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**Quantum Information and  
Computing**

**Prof. D.K.Ghosh  
Department of Physics IIT Bombay**

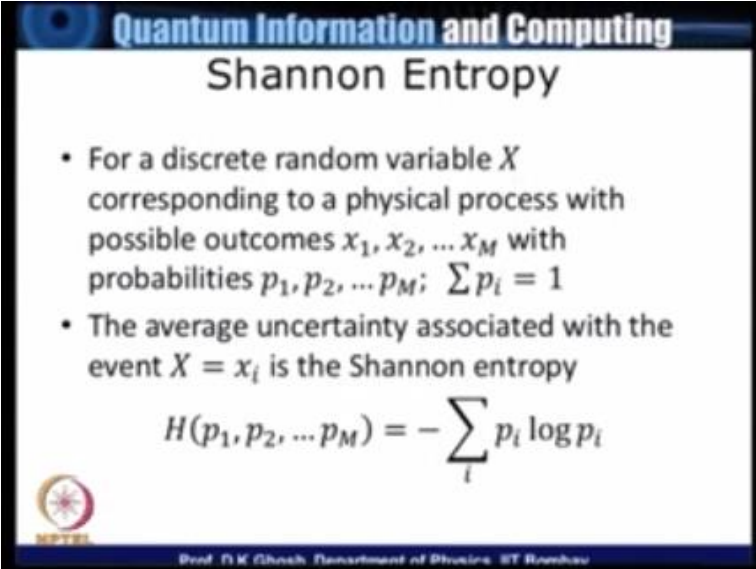
**Modul No.07**

**Lecture No.37**

**Von Neumann Entropy**

In the last lecture we have been talking about Shannon entropy which is basically a classical information system which tells me what is the optimal compression that I can get.


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**Quantum Information and Computing**  
**Shannon Entropy**

- For a discrete random variable  $X$  corresponding to a physical process with possible outcomes  $x_1, x_2, \dots, x_M$  with probabilities  $p_1, p_2, \dots, p_M$ ;  $\sum p_i = 1$
- The average uncertainty associated with the event  $X = x_i$  is the Shannon entropy

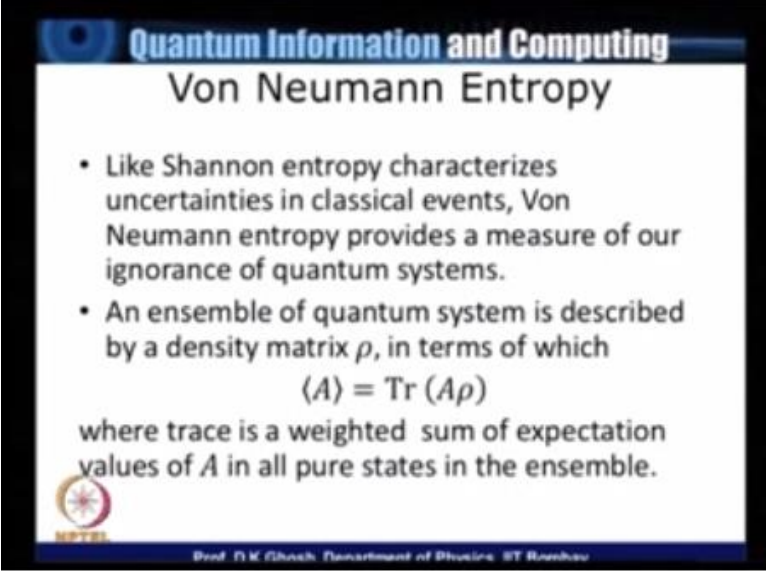
$$H(p_1, p_2, \dots, p_M) = - \sum_i p_i \log p_i$$

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And if just to summaries what we said is supposing we have a random variable  $x$  corresponding to a physical process with has possible outcome which has possible outcomes  $x_1, x_2$  up to  $x_M$  with respective probabilities  $P_1, P_2$  up to  $P_M$  subject of course  $\sum P$  to 1 we said that the average uncertainty associated with the event is what is called as the Shannon entropy and has an expression which is  $\sum_i P_i \log P_i$  we mentioned that traditionally it is conversational to take the base of the logarithm to be equal to 2.

The second point that we notice is that this definition does not depend up on the value that the random variable takes but rather or then on probability with which a particular event occur. We also said that Shannon entropy gives a lower bound on the uncertainty associated with an event and if the length of a uniquely describable prefix code happens to be  $n_i$  with the probability  $P_i$  then the length of the word should be  $\geq$  the Shannon's probability function  $H$ .

