Theory of Group for Physics Applications

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Lecture - 01
Introduction

Let me begin by saying something about what is group theory.

I think most of you have undergone courses in linear algebra, matrix matrix in them so on. So, we will be heavily relying on those matrix. So, you have to know matrices very well. But that is essentially all the math you really know to begin with. And there is a lot of conceptual component of group theory and that is what we will try to cover as well in addition to all the methods.

So, to begin with the origins of group theory are in two very broad areas of interest. One had to know permutations, and this gives rise to discrete group as we will see you have some n objects and you permute them, then each of the operations of permuting can be considered as an algebraic element in a set and then you can combine two of these because two permutation gives rise to a new permutation. So, there is an algebra obeying by permutations. And that gave rise to the most essential kind of group and these are called discrete groups. And as we know the study of quantum mechanics I mean molecules and lattices, it relies heavily on the shape of the molecule.

So, when molecule has some symmetries like ammonia molecule has 3 nitrogens. So, you can rotate the nitrogens into each other and nothing changes. So, all the physics that governs ammonia molecule incorporate those symmetry, all the transitions that can happen in an ammonia molecule they also automatically inherit the symmetry that is in the molecule itself.

So, in quantum mechanics of molecules and as well as lattices where again it is discrete because depending on the lattice geometry you can rotate the (Refer Time: 02:26) in discrete number of ways. So, in this form in the form of discrete groups group theory enters quantum mechanics.

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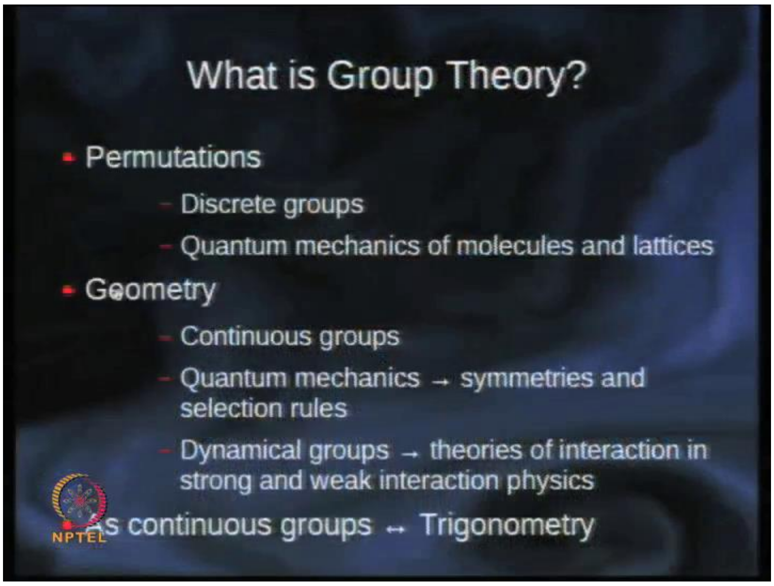

And the second origin of groups is in geometry and that is in some sense much richer because it has to do with rotations first of all. You must have been thought very early when you learnt rotations, that rotations in 3 dimensions do not commute you rotate one way and then rotate around another axis then you, but if you instead rotated about second axis first and then try to rotate about first you do not get back the same rotation. Only if it is rotation about a fixed axis do the rotations commute, but if you do 3D rotations then you do not operations do not commute.

So, that is our most commonly encountered example of groups because we are doing rotations all the time, and that is why I have added this last line that as continuous groups in the form especially of the rotation group it is essentially like a generalization of trigonometry. And I think everyone should know these essential aspects of rotations. It is just that we cannot introduce them any earlier because it does require knowing matrices and that is why it has to be postponed in studies, but it is something once you get used to it is as natural and common as you think in terms of trigonometry, when you think of angles and directions and distances and so on.

