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Quantum Information and Computing

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Modul No.02

Lecture No.6

Multiple qubits and Entanglement

In the last lecture we had introduced the concept of a qubit, which corresponds to the usual classical bits namely 0 and 1. And we had pointed out that unlike the classical bit which can at a time take the value either 0 or 1 in case of quantum bits it can take a linear super collision of that. And in fact it is this linear super collision which provides the quantum computing with enormous parallel computing capability, what we do today is to extend this concept to multiple qubits, in fact all that we will do is to talk about two qubits cases in somewhat detail and multiple qubit is just an ordinary extension there off. So the idea is the following.

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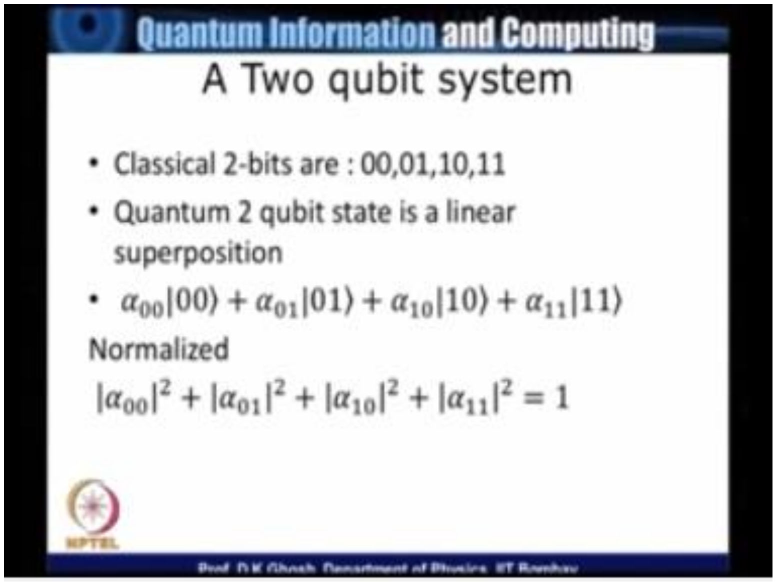

That if you go to the corresponding classical situation, then your bits are.

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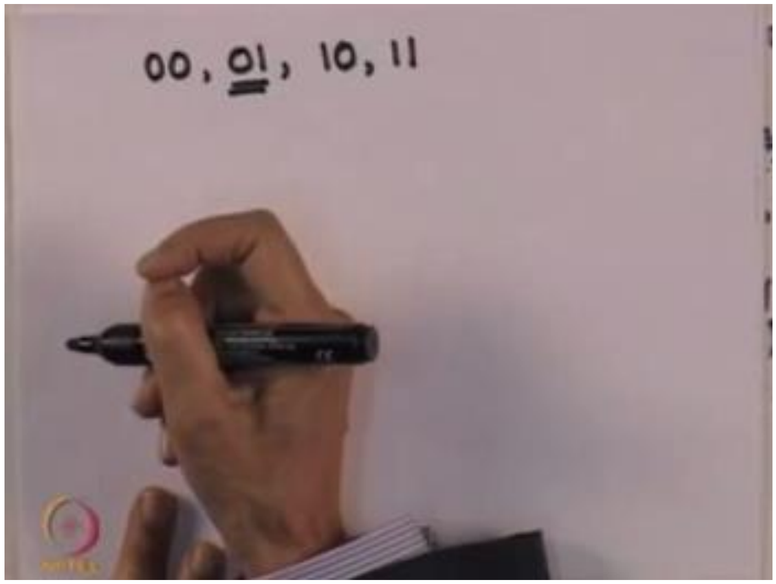

As you know 00, 01, 10 and 11, now in this case the corresponding quantum state is to be regarded as a state which is a direct product of the state, for example in this case direct product from the state 0 and state 1, this could for example refer to the qubit state of two different particles or whatever.

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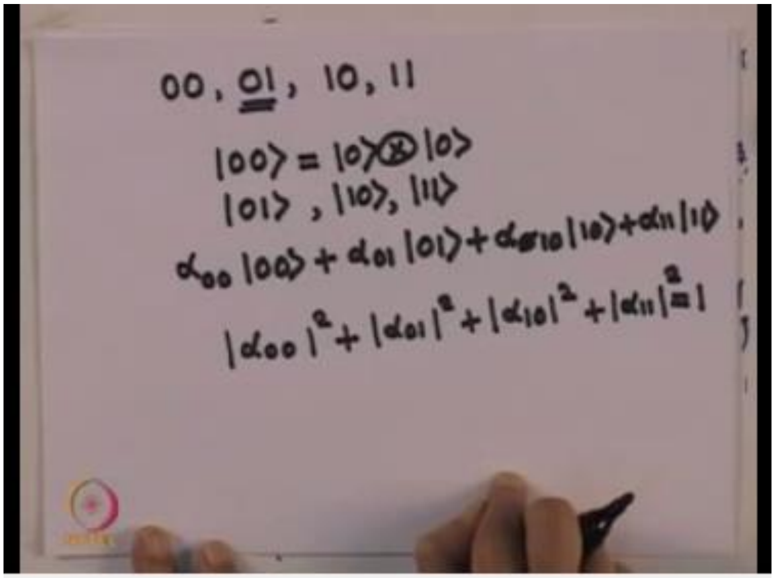

So therefore instead of these things what I have is basically a state like 00 this will be the short turn notation for writing down 0 direct product with 0 and likewise of course 01, 10 etc. Like in the one qubit case a general state of two qubits then will become a linear combination of these things which let me write it as α_{00} $|00> + \alpha_{01}|01> + \alpha_{10}$ $10>$ and $\alpha_{11}|11>$ like in the one qubit case I can normalize this thing such that $|\alpha_{00}|^2 + | \alpha_{01}|^2 + | \alpha_{10}|^2 + | \alpha_{11}|^2 = 1$.

Another question is this, that there is an input into way in which the two qubit state would differ from a single qubit state and that is because one can make a measurement of either the first bit or the second bit. So if you for instant do the measurement on the first qubit and you get a value 0 then it means.

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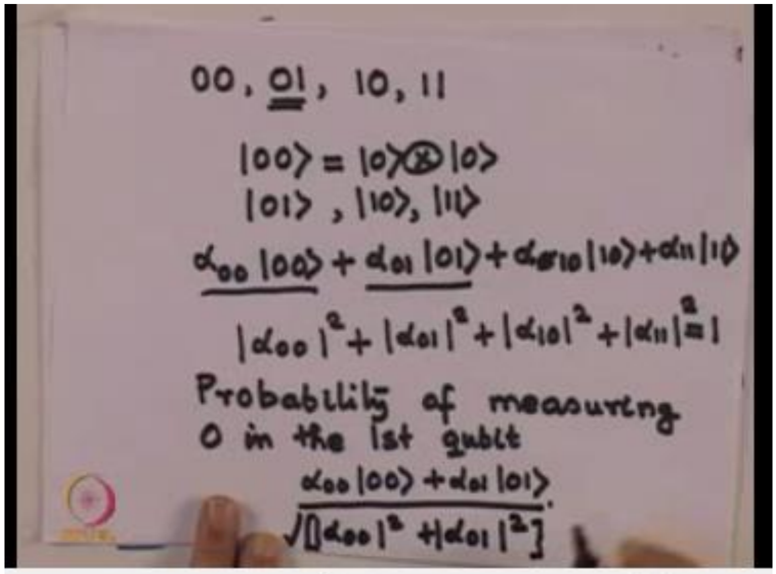

That the state of the two qubit was either this or this because these are the only two states which have 0 in the first position. Now if that happens then we can say that the probability with which I measure, probability of measuring 0 in the first qubit, is always given by $\alpha_{00}\ |00 > + \alpha_{01}|01>$ okay. Well basically the probability would be given by this square plus this square divided by the whole thing which we have of course normalized be equal to one.