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The slide is titled "Quantum Information and Computing" and "Von Neumann Entropy". It contains two bullet points: "Like Shannon entropy characterizes uncertainties in classical events, Von Neumann entropy provides a measure of our ignorance of quantum systems." and "An ensemble of quantum system is described by a density matrix  $\rho$ , in terms of which  $\langle A \rangle = \text{Tr} (A\rho)$  where trace is a weighted sum of expectation values of  $A$  in all pure states in the ensemble." There is a small logo in the bottom left corner and a footer at the bottom that reads "Prof. D.K. Ghosh, Department of Physics, IIT Bombay".

**Quantum Information and Computing**  
**Von Neumann Entropy**

- Like Shannon entropy characterizes uncertainties in classical events, Von Neumann entropy provides a measure of our ignorance of quantum systems.
- An ensemble of quantum system is described by a density matrix  $\rho$ , in terms of which
$$\langle A \rangle = \text{Tr} (A\rho)$$
where trace is a weighted sum of expectation values of  $A$  in all pure states in the ensemble.

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What Von Neumann did is to generalize this to the case of quantum systems and it did it by an argument which we will not go to but what happens is like Shannon entropy characterizes uncertainties of classical event, Von Neumann entropy provides a measure of our ignorance or

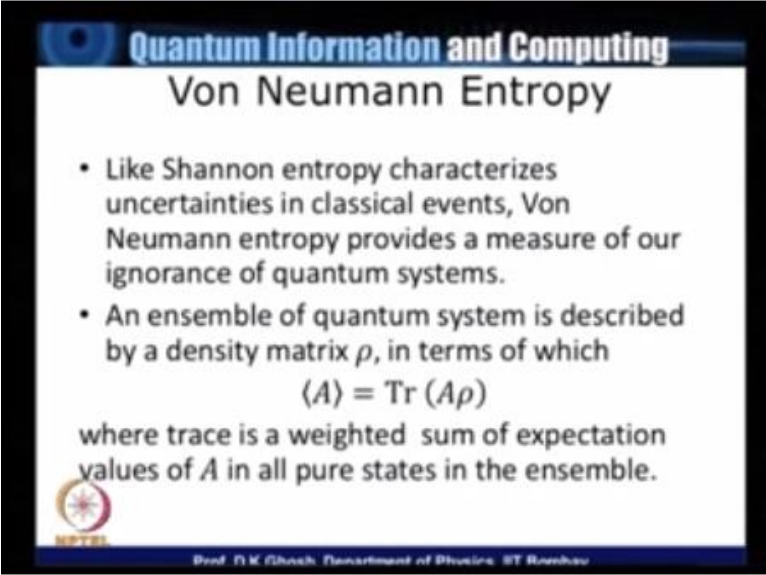
uncertainties of the quantum systems like in Shannon case the situation that would look at is not that a pure system but of an ensemble described by a density matrix  $\rho$  we have had.

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Quite a bit of a discussion on the property density matrix.

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**Quantum Information and Computing**

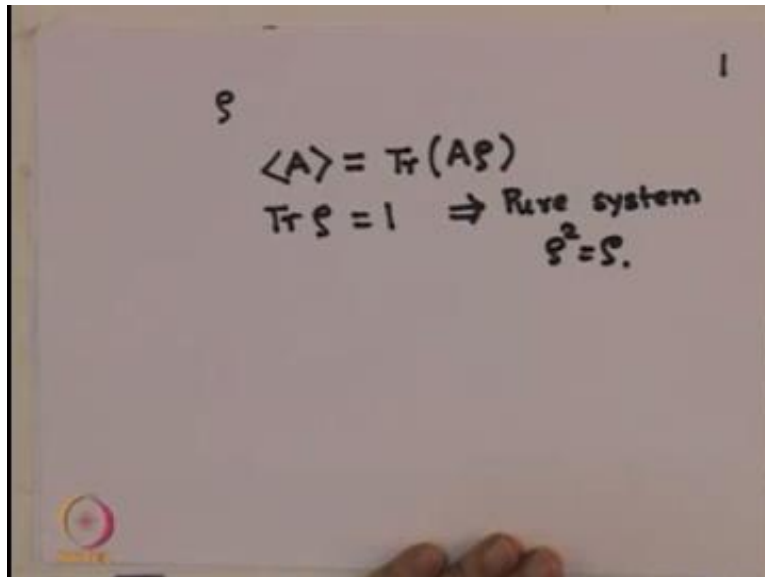
## Von Neumann Entropy

- Like Shannon entropy characterizes uncertainties in classical events, Von Neumann entropy provides a measure of our ignorance of quantum systems.
- An ensemble of quantum system is described by a density matrix  $\rho$ , in terms of which
$$\langle A \rangle = \text{Tr} (A\rho)$$
where trace is a weighted sum of expectation values of  $A$  in all pure states in the ensemble.

