you have $_{k}$ boxes of size k those also could be written in any order it should not matter, so k^{ν_k} also. So, if you take n! and divide out by these factors then you get the number of independent elements you will have in this particular class. So, that is called order of the class, alright.

So, now I come to the more exciting part which is that with all this knowledge I am with all this knowledge mathematician's got very excited, and they said can we find out all the possible groups there can be because S_n gives you so much control. So, it is called classification. To go towards classification one first defines what is called as simple group, ok. So, a group is called simple if it has no normal or invariant sub group.

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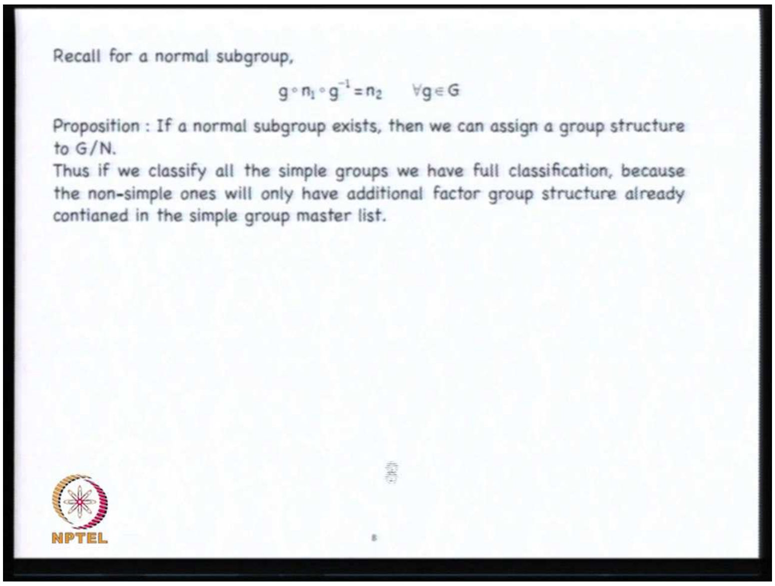

Now, you remember invariant subgroup is of this type. So, recall and normal or invariant subgroup is, I should have written capital N is such that if you do conjugacy transformations on it you keep getting back N itself, right. So, this is the definition of, so you can have a subgroup. But typically if you carry out a conjugacy transformation and you may get some other copy of and which may not even be a subgroup you just get another set of same size in the large group G. But if it happens that for all $g \in G$ any one member of the normal subgroup is transformed into itself then that is called a normal subgroup.

Now, one can see the following thing. If a normal subgroup exists like this then we can construct a factor group you can you can then construct a group to be G dividend into equivalence classes of N. So, well here I am really jumping through a few things I should have spent more time, but I want to get onto something that. So, we will come back and prove the theorem, ok.

If you have a normal subgroup then you can create a factor group such that G/N itself as a structure of a group. The equivalence classes of G under N the copies of N in G there will be all such. So, for that one has to define the inverse relation that is transformations of g elements under the operation of n then one can define. So, if $g_1 n = n g_2$ then g_1 and g_2 are supposed to be in same conjugacy class, right.

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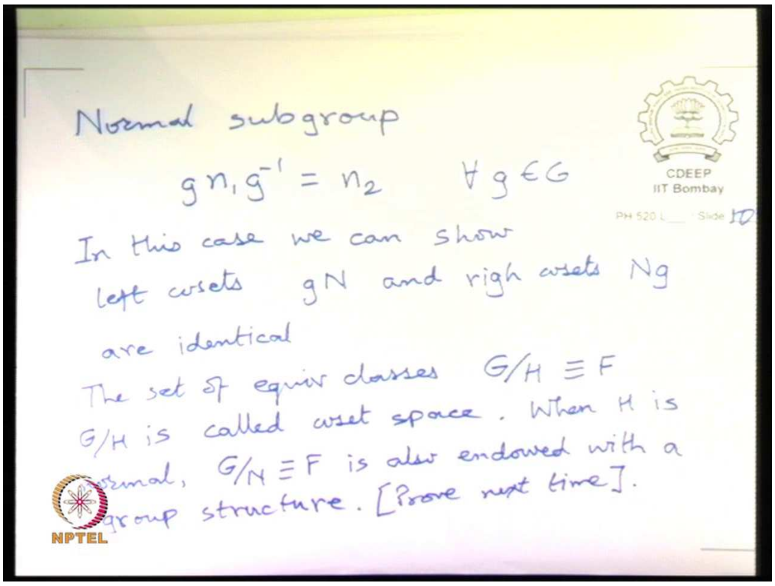

In this case we show that the left cosets, and remember the cosets cosets are constructed from subgroups by left coset is gH right cosets are Hg. So, left cosets and right cosets Ng are identical. I do remember that if you had a subgroup H if you hit it on the left with all the group elements then the subgroup gets mapped in to other copies. And these are called left cosets of it establishes a equivalence relation you have to say $g_2^{-1}g_1$ belongs to N the etcetera.