The poster measurement state of this system would then be this state normalized appropriately mainly this divided by $|\alpha_{00}|^2 + |\ \alpha_{01}|^2$ and of course you have to take a^2 root thereof. Now there is a very important way in which the two qubit state and of course in principle the multi qubit states they differ from a single qubit state. So we mentioned already that a general state is required like.

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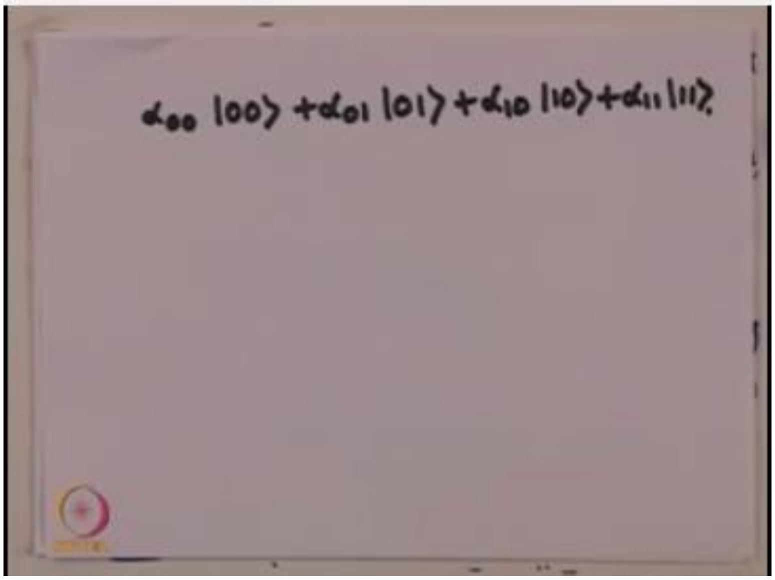

α_{00} |00 > + α_{01}|01> +α_{10} 10> and α_{11}|11> number point is this that depending upon the values of these constants α which in general are complex, I can obtain different types of states, if you consider for example state like the following, the if you refer to the slide.

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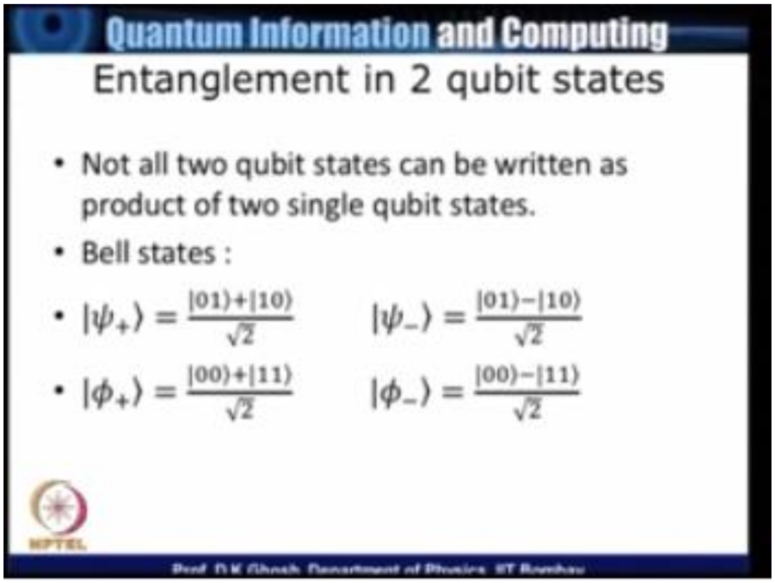

I have given the set of four states quick because of their importance of quantum computing are known as the bell states, but let me illustrate this with a single situation, so for example let us talk about a state which is written there as ψ_+.

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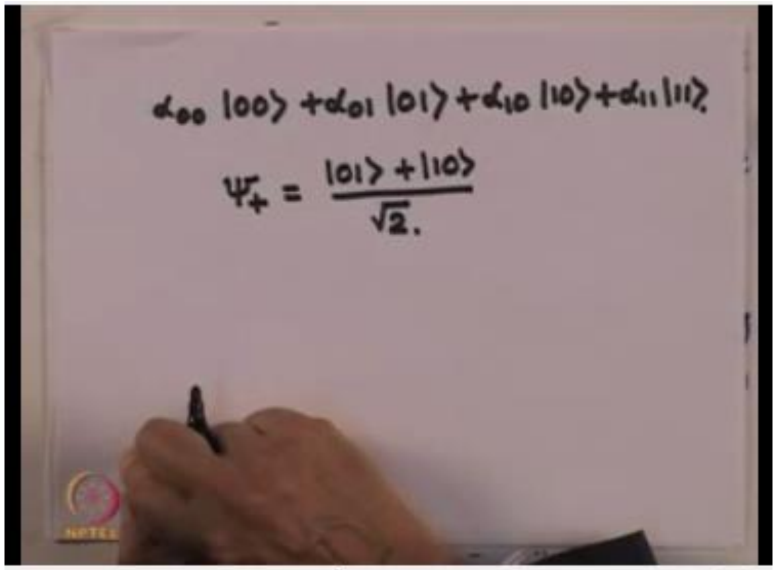

Which is written as |01> + |10> and normalized to divided by square root of 2, another question is this, that this state is different from the corresponding single qubit state in the following way that suppose I make a measurement of the qubit number 1, now of course I have two possibilities, I can either get 0 or get 1 and with equal probability, but notice that when I measure the state 0 this state of the second qubit is automatically determined, the if for example.