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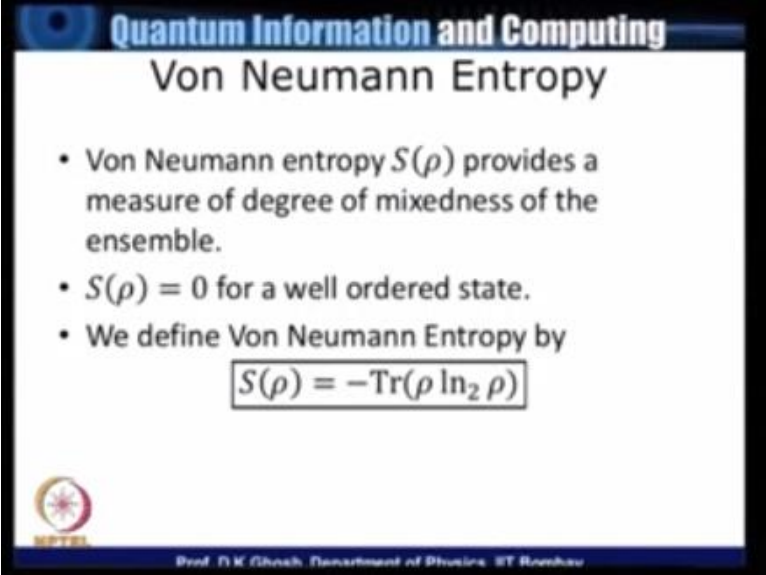
And we have said if I have an observable  $A$ .

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The expectation value of the operator  $A$  is given any  $\text{Tr}$  of  $A$  with  $\rho$  where  $\rho$  is the density matrix and that the density matrix as a property that  $\text{Tr}\rho = 1$  for a pure system we had seen that  $\rho^2$  must be also  $= \rho$ .

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**Quantum Information and Computing**

## Von Neumann Entropy

- Von Neumann entropy  $S(\rho)$  provides a measure of degree of mixedness of the ensemble.
- $S(\rho) = 0$  for a well ordered state.
- We define Von Neumann Entropy by

$$S(\rho) = -\text{Tr}(\rho \ln_2 \rho)$$

  
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So Von Neumann entropy  $S(\rho)$  provides a measure of the degree of mixedness in a ensemble so firstly I require for a pure system like in case of Shannon entropy for a pure system.

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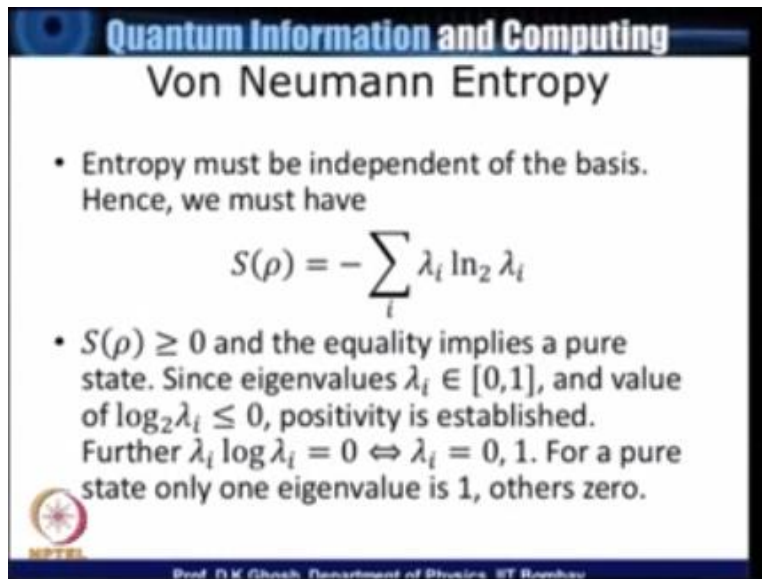
$$\rho$$
$$\langle A \rangle = \text{Tr}(A\rho)$$
$$\text{Tr} \rho = 1 \Rightarrow \text{Pure system}$$
$$\rho^2 = \rho.$$

For a pure system  $S(\rho) = 0$

$$S(\rho) = -\text{Tr}[\rho \log_2 \rho]$$

$S(\rho)=0$  Von Neumann took his use of Shannon discussion of entropy and he defines the entropy if it is now carries his name as equal to minus remember it was  $-\sum_i T_i \log P_i \log P_i$  so this is  $-\text{Tr}[\rho \log \rho]$  and as before I can take the base of logarithm what I want but they as I said the traditionally you will take the base is 2.