In the case when N is a normal subgroup it will turn out that the left and right cosets are identical and then we can construct this a set of equivalence classes G/H. So, here it is actually the usual slash, but here it has a different meaning it is the group theoretic or set theoretic slash which says all the set of all elements in G equivalent up to equivalence relation defined by H. So, this is called factor group the well first of all it is called coset space.

When, H is N, G/N = F is also endowed with a group structure. So, this will prove next time. If there is a normal subgroup then essentially we can form this factor groups, but there, but therefore, if you define a simple group is one that has no normal subgroup then that is a new unique kind of group, ok. So, thus if we classify all the simple groups then we have full classification because the non simple ones will only have additional factor group structure which is already contained in some previous simple group. So, the problem of understanding all the groups boils down to understanding only the ones that are simple that do not have a normal subgroup, if there then anyway you have a description for that.

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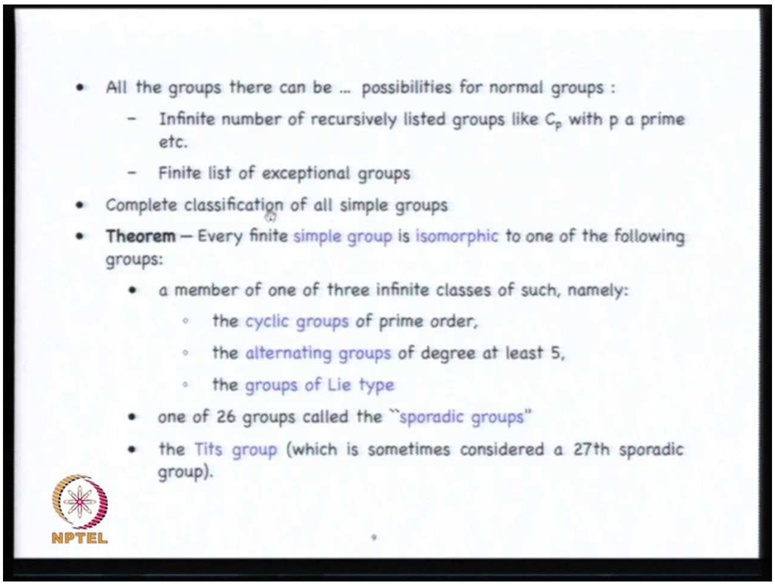

So, what are all the groups that can be? Turns out that here I have written some general thing it is a possibilities for a normal group are, first of all you can think of the obvious possibility if I have a just cyclic group, but of prime order then such a group never as a subgroup because if it is a prime order group and it is cyclic. So, it is a commutative it is just p-th of unity, where p is a prime such a group has no subgroup because it has no factors. So, you cannot make any smaller group out of it.

So, clearly the list is endless that much we have proved already. The list of possible finite groups discrete groups is endless because primes are endless. Everyone knows primes are endless you know proof of Euclid a proof is due to Euclid that the prime because all you do is multiply all the previous primes and add 1 to it, you obviously, cannot divide by any of the previous prime. So, it is a new prime.

So, you can always construct a bigger prime given any number of simple proof is due to Euclid. So, if you have a p-th root of unity those elements; obviously, form a new group. So, the list is certainly endless and the complete classification is like this, every finite group is a isomorphic to one of the following. So, this is actually cut and pasted from Wikipedia, you can read in Wikipedia which is good enough for most of these things.

The member of one of the three infinite classes of such namely either cyclic groups of prime order or alternating groups of degree at least 5. So, these are the A_n groups that that is another category and is at least 5 is interesting because this is what Gelmann actually manage to actually prove that is why polynomials of degree 5 and higher do not have explicit closed form solutions. So, the fact that 5 and higher alternatively proved that that is what enters into trying to get solutions in the complex plane, ok. And then there is something called group of Lie type there is some list finally, what remained was some 26 kind of groups called Sporadic groups and there is something called Tits group which was something very unusual you can consider it 27 unusual groups. So, these are set of groups which are sort of long list infinite list, but then there is a set of bunch of unusual things that are called Sporadic groups that makes up everything.