So, continuous groups are extremely important and very useful in daily life, but they also enter in quantum mechanics again also in transitions of atoms and molecules, but this time through the rotation group, through the angular momentum algebra. And through the fact that particles carry spins photon has spin one and electronics spin half and so those rotational symmetry is mixed with various symmetries of the molecules and lattices and therefore, the continuous groups also enter quantum mechanics in a slightly different way the calculations remain moral as the same its the quantum mechanical calculations, but they give rise to somewhat different features in quantum mechanics then those that we get simply from the size and shape.

So in fact, we can say at this point since I mention trigonometry that pretty much when you go to quantum mechanics where things are reasonably intangible you cannot really feel or size of any molecule or atom individually, it is the symmetric group of that molecule or that lattice that really substitutes for what we call shape. So, the geometry of shape and size of things is more or less completely subsumed by group theory, the group symmetry group of the objects you are trying to study. So, in that sense it is a very very important aspect of quantum mechanics you have to give up the conventional picture you still hold most of the time even in chemistry everywhere you try to think of some kind of a fuzzy ball or pyramid and so on. But eventually it is the group theory that is your secure handle on how quantum mechanical objects obey behave.

And then in the last one century we have learnt something very very surprising and very precious which is that group theory, group symmetries the abstract groups also determine the kind of particle interactions we have including electromagnetism. So, electromagnetism as well as the weak nuclear force and strong nuclear force their interactions not just their shapes, the interactions that the particles entering into those theories have they are also determined by mathematical theory of groups. So, hopefully in the last part of the course that is where that is what will try to cover. So, again to highlight it is essentially almost as if you are giving up your traditional view of geometry and shapes and sizes and the mathematics of groups takes over when you go to quantum mechanics. Of course, even in classical systems which have symmetries will have some applications of group theory, but in the advanced physics this is what is really going to be relevant. ok.

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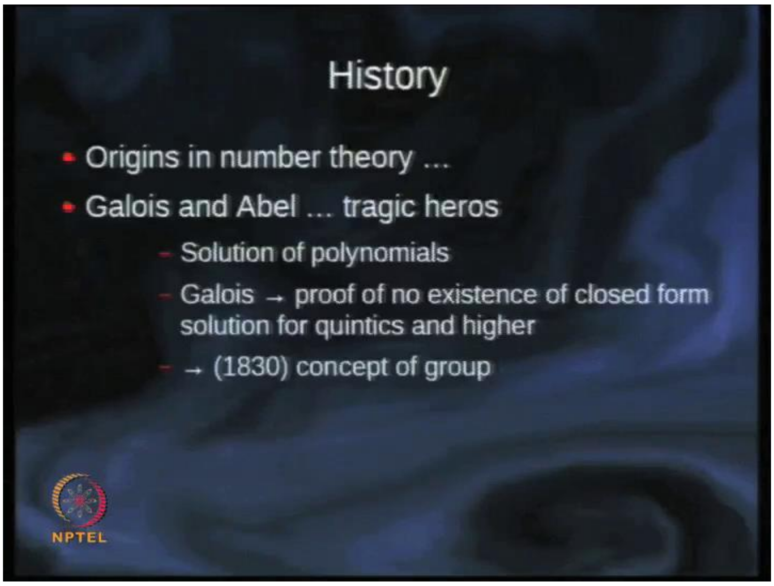

So, a little bit about history you can read Wikipedia. Lot of people try to question against reading Wikipedia I said that if you are a discerning reader if you generally are clever enough to figure things out Wikipedia is a great source because the to the extent that there may be limit some information may be limited or garbled you can usually sense it ok, so you can filter those things out. But the kind of information it has, and the quick the quickness with which you get it, and the depth to which you get it because of the love of labour of thousands of contributors to Wikipedia I think it is a great source of most things would like to know.