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**Quantum Information and Computing**  
**Von Neumann Entropy**

- Entropy must be independent of the basis.  
Hence, we must have

$$S(\rho) = - \sum_i \lambda_i \ln_2 \lambda_i$$

- $S(\rho) \geq 0$  and the equality implies a pure state. Since eigenvalues  $\lambda_i \in [0, 1]$ , and value of  $\log_2 \lambda_i \leq 0$ , positivity is established. Further  $\lambda_i \log \lambda_i = 0 \Leftrightarrow \lambda_i = 0, 1$ . For a pure state only one eigenvalue is 1, others zero.

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So let us look at what are the properties of this system like this. First thing is since entropy must be independent of the bases in which you are calculating because is that  $\text{Tr}$  so therefore I could go over to a representation in which  $\rho$  is diagonal.



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The image shows handwritten mathematical notes on a whiteboard. At the top left, the Greek letter  $\rho$  is written. Below it, the expectation value of an operator  $A$  is given as  $\langle A \rangle = \text{Tr}(\rho A)$ . The next line states  $\text{Tr} \rho = 1 \Rightarrow$  Pure system, with  $\rho^2 = \rho$  written below it. The text "For a pure system  $S(\rho) = 0$ " is written. Below that, the entropy formula is given as  $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ . The final line shows the entropy as a sum over eigenvalues:  $S(\rho) = -\sum_i \lambda_i \log \lambda_i$ .

And then my  $S(\rho)$  would then be given by  $-\sum_i \lambda_i \log \lambda_i$  there are few properties of this which we will discuss. So of course from the definition it becomes clear that your  $S(\rho)$  is equal to 0 for a pure system and the reason is the following if you take.

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- 2 -

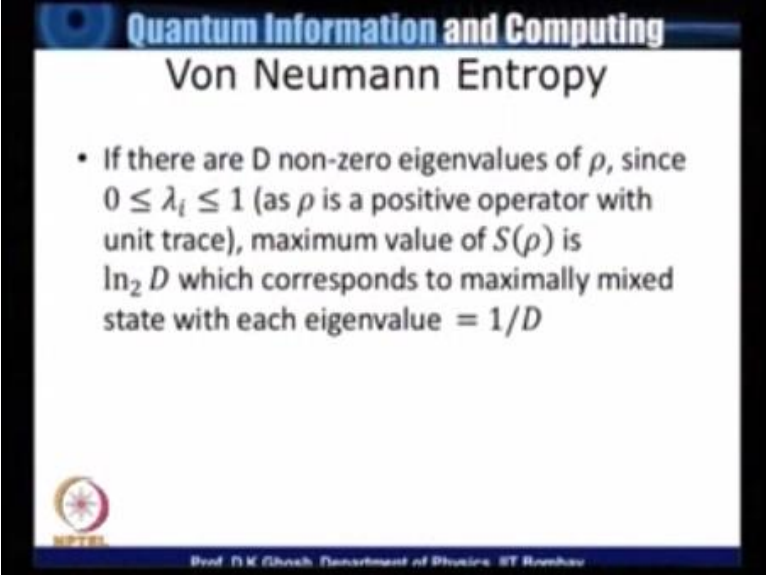
$$S(\rho) = -\sum_i \lambda_i \log \lambda_i$$

For a pure system  
 $\lambda_i = 1$  for specific  $i$   
 $= 0$  everything else

$$S(\rho) = 0$$

$S(\rho)$  to be equal to  $-\sum_i \lambda_i \log \lambda_i$  recall that for a pure system one of my  $\lambda_i$  is must equal to one and all other is 0. So therefore, I get  $-1 \log 1 - 0 \log 0$  but  $0 \log 0$  is to be understood in as a limit which is equal to 0 so therefore, for a pure system may be a particular  $\lambda_i = 1$  for specific  $i$  and is equal to 0 for everything else. So therefore,  $S(\rho) = 0$  that is one property which we wanted and that is fine.

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The slide features a blue header with the text "Quantum Information and Computing" and "Von Neumann Entropy". The main content is a bullet point explaining the maximum value of the von Neumann entropy  $S(\rho)$  based on the number of non-zero eigenvalues  $D$  and the constraint  $0 \leq \lambda_i \leq 1$ . A logo is visible in the bottom left corner, and the footer identifies the speaker as Prof. D.K. Ghosh from the Department of Physics at IIT Roorkee.