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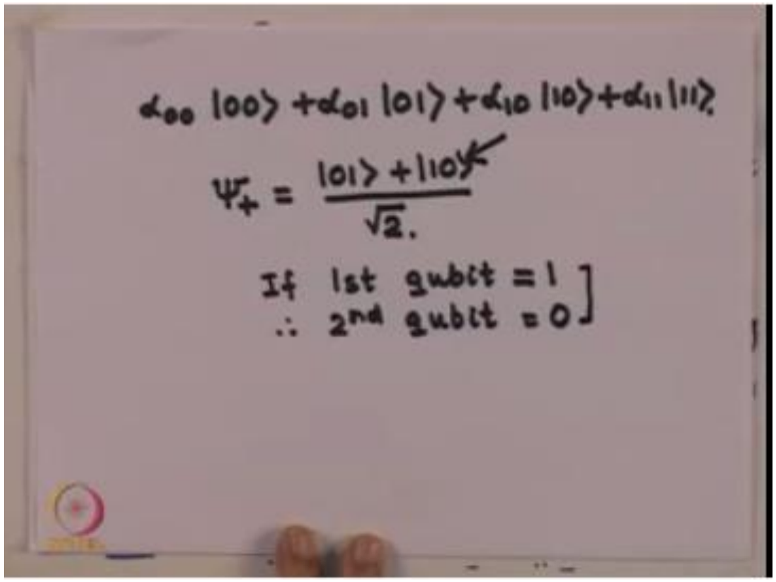

Measurement of if first qubit happens to be 1, then this must have been the state from where it came from and therefore second qubit will then be naturally equal to 0. Now this is very peculiar because I have not really measured the second qubit, absently measured the first qubit but because the first qubit being equal to 1 comes only in that combination, the state of the second qubit is immediately decided and this thing is known as entanglement.

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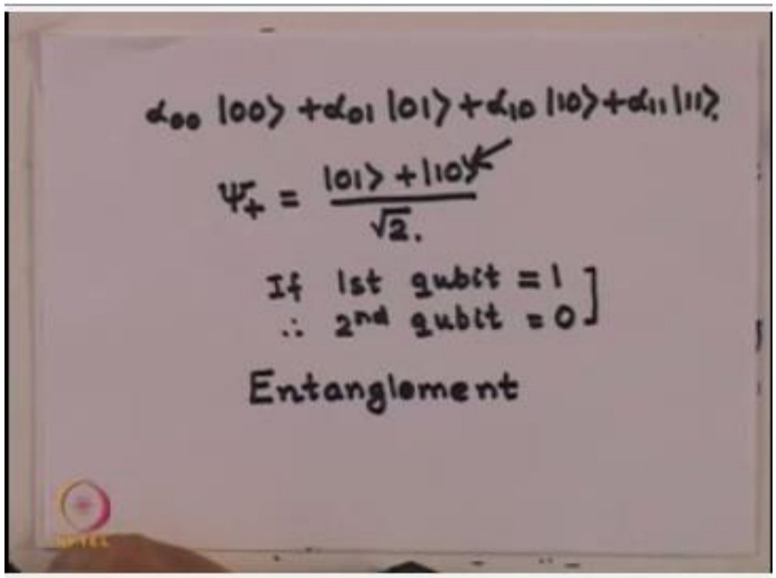

We will have lot more to say about entanglement as we go along in the course, the. basically the point to the following that not all two qubit states can be written as a product of two single qubit states and these are well states that I have given you are examples of that. Now let us translate this into our usual notation.

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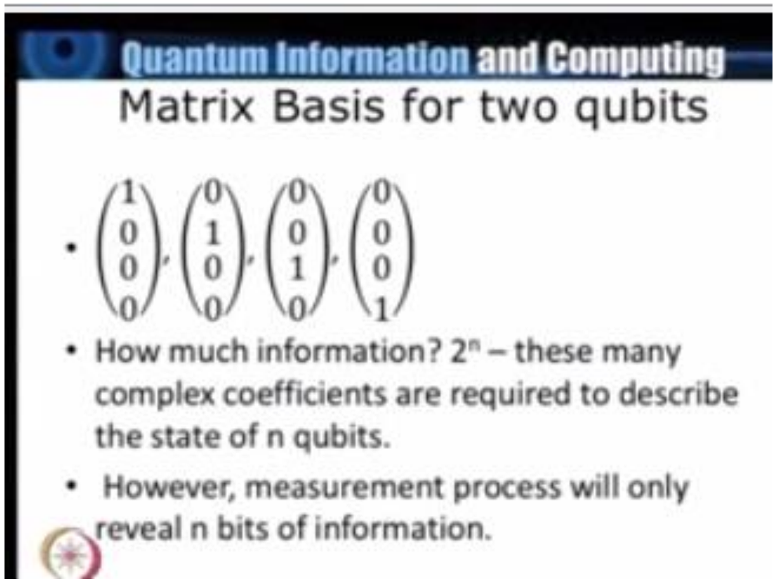

For the matrix notation for the qubits, now look at what is that we are talking about see the basics for the single qubit.

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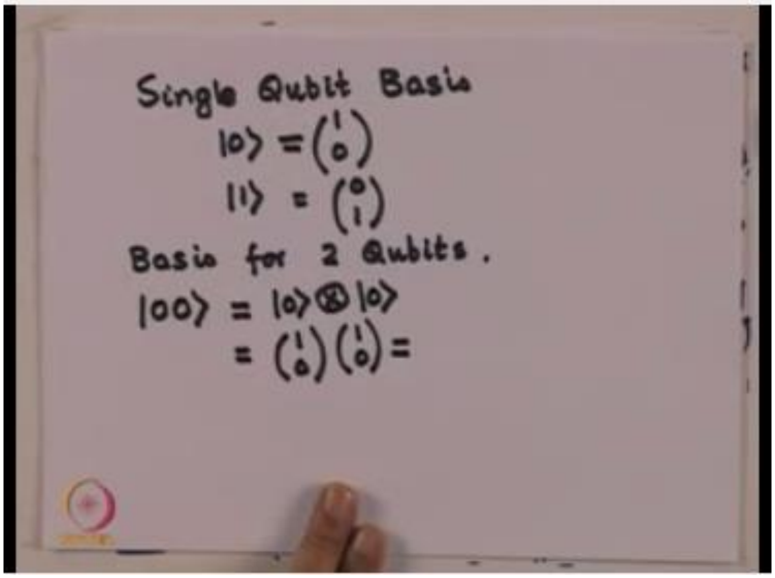

As we know the state 0 was 10> and state 1 is 01>, now I can use this to define a basis for two qubits or for that matter for any number of qubit. So in this notation the state |00> which is the short hand notation for 0 state direct product with 0 and so therefore I have to simply multiply the matrix 10 with 10 itself, now notice is this is not a usual matrix multiplication but it is what we call as a chronicle multiplication or chronicle product. Now so the way it happens is.

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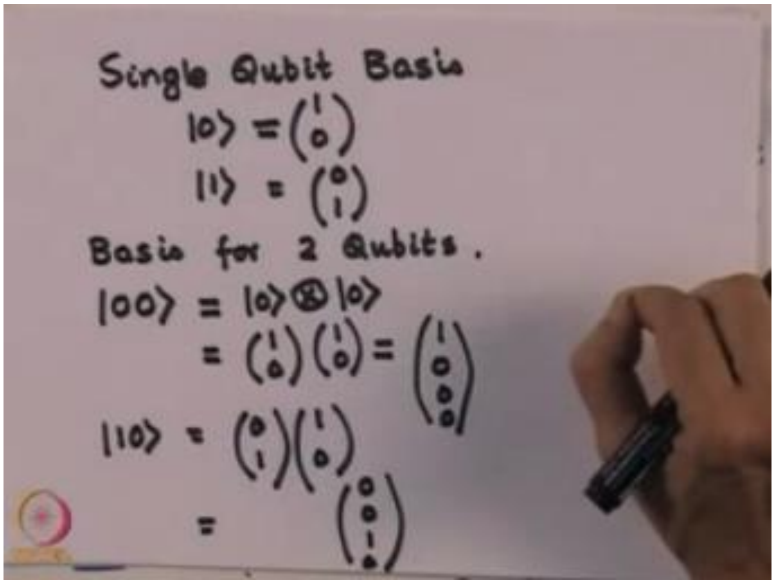

So you multiply first 1 with this whole thing, so you get 10 and 0 with this so I get 00. And likewise I will just illustrate one more. For example, if you want 10 I know the state 1 is given by 01 and state 0 is given by 10 so therefore this product is going to be 0010. So that provides us a basis of for if you look at this slide.