And so you can read the Wikipedia page for history of group theory, but they say origins are in number theory Euler and others which I do not know much about. But there are somewhat colorful stories of these two mathematicians from 19th century Evariste Galois and Henrik Abel unfortunately both of them died early Abel in his late 30s and Galois in late 20s. Abel I think out of illness, but Galois out of more colorful thing of some quarrel over love. So, he got into some duel and then got killed, but before he died he was furiously writing because he knew he was not strong contender to survive in a dual, he was writing up whatever he had to say about a solution of a quintic the fifth order.

So, from the Greek times when you a close, so if you are given a polynomial of second degree quadratic equation. If you know the coefficients a b c you can write the general solutions the two roots that you get. About cubics also such formulae exists, but higher than cubic nobody knew. So, it turns out that after much struggle they knew the formula for the quadratic polynomials, the solution the 4 solutions of a quadratic polynomial, but Galois set out to prove that in general as a matter of principle it was never going to be possible to find the solutions in close forms in terms of just the given coefficients of the 5 roots of a quintic or all the higher order polynomials. So, it is somewhat like trisecting a general angle you cannot do it by simple rules of geometric construction.

Similarly here roved that as a matter of principal it was not going to be possible to find the solutions of quintic and higher order polynomials. So, that was his great contribution and this was around 1830s it turns out that he was also the first one to emphasize that there is a structure of a group you know the precursor did not quite emphasize the an algebraic structure to the group, but apparently Galois was the first to emphasize that there is there is such a structure. But I think it was almost 30 or 40 years after his death that other mathematician who translated who rewrote his work and re-publicized his work could understand what you was actually trying to say.

So, started then by late 1800s it had become a very important tool and all the grades of 90th century mathematics (Refer Time: 11:31), Wirestress, Philips Klime mostly who all use groups in various different ways. And it was Philips Klime who enunciated it like a you know like a religion that everything has to be explained in terms of groups, all mathematics was going to be recast in the form of statement group statements which was also very ambitious, but that is how history goes. Interestingly enough I should have added here itself it was only around the late 1800s and more or less by the year nineteen hundred that man called Sophus Lie whose spelling is like a Lie, Lie. He was trying to see if differential equations can be solved by the symmetries.

So, you know that you can picturize certain equations as if they are on some manifold you know. So, if you think that something is moving on a sphere then the fact that the geometrical it is a sphere will simplify your thinking about it. So, Lie was out to work out a way that given any differential operator try to first find out the underlying geometric space its symmetries and then maybe all the solutions will be easy to find out. I do not know somehow that particular goal has not been achieved, but in the processee classified what we call classification we characterized what a continuous group is all about. So, Lie groups which are continuous groups as against discrete groups, groups that are parameterized by a continuous values of parameters they became very important after the year 1900s and advent of quantum mechanics suddenly saw it flower as a great method for physicists.

So, you will be surprised, but physicists were very illiterate mathematically until after until 20th century. I guess the only thing they really used extensively was calculus and differential equations because that is what essentially Newton had solved dynamics using that and the motions of planets, and motion of mechanisms they all use differential equations. But all them more advanced conceptual aspects of mathematics (Refer Time: 14:20) to physicists, including vector calculus you know the concept of a vector the compact notation for vector groups even matrix algebra. You will be surprised to know, but when Heisenberg formulated the first principles of quantum mechanics he did not know matrices. So, he wrote out his formulae as a_{ij} , b_{jk} and so on, and when Max born his teacher saw it he said oh I remember some mathematicians giving something talking something like this in a seminar and so then they went and then dug up the thing and then they understood that it is really matrix algebra.

So, as late as 1920s the pioneers of quantum mechanics did not know matrix algebra. So, you and me are much luckier. So, it is using this that one really that once these techniques entered physics it was much easier to import the more advanced mathematical methods. And it was a using Wigner who was the pioneer of introducing group theory methods in quantum mechanics, and then writing extensively systematic books that exposed group theory to physicist. In fact, I think it was Wigner who said that the applicability of groups essentially some arcane things that mathematicians were doing suddenly becoming so applicable to physics was very unreasonable applicability of mathematics to physics, but all it means is that mathematician at zero d non concepts that were actually quite relevant to understanding physics.