**Quantum Information and Computing**  
**Von Neumann Entropy**

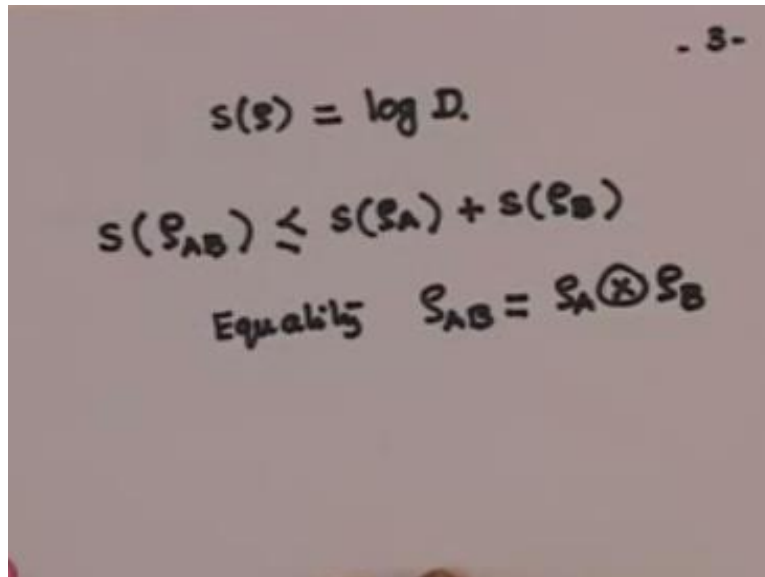
- If there are  $D$  non-zero eigenvalues of  $\rho$ , since  $0 \leq \lambda_i \leq 1$  (as  $\rho$  is a positive operator with unit trace), maximum value of  $S(\rho)$  is  $\ln_2 D$  which corresponds to maximally mixed state with each eigenvalue  $= 1/D$

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Now what is the maximum value of the entire this requires a detailed mathematical proof I will not go through it, but since I said I will give an intuitively which is very simple thing to understand since we said that entropy is the measure of the degree of mixedness of the system supposing I am  $D$  dimension they have a  $D$  dimension then if there are  $D$  number of non-zero Eigen values and obviously the maximum mixedness comes for the case where each of the  $D$  Eigen values is equal to  $1/D$ .

And that is because the sum of the Eigen value of the density matrix must be equal to 1 because  $\text{Tr of } \rho$  is equal to 1 so therefore I must have the  $S(\rho)$ .

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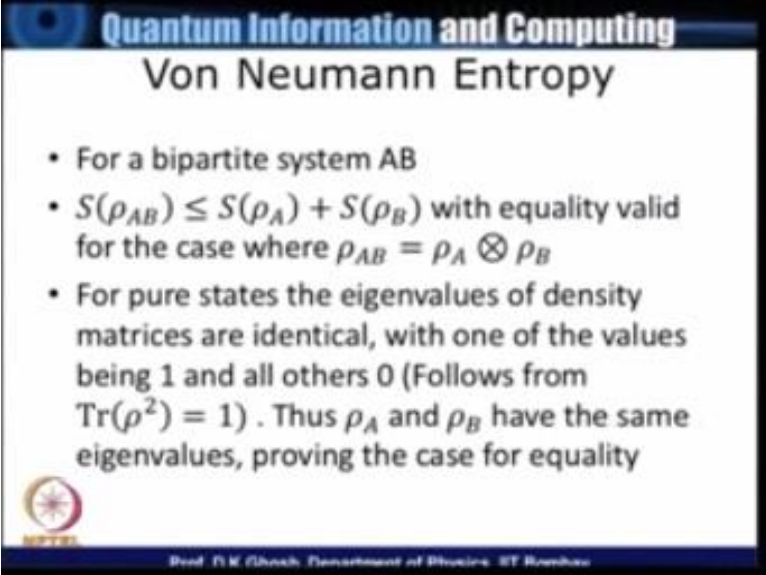
The image shows a slide with handwritten mathematical notes. In the top right corner, there is a small number '- 3-'. The main content consists of three lines of text:

$$s(s) = \log D.$$
$$s(p_{AB}) \leq s(p_A) + s(p_B)$$

Equality  $p_{AB} = p_A \otimes p_B$


Maximum log of D.

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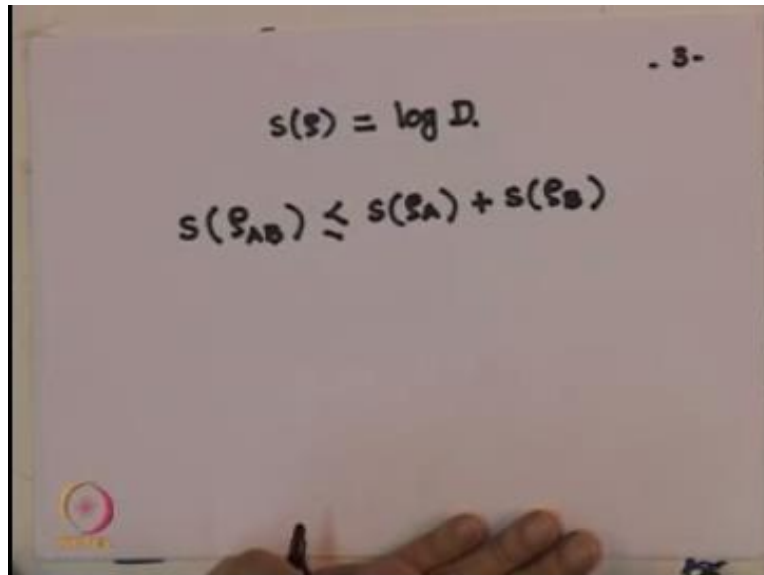
**Quantum Information and Computing**  
**Von Neumann Entropy**