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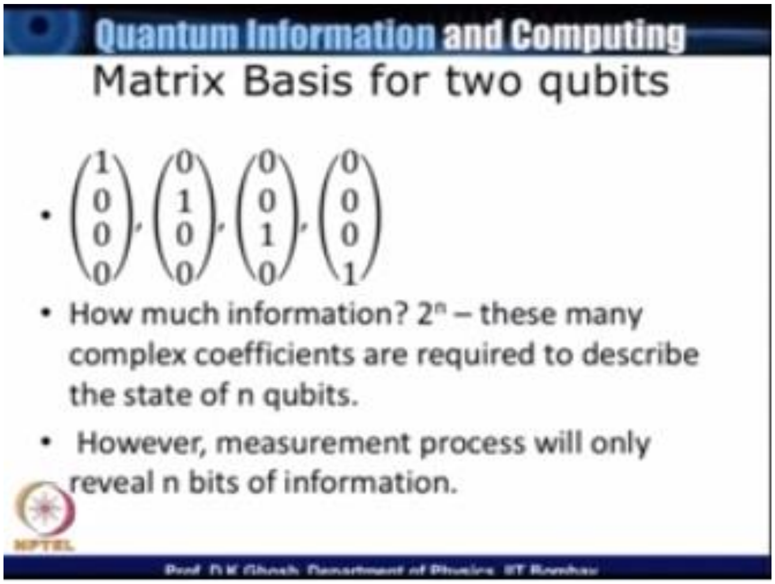

The slide tells you that there are the states here 1000 there 0100, 0010, 0001 and any two qubit state which in principle is just as I wrote α000, α01, α10 and α11 can be written as a linear combination of this basis. Now how many, how much of information is there in this type of writing, so firstly of course you realize there are supposing I have a n qubits so there are 2^n number of, you know complex coefficients I have because if I have n qubits then there are 2^n possibilities of and the value being either 0 or 1.So that much, but however as we will see later.

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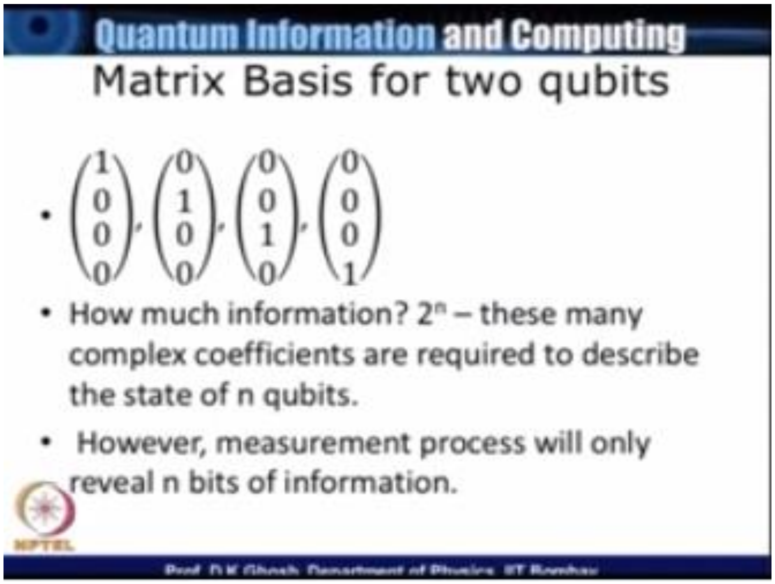

That most of such information remains hidden and when we make the measurement we will really get n bits of information.

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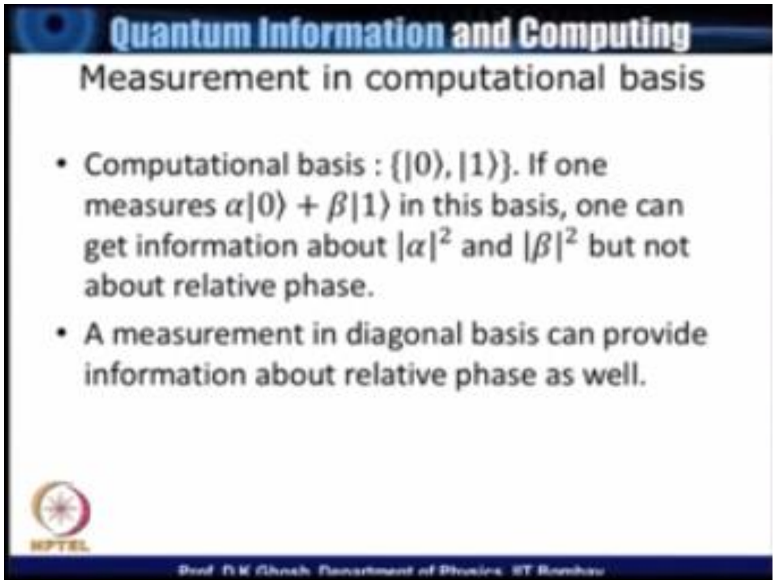

The next point that we are going to talk about is the process of measurement. Now the process of measurement is a little interesting thing, you remember that we have been talking about preparing a state in what we have called as the computational basis. A computational basis is basically writing a state as a linear combination of 0 and 1, but as we have seen earlier that a basis need not be 10 and 01.

For example, for a single qubit state you could equally well have a basis which is 11 normalized to 1 over square root of 2 that is one over square root of 2 11, and one over square root of 2 1-1 and again any state AB can be written as a linear combination of this two states. Now so this particular basis which I have talked about is what is known as a computational basis.