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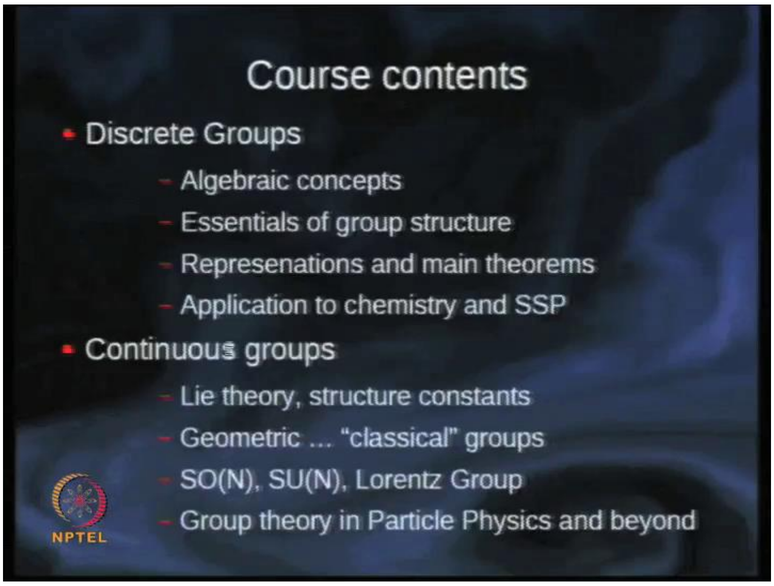

We will as I have been saying split the course roughly into two more or less independent parts, first half up to semester will essentially be dealing with discrete groups and in second half with continuous groups. We will begin with algebraic concepts some of which you may know like vector spaces, fields, rings and so on, and then the main parts of group theory, then representation theory is the important part, and the main theorems that enter the discrete group theory that go by Schure and others names of sure and then some applications to chemistry and solid state physics this is my hope I am not completely sure that we will do it in any detail, but certainly will try to cover some of it.

And then we will move on to continuous groups where we do Lie group theory a core concept of lie group theories generators I should have added here generators and structure constants. And then we cover the various classical groups which are SO(N), SU(N) which you will notation which you will understand very soon it is very simple actually. It just means orthogonal matrices of size n and special just means determinant = 1 and similarly these are complex their unitary matrices of size n always square and the determinant= +1 and Lorentz group which is the group of special relativity it is neither of these not a simple as this, but it is quite similar. And then I will try to cover some applications to particle physics and some of the modern concepts where it is getting used.

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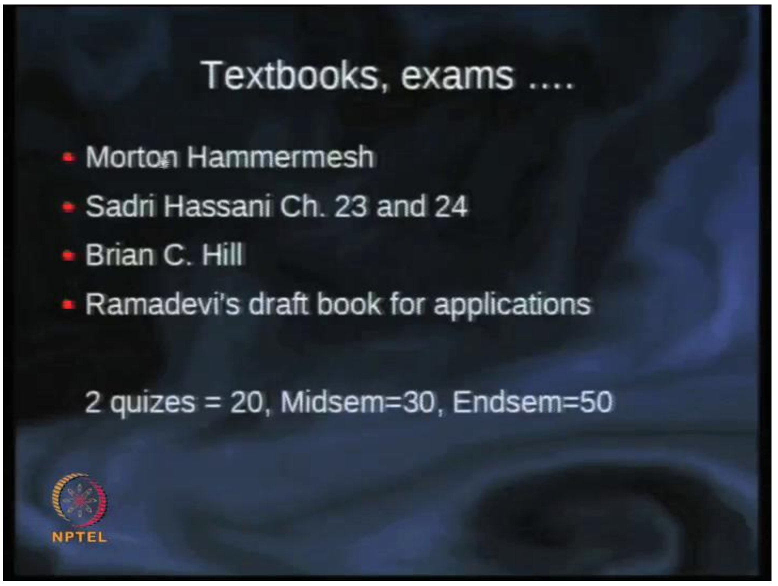