- For a bipartite system AB
- $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$  with equality valid for the case where  $\rho_{AB} = \rho_A \otimes \rho_B$
- For pure states the eigenvalues of density matrices are identical, with one of the values being 1 and all others 0 (Follows from  $\text{Tr}(\rho^2) = 1$ ). Thus  $\rho_A$  and  $\rho_B$  have the same eigenvalues, proving the case for equality

  
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There is another property. Which is valid for a bipartite system multi qubit systems we have discussed and so what we are saying is this.

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A photograph of a whiteboard with handwritten mathematical equations. The equations are:  $s(S) = \log D.$  and  $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$ . The whiteboard has a small logo in the bottom left corner and the number '- 3-' in the top right corner. A hand is visible at the bottom of the frame, holding a pen.

$$s(S) = \log D.$$
$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

That entropy for a composite system  $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$  once again I will not be proving these, but I will illustrate their validity via an example.

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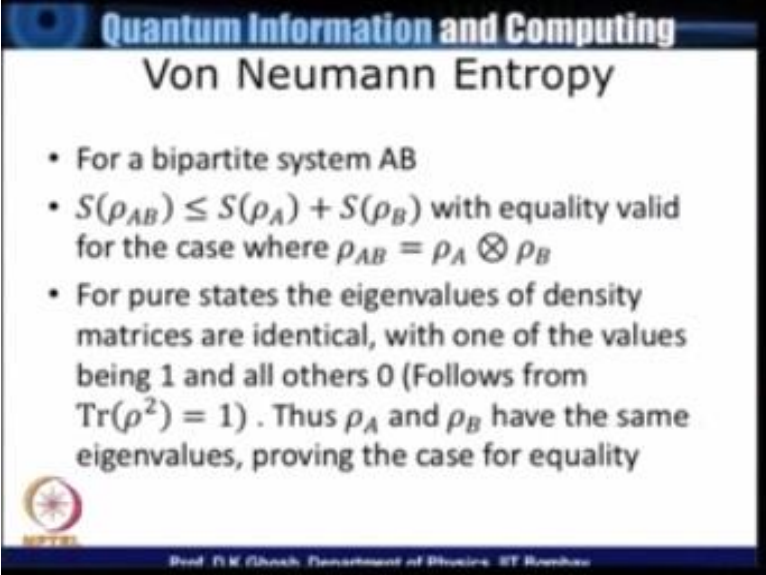
The image shows a whiteboard with handwritten mathematical formulas. At the top right, there is a small number '- 3-'. The main text consists of three lines: the first line is  $s(S) = \log D$ ; the second line is  $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$ ; and the third line is 'Equality  $\rho_{AB} = \rho_A \otimes \rho_B$ '. A hand is visible at the bottom of the whiteboard, and there is a small logo in the bottom left corner.

$$s(S) = \log D$$
$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

Equality  $\rho_{AB} = \rho_A \otimes \rho_B$


The equality holds if my  $\rho_{AB}$  for the multi qubit case or two qubit situation in this case is  $\rho_A \rho_B$  factorize. Now you can see why this at least should be true.

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**Quantum Information and Computing**  
**Von Neumann Entropy**