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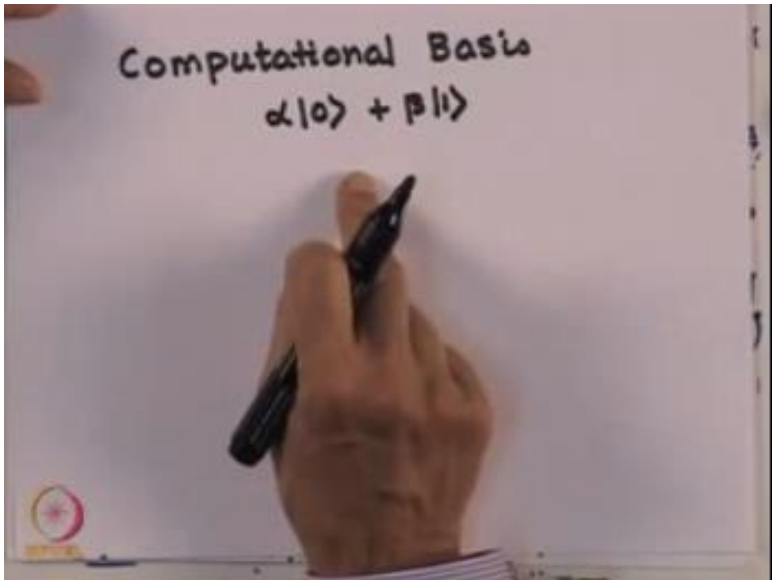

This is just given special name is called a computational basis. Now so therefore, in the computational basis a single qubit state is α0+β1 of course, you can always normalize it. And what we know is that in this basis we can get information about the modulus of α^2 and modulus of β^2. And in the last lecture we have pointed out that if there is a relative phase between them then a measurement in the computational basis is not going to help, but if you do a measurement and what we called as the diagonal basis it helps in finding out some information about the relative phase.

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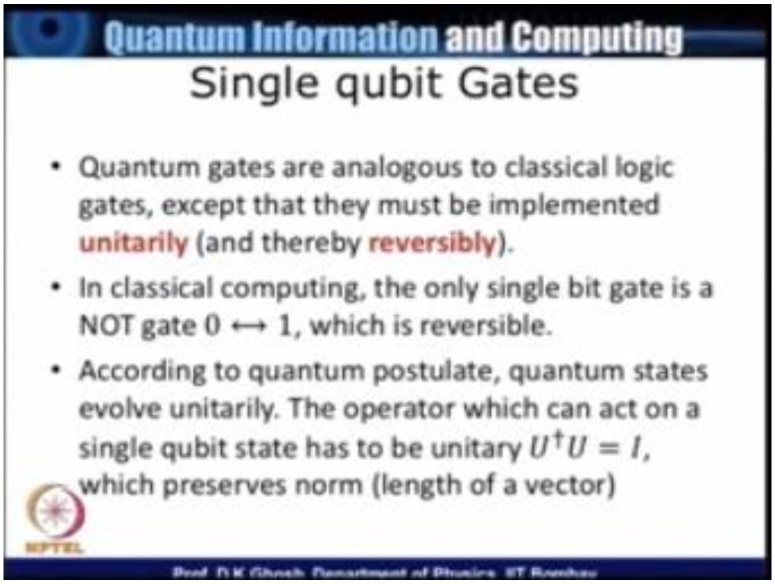

So having talked about the one and two qubits let me now get to the basics of quantum computation. Now before I do that let me recollect for you, that how do you do classical computation. Now remember that in classical computation also what we have are what we call as the classical logic gates. Now you do not actually have separate gates for depending on what is the number of qubits you have. So basically there are essential components. For example, you could have gates corresponding to OR, AND, NAND etc.

There are some theorems which will tell you that some of those gates are universal, in terms of which you can express all other gates. But the basic point is that you need to define a few elementary gates in order to make an element which can run a computer. Now this is exactly the situation here accepting thing there are some other issues, the unlike classical computation, the quantum computation must be performed reversibly and the corresponding gates that you have, they must perform operations on a state unitary.

So let us now make a mathematical statement out of that, the mathematical statement is that I have in a typical computer logic flow, I have an input coming in, certain operators work on it which is what is represented by the logic gates and I get an output and in case of classical computation the inputs are bits could be for example n bits and outputs are also bits could be m bits. Occasionally we require additional registers for doing operations easily and those will be called as ancillary register or as is the common practice in quantum computation we will be calling them ancilla.

Now the thing now is this, that what I have instead of classical bit is a quantum state which we have written in terms of qubits either single qubit or multiple qubits. Now when I do an operation on a qubit or qubits I want the I want the output to be qubit or qubits that is a quantum state gets converted into another quantum state, but this operation must follow laws of quantum mechanics, and the operators which are required for this must be unitary operators, if you recall we define unitary operators as those which satisfy.

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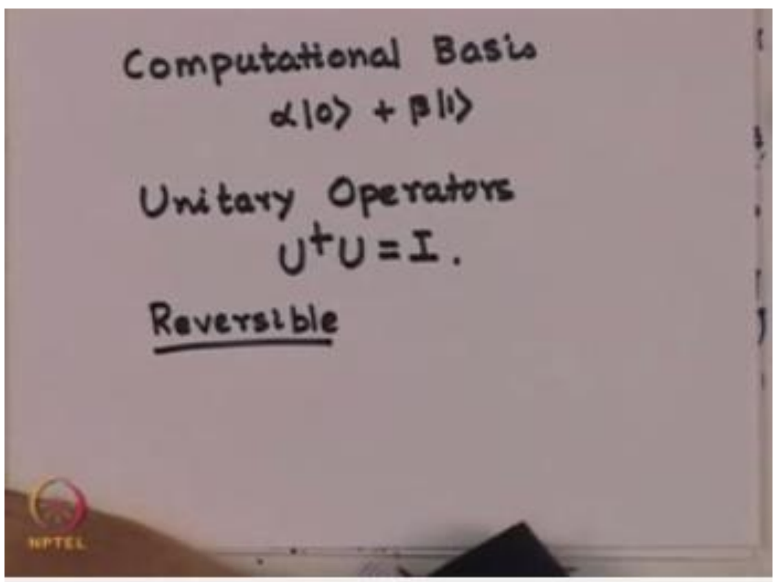

Unitary operators they satisfy $U^{+}U$ is equal to the identity. So we need reversible computation we have pointed out that normal classical computing is not reversible, you can make them reversible by storing a lot of garbage information, but that is extremely energy inefficient. A quantum computation on the other hand has to be reversible. So let us look at what are the elementary gates that we require for making a quantum computation work.

The one point like we point out that in classical computation also there is a gate which is reversible and that is just a NOT gate.

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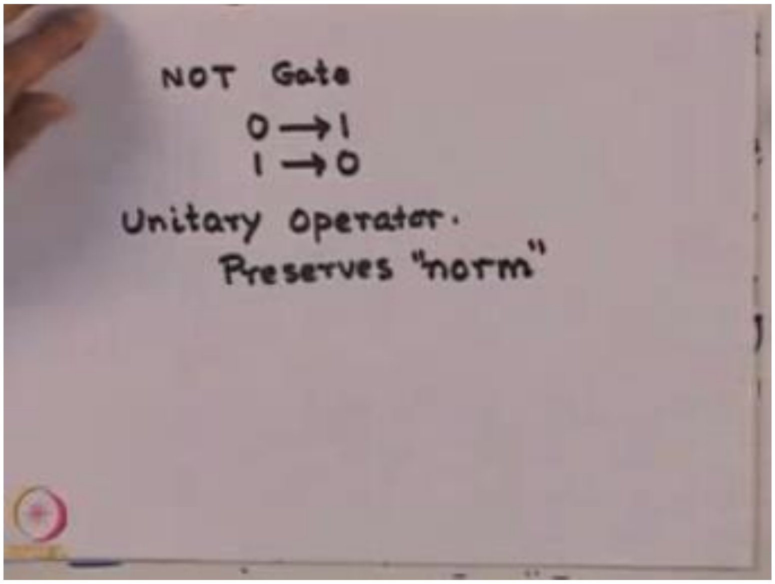

The NOT gate is reversible because a NOT gate 0 goes to 1 and 1 goes to 0, and so if you know the output you can also find out what the corresponding input was okay. So this is now what is a unitary operator, an unitary operator has this property that it preserves norms which we have talked about it in earlier lectures. So basically it preserves the length of a vector, last time we talked about the representation of a single qubit on a Bloch sphere, so since all the single qubits have a representation on the Bloch sphere.