The text books there is a book by Morton Hammermesh. I personally find this book very difficult to use because it is somewhat verbals and it is given as a discourse not sectioned up properly. The opposite extreme is of course, the mathematical language where the just lemmas and theorems that is also very difficult to read, but Morton Hammermesh completely the opposite extreme of just huge discourses which wind through very specific and special things and very general things are all interspersion mixed up and find it difficult to read, but the book has everything. So, and explain very nicely. If you read that specific topic there by itself it is very clear what is going on with examples and everything. So, in that sense it is a good book.

There is a book called mathematical methods, it is like one of the standard mathematical physics books there is one very common one which I do not want to name which is very very badly written, but this book has all mathematical physics you know like the M.sc course mathematical physics part 1 and part 2 they are all covered here.

Sadri Hassani has two books, so I think the more advanced book he wrote first it is a very beautiful book, if you its Springer has an Indian edition for about 700 rupees it is a fat book if you can get it its worth keeping. So, you can say it is in the class of crisic, but its more advanced than crising and more oriented to physics. It has it has some chapters on group theory I i think there are more chapters maybe up to 26 or so. So two chapters on discrete groups and to one continuous groups I think the book is extremely well written and very compact with histories of personalities and so on. So, I very strongly recommend this ok.

Then there is a mathematics book Springer book by Brian C Hall, this book has some title like group theory and representation theory. This book is a very good mix between mathematics and physics it does not do any physics applications, but it is a mathematics book written such that physicist can read it. So, it has all the mathematical riger about it is systematic and it is readable to physicist, but it has much more than you will want to know for physics. So, that is the; it is a bit expensive.

There are of course, some very expensive books written by physicists as well, but I will not mention them here to clutter things. So, these 3 are the primary sources I will be using. Additionally Professor Ramadevi she has taught this course before and her lectures are already in the form of a book the book is yet to be published, but she has much better coverage of the applications to molecular physics and solid state physics and with her permission I may share some lectures out of her book.

So, we will be using this, ok. So much for the preamble. So, I think we will now go to main aspect of group theory I will start the course itself, ok.

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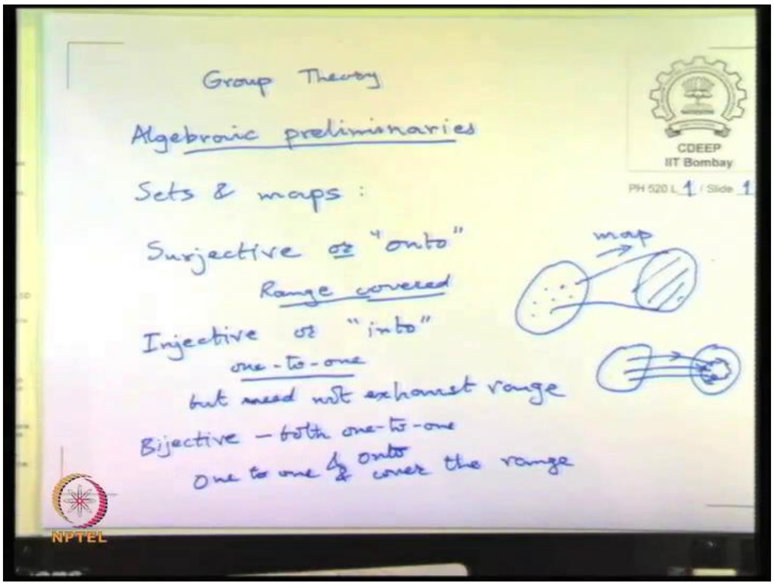

So, let us begin by the algebraic preliminaries. So, let us begin at the level of sets and maps. Mathematician classified maps in 3 versions, one is a surjective or onto this is where the map covers the range you know this domain and range and so range is completely covered. So, you have these, you may have some points here, but the entire set the range is covered ok, this is the way the map is. So, this is called a surjective map and it is called onto there are injective maps or into which are one to one ok. So, they are one to one, but they need not cover the range.