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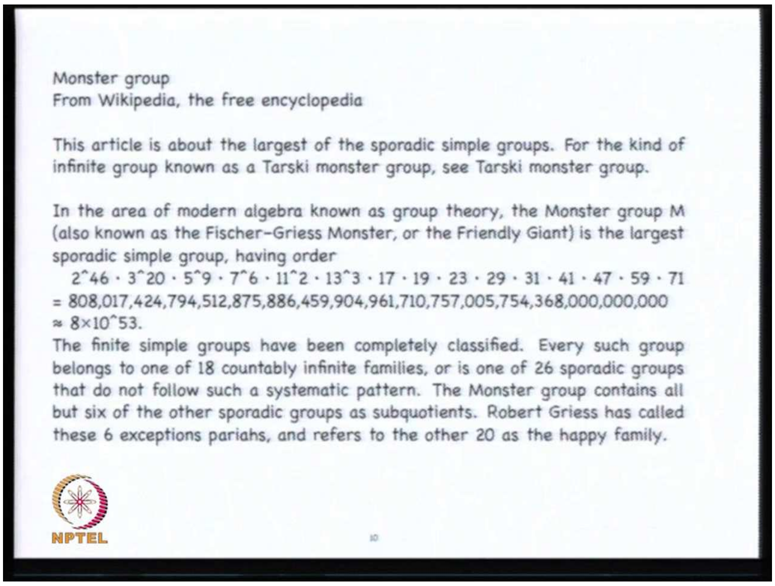

And the biggest group is called the Monster group. So, the largest sporadic group of simple type called Tarski Monster group. The area of modern algebra monster group m is the largest Sporadic group having it the order of the group is, this is the order of the group which is approximated as 10^{53}, but that prove that this is the largest possible Sporadic group you can help there is nothing more.

This exploration was done by using a computer, ok. So, in the 90s, mid 90s and if you read this article in Wikipedia it is worth reading because it says that it took almost 200 years of continuously working not 200, but whatever can be because Gelmann is 18 something right so, but 100 plus years of this guess that it should be possible to classify and the program to prove started and various people proved various pieces of the thing. In the end in 90s with the arrival of first computers some smart guys said the remaining thing we can figure out by writing an algorithm and then they cracked it and showed that there is no bigger size simple group left. This shocked many mathematician's because they thought that mathematics was pure thought and if a machine could find it then it kind of was an insult to them, but they have learn to live with it now.

So, finite simple groups have been completely classified each such group belongs to one of the 18 countable infinite families, one of the 26 Sporadic groups that do not follow such a systematic pattern. The monster group contains all, but 6 of the sporadic groups as its as sub portions Robert Griess has called the 6 exceptions pariahs and refers to the other 20 as the happy family, ok. So, this is the status of group theory at present it is quiet good to see that these exist. We are at the end of the class, but I will leave you with what we will do next time.

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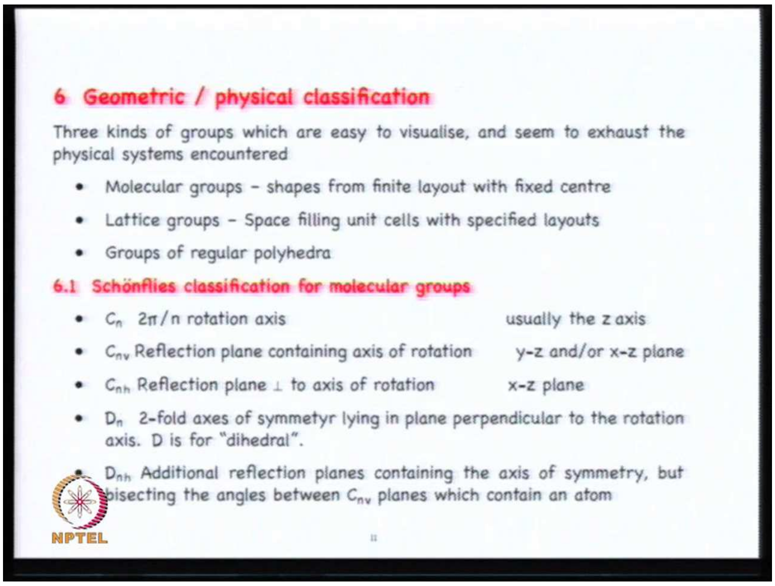

We are not going we are not interest in the mathematicians classification, but we are interested in the geometric or physical classification, and from the point of view of physics there are again 3 broad classes of groups.

There are molecular groups which chemists use where there is essentially a stand alone thing. So, these are all these are called point group these 2 classes of groups. There is a shape a molecule of some kind or there are lattice groups which are space filling unit cells with each unit cell having a specified layout that layout will be similar to molecular group, but then there is a repetitional there are shift operators as well and then there are groups of regular polyhedra. Because, but this exhaust pretty much what we would be interested in from physics point of view.

And there is a nomenclature that list all of them. We will only do the ones that chemistry the first type molecular groups and its classification the others become too involved nomenclature becomes involved and proving its utility. So, the utility of the lattice groups is essential in X-ray spectroscopy which most of the EP students will learn I mean in both EP and M.Sc students will probably learn.

So, we will not actually going and their group theory per said does not play that much role it is the geometric structure. But for molecular groups we will see that there is some interesting result from group theory that applies to it.