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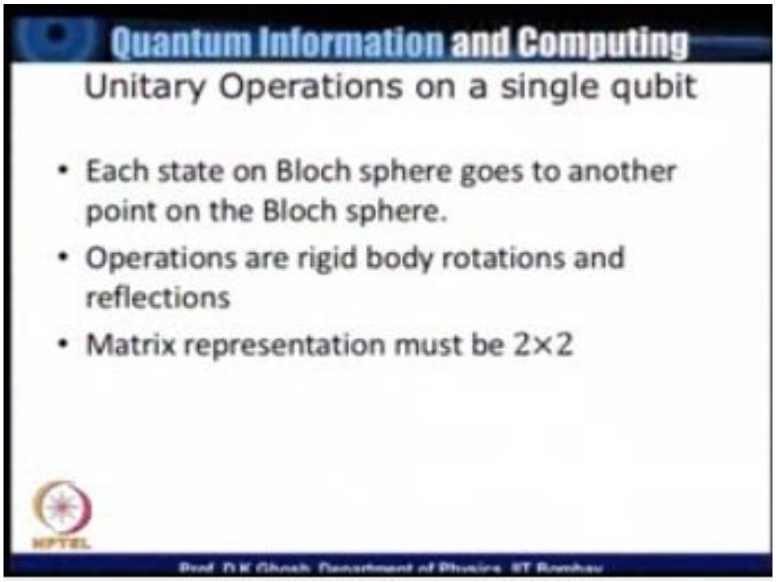

So if I am converting a single qubit by applying unitary operator from one state to another state both these lie on the Bloch sphere and Bloch sphere as you have seen is a sphere of unitary basis. So therefore, the operations correspond to transforming one point on the Bloch sphere to another point keeping the length constant which is the radius of this sphere and so therefore, geometrically they correspond to rigid rotations on the unit sphere and of course the reflections.

So any point can be transformed into any other point by a sequence of operations which are either rotations or reflections or both. Now if I now go to a matrix representation of these operations since my input and output both must be of this form.

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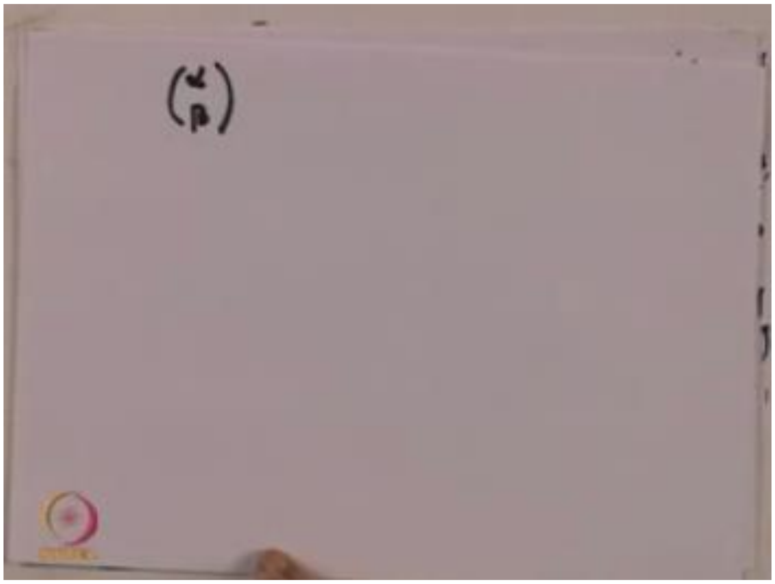

The operator which acts on such a state giving me a singular looking state then meaning they are giving me a matrix which has two rows and a single column has to be a 2/2 matrix.

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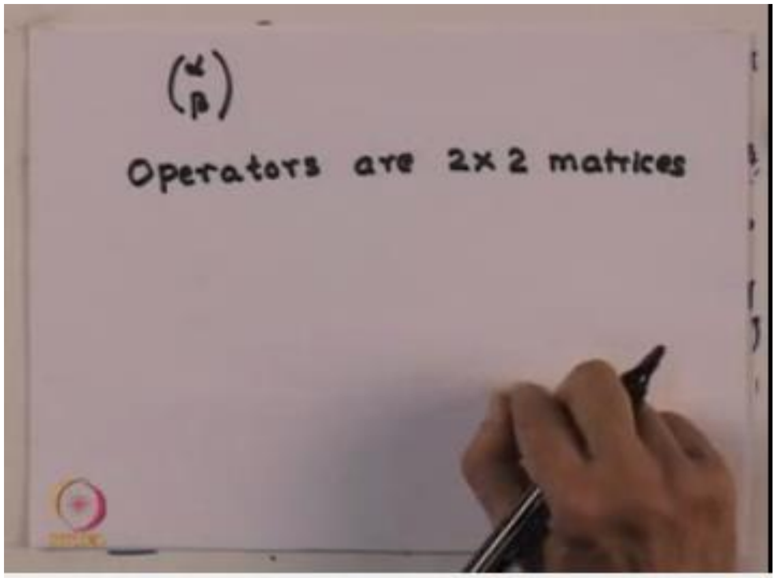

So operators which acts on this or 2/2 matrices, now it is fairly straightforward to show that a 2/2 matrix in general.

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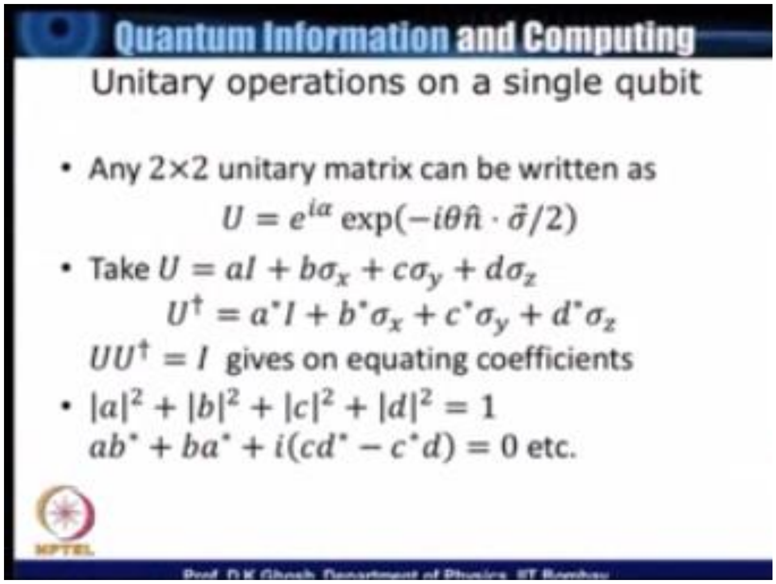

Can be expressed as there is a general form.