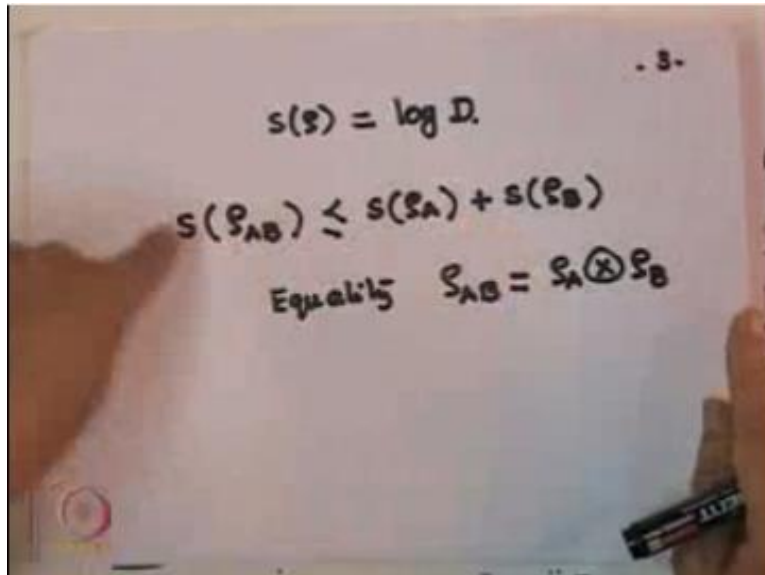
- For a bipartite system AB
- $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$  with equality valid for the case where  $\rho_{AB} = \rho_A \otimes \rho_B$
- For pure states the eigenvalues of density matrices are identical, with one of the values being 1 and all others 0 (Follows from  $\text{Tr}(\rho^2) = 1$ ). Thus  $\rho_A$  and  $\rho_B$  have the same eigenvalues, proving the case for equality

  
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So you know that for pure states the Eigen values of the density matrices is an identical with the one of the values being equal to 1 and all others being equal to 0.



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The image shows a whiteboard with handwritten mathematical formulas. At the top right, there is a small number "- 3-". The main text consists of three lines: the first line is  $s(\rho) = \log D$ ; the second line is  $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$ ; and the third line is "Equality  $\rho_{AB} = \rho_A \otimes \rho_B$ ". A hand is visible on the left side, pointing towards the second line, and another hand is on the right side, holding a black marker.

Just  $\rho_A \rho_B$  must have their same Eigen values so therefore equal this is the standard mixing theorem that is when you mix this system SA and SB the  $S(\rho_A \rho_B)$  becomes less than equal to this. As I said there will be very few regress proof I will get for the Von Neumann entropy let me give you a few examples for this consider a pure state  $\rho$ .

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$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad S(\rho) = 0$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$S = |\psi\rangle\langle\psi|$$

$$= \frac{1}{2} \left[ \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

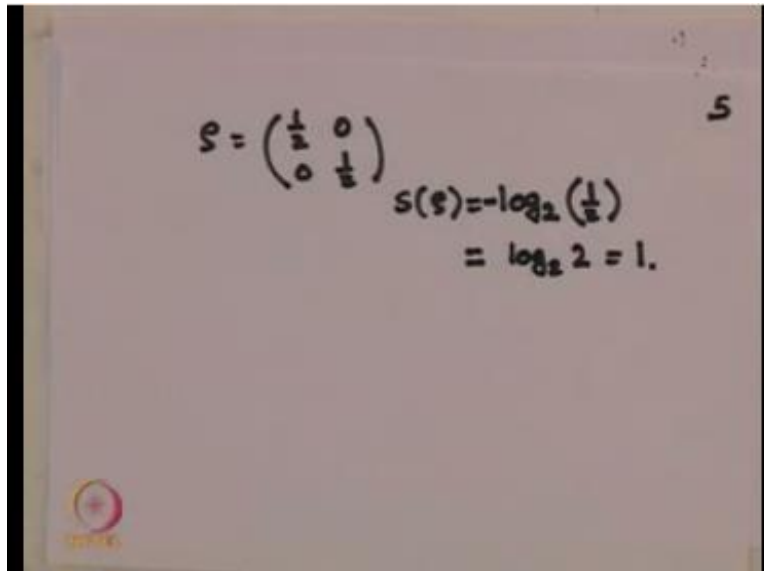
$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S(\rho) = 0.$$

So I know that the Eigen values are 1 and 0 so supposing I go we are representation in which design so obviously this we so I know of course  $S(\rho) = 0$ . Now consider a state  $|\psi\rangle = 1/\sqrt{2} (|0\rangle + |1\rangle)$  remember the diagonal states that we had talked about. So my rho here so this is the pure state is  $|\psi\rangle\langle\psi|$  and this is equal to if I go over to the matrix representation I have  $1/\sqrt{2}$  twice at the herein so it is  $1/2$  remember that  $|0\rangle$  is 1, 0 and  $|1\rangle$  is 0, 1.

Now the corresponding bra's are 1, 0 row etc. So I get 1, 0 + 0, 1 which is nothing but 1, 1 and this is of course 1,1 because this is just the diagonal bases. Which then is  $1/2$  if we multiply these two you get 1, 1, 1, 1. So you can see immediately that the Eigen value of this matrix will be 1 and 0 so once again  $S(\rho) = 0$ . But consider a next cube, for example the maximum Linux state which we have discuss several times has supposing.