So, it may cover only part of the range, and finally, bijective which is both surjective and injective. So, they are one to one and cover the range fully, ok. So, these are the 3 categories of maps sometimes were one wonders because the discussion is not sort of symmetric between surjective, and injective because injective is indeed one to one it is not that the domain is covered it is not something like that but turns out that these are the ideas that are most relevant to all the algebraic discussion. So, this is about sets and maps.

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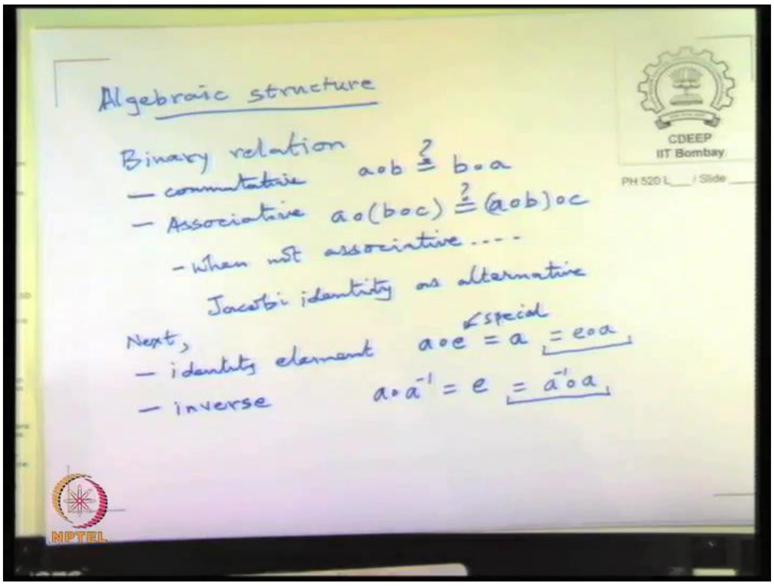

The next thing we talked about is inclusion of algebraic structure. So, within a set you may have what are called binary relations and to my knowledge I think the mathematicians have sorted out that algebraically, it is not very fruitful to talk of trinary or quarternary relations at simultaneously involved too many objects you can break it down always to something that is binary. And as far as binary relations go there are specific very general kinds of properties, firstly, about whether it is commutative or not. So, one thing is commutativity that is the binary relation is if you have ab = ba or not associated. So, let me write down over here a o b = b o a, and then associative means the when there are 3 objects involved the sequence in which you do the combination.

So, whether doing bc first and then applying a although the sequence order is same you apply from the left, but still the sequence in which you combined them is different whether it is or not decides, whether the binary operation is associative or not. It turns out that there are binary operations that are not associative. So, the first one commutativity also is as you know matrix algebra is not is not commutative matrix product ab ≠ ba in general, but matrix product is associated. But there are algebras which are not associated; it turns out that when not associative. So, you may realize that the moment you give up such control you lose the value of the utility of such a thing, thus more generic if you just a set of points it is not very useful by itself unless it has algebraic structure and so on.

Similarly, within algebraic structures the less the fewer the axioms the less control you have. If you do not have associativity it turns out that there is an interesting alternative which is called Jacobi identity. So, we will see that a little bit later you any how probably studied Jacobi identity in classical mechanics or somewhere we will come to it. So, associativity or alternatively Jacobi identity are very useful ingredients in a algebraic structure.