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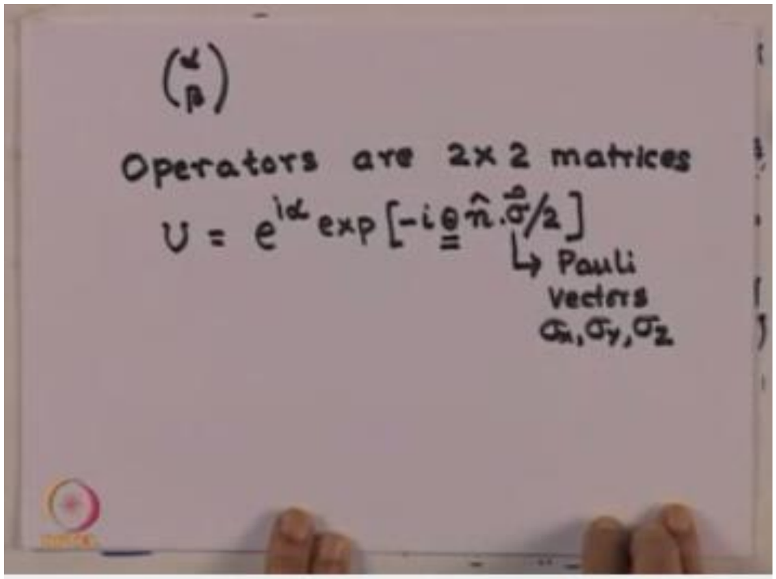

$e^{i\alpha}$ α is the real exponential of $-i\ \theta$ n.σ/2 where σ is our Pauli vectors having components σ_x, σ_y, and σ_z θ is a real number so is α and n is a direction.

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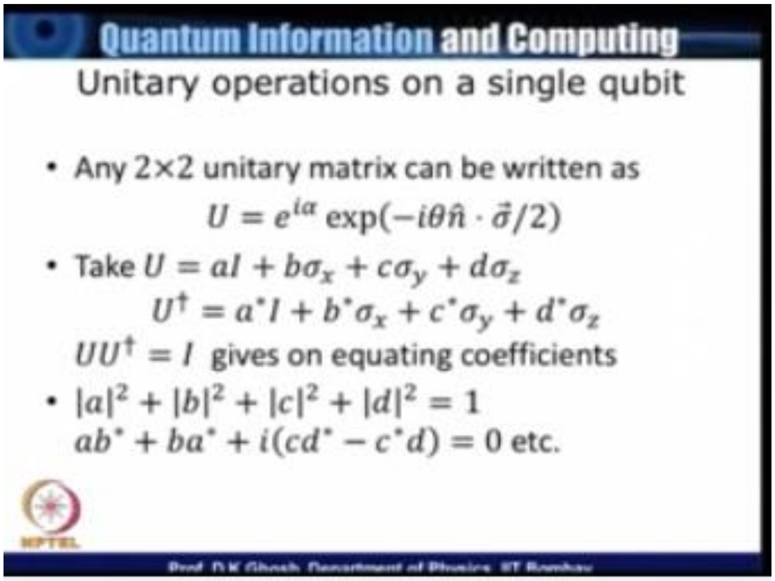

So basically this is what we want to prove and in order to prove this is fairly straightforward algebra I would urge you to work it out a detail but let me what about tell you some way of doing this. Now remember I have said u is given by a 2/2 matrix, so supporting U has this structure, supporting u is a, b, c, d, remember a, b, c, d are complex. So therefore, my U^+U you can write this so remember the U^+ will have a*, b*, c*, d* you can just multiply them and realize the following.

I can always express a matrix, so a matrix can be always expressed, a linear combination of identity matrix σ_x, σ_y, and σ_z. So what you do is this you essentially write down U in terms of this remember σ_x, σ_y, and σ_z.

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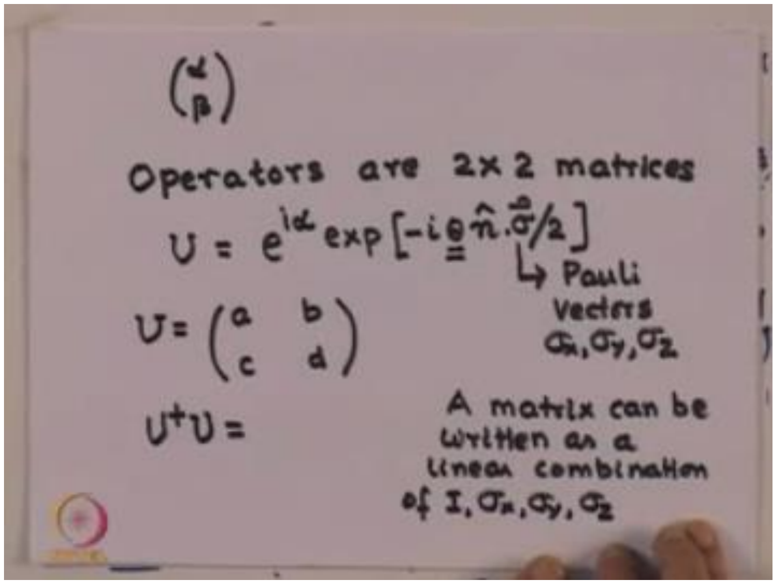

I have given this structure several times, but let me repeat again.

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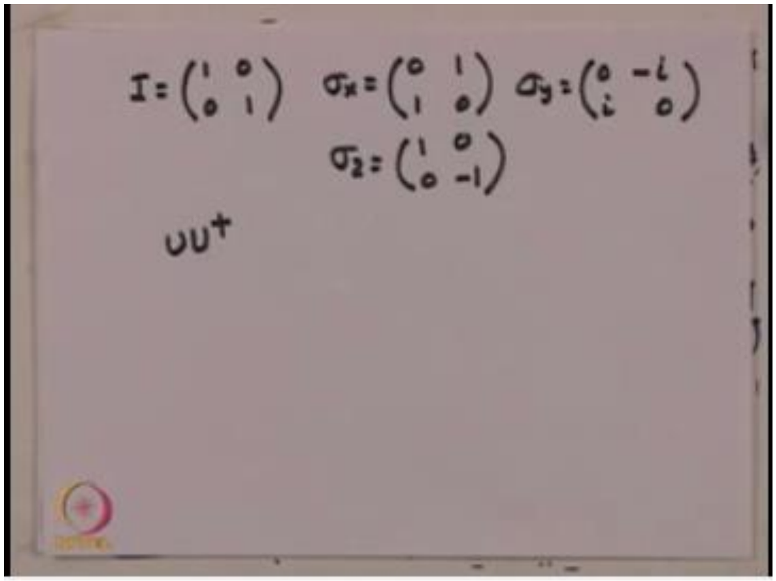

So I have got I of course identity which is (1, 0, 0, 1) σ_x is (0, 1, 1, 0) σ_y is (0, –i, I, 0) and σ_z is (0, 1, 0, -1). So you write this u in this form and find out what is UU^+. So this is basically a straightforward multiplication. So therefore, these expansions we will have constant term which you can trivially see to be moda^2+modb^2+modc^2+modd^2 and terms which are proportional to σ_x, σ_y and σ_z.