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The image shows a whiteboard with handwritten mathematical work. On the left, a probability distribution  $\mathcal{P}$  is defined as a vector:  $\mathcal{P} = \left( \frac{1}{2} \ 0 \right)$ . To the right, the entropy  $S(\mathcal{P})$  is calculated as  $-\log_2\left(\frac{1}{2}\right)$ , which simplifies to  $\log_2 2 = 1$ . A small number '5' is written in the top right corner of the whiteboard. A logo is visible in the bottom left corner.

$$\mathcal{P} = \left( \frac{1}{2} \ 0 \right)$$
$$S(\mathcal{P}) = -\log_2\left(\frac{1}{2}\right)$$
$$= \log_2 2 = 1.$$

$\mathcal{Q} = \frac{1}{2}, 0, 0, \frac{1}{2}$   $S(\mathcal{Q})$  you can easily immediately  $\log_2(1/2) - \frac{1}{2} (\log 2) \frac{1}{2}$  so that adds up to one, so this is equal to simply  $\log_2 2 = 1$ . Now let me give you an illustration, of that inequality that I talked about. Let us consider the following situation.

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$$|\psi\rangle = \cos\theta |00\rangle + \sin\theta |11\rangle$$

$$\rho^A = \text{Tr}_B(\rho_{AB})$$

$$\rho_{AB} = (\cos\theta |00\rangle + \sin\theta |11\rangle)(\cos\theta \langle 00| + \sin\theta \langle 11|)$$

$$= \cos^2\theta |00\rangle\langle 00| + \sin^2\theta |11\rangle\langle 11| + \cos\theta \sin\theta (|00\rangle\langle 11| + |11\rangle\langle 00|)$$

Consider a state which is  $\cos\theta |00\rangle + \sin\theta |11\rangle$  2 qubit state, well since this is the pure state I can immediately calculate what is the entropy of these two states and find out that the entropy = 0. So let us look at what happens to my  $\rho$  here. So in this case look at what is  $\rho^A$  okay like this is my bit A this is my bit B, remember how I calculate  $\rho$ , I calculate  $\rho^A$  by taking a partial trace over B of  $\rho_{AB}$ .

And what is  $\rho_{AB}$ ?  $\rho_{AB}$  is simply this get multiplied by the corresponding value. So let us write  $\rho_{AB} = \cos\theta |00\rangle + \sin\theta |11\rangle$  multiplied by  $\cos\theta \langle 00| + \sin\theta \langle 11|$ . So I can write down this as  $= \cos^2\theta |00\rangle\langle 00| + \sin^2\theta |11\rangle\langle 11| + \cos\theta \sin\theta (|00\rangle\langle 11| + |11\rangle\langle 00|)$ . Now if I take a trace of let us say B remember what we said, so B is this second bit, we said that when you take a trace of a ket followed by a bra all that you get is the product of the bar with the cketat, do you know product of bra with a ket. So if I am taking a trace.

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$$\begin{aligned}
 |\psi\rangle &= \cos\theta |00\rangle + \sin\theta |11\rangle \\
 \rho^A &= \text{Tr}_B(\rho_{AB}) \\
 \rho_{AB} &= (\cos\theta |00\rangle + \sin\theta |11\rangle)(\cos\theta \langle 00| + \sin\theta \langle 11|) \\
 &= \cos^2\theta |00\rangle\langle 00| + \sin^2\theta |11\rangle\langle 11| \\
 &\quad + \cos\theta \sin\theta (|00\rangle\langle 11| + |11\rangle\langle 00|) \\
 \text{Tr}_B \rho_{AB} &= \cos^2\theta |0\rangle\langle 0| + \sin^2\theta |1\rangle\langle 1|
 \end{aligned}$$

I am going to get equal to  $\cos^2\theta |0\rangle\langle 0|$  the B that is spaced out but I have 0 here and a 0 here so that I get  $1 + \sin^2\theta |1\rangle\langle 1|$  these terms will give a zero because when I am taking a trace over B I have trace out 0, 1 which is the scalar product of 1 with 0 or in this case is 0 with 1 both of them become equal to 0. So if I now, now that I have got a one qubit situation I can express  $\rho_A$ .

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The image shows a whiteboard with handwritten mathematical expressions. At the top right, there is a small number '7'. The main content consists of three lines of equations:

$$S_A = \begin{pmatrix} \cos^2 \theta & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$
$$S^A = -\cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)$$
$$= -2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$$

A hand holding a black marker is visible at the bottom of the whiteboard, positioned as if writing the final line of the equation.

Which is by this, so this is nothing but  $\cos^2 \theta$  and  $\sin^2 \theta$  which is of course a proper density matrix because you can see that the  $Q_A$  adds up to 1. So what happens to my entropy in this