Next are we have of course, the question of whether there is identity element. So, the things that Jacobi identity kind of bifurcated this point they do not have identity element usually, but the Jacobi identity is a very powerful thing so, but returning to things that are commutative associative and so on. The next thing is whether there is an identity element in the group and then the idea of inverse. So, identity means that a o e = a there is a special element e special. So, that combined with any a it gives back a and one may sometime insist that it is this, but it turns out that this is sort of redundant if there is a left identity you can prove most of the time that it is also the right identity.

Similarly the inverse is that $a \circ a^{-1} = e = a^{-1} \circ a$. So, to define inverse you need the concept of identity. There is some inverse element which when combined gives identity and again we can ask whether the left inverse is same as right inverse, but I think it can be proved that you do not need to specify it, but if you just want to make life easy for you say that in advance that is both left and right. So, these are the very generic features of an algebraic structure.

So, these are the very general features of some algebra of an algebraic structures some binary operation that a certain features. But before we go on let me just go back to say something about the maps. So, when you have a map between two sets that has some algebraic structure on them binary algebraic structure.

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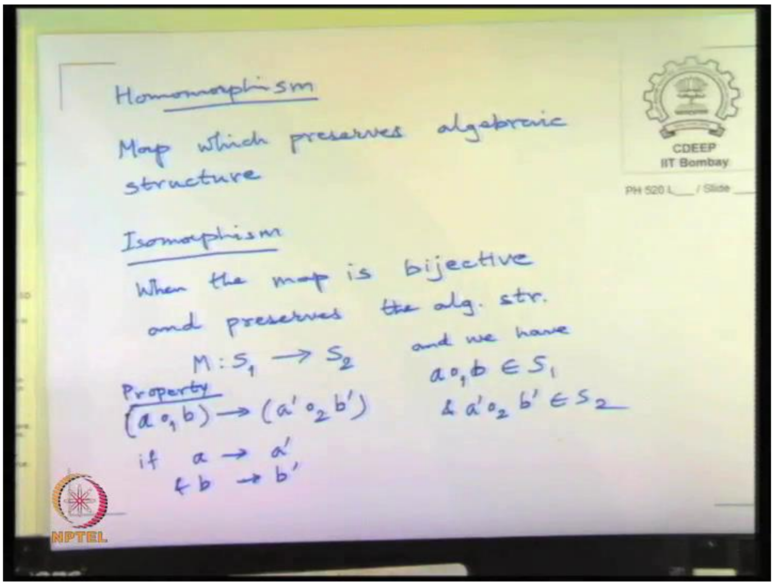

If the map is such that it preserves this binary structure then it is called homomorphism if the map $M : S_1 \longrightarrow S_2$. So, map which preserves algebraic structure. There are further refinements things called epimorphism and so on which I will not get into right now. But let me just say that there is important concept of isomorphism which is when two sets have a bijective mapping between them and that bijective mapping also preserves the algebraic structure then the two sets are isomorphic. You have to sets you have one to one and onto map between them. So, you have essentially a one to one identification and whatever algebra that set S_1 obeys it is reflected in set S_2 under that map. So, then such a map is called a bijection or isomorphism.

So, I guess I did not said more specifically what is meant by preserving algebraic structure is that if I have $M : S_1 \longrightarrow S_2$, and we have algebraic structures $a \circ_1 b \in S_1$ and $a' \circ_2 b' \in S_2$ Then m is such that the map of. So, we need some notation. So, $(a \circ_1 b) \longrightarrow$

$(a^{'} \, o_2 \, b^{'})$ if $a \longrightarrow a^{'}$ and $b \longrightarrow b^{'}$. So the property of the map which is this arrow is that if the map takes $a \longrightarrow a^{'}$ and $b \longrightarrow b^{'}$ from $S_1 \longrightarrow S_2$ then the map also ensures that the composition of a with b lands exactly at the point which is composition of $a^{'}$ with $b^{'}$. So, this is called a isomorphism. And it is both true forward and backward because it is a bijective map.

So, these are the most bare essentials algebraic concepts which we will use every once in a while.