There are these products terms which will come, for example there might be a term which is σ_x x σ_y, but by the property of the poly matrices that is nothing but I times the following. So convert all these things into that. Now having done that you can then show easily that this is the way one can write any 2/2 matrix.

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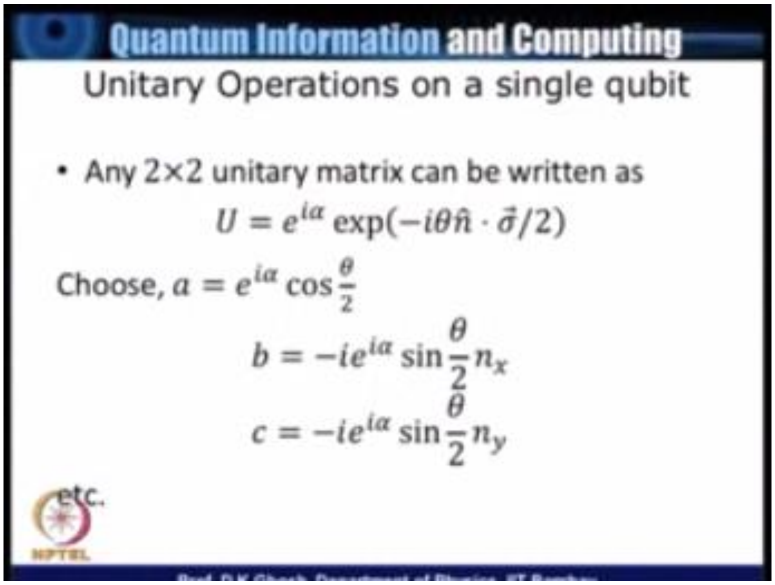

And that sort of tells you what is this, so the I am just leaving this as an exercise for you.

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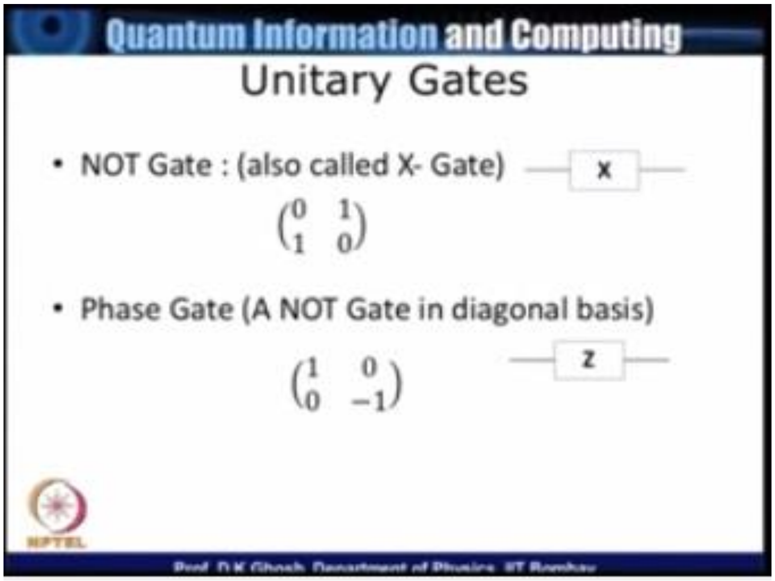

And you can try it.

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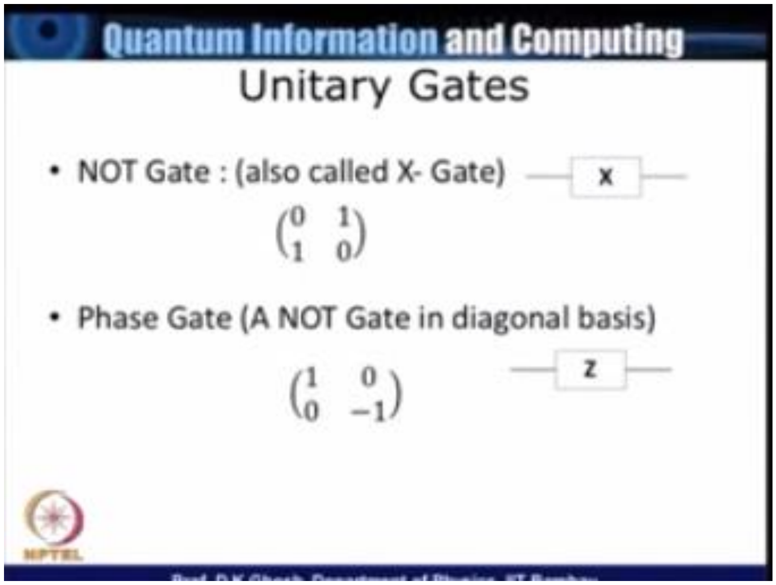

So look this let me then go over to one qubit gate. So we have already seen that the structure of one qubit gates are those which correspond to rotations and reflections on the Bloch sphere. So I will start with for instance, in NOT gate.

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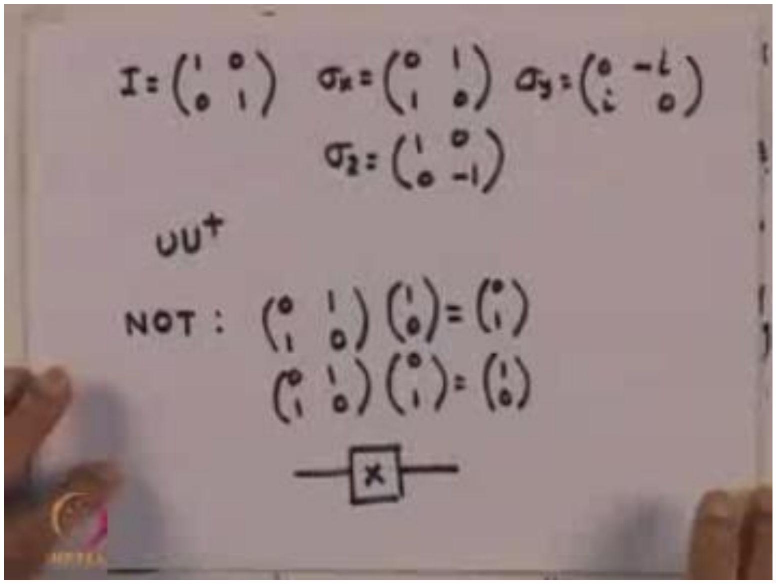

Remember that NOT means converting 0 to 1 and 1 to 0. And this is trivially done by the Pauli's X matrix, you can check immediately this acting on 10 gives me 01 and this acting on 01 gives me 10. And this in our notations for a quantum gate will be simply written like this. So this is one of the gates of the single qubit.

We have already said that there are other possibilities because this should be a collection of the rotation and reflections on the Bloch spheres. In the next lecture we will continue talking about the gates that possible in one qubit and two qubit cases and then carry on to a discussion on of quantum circuits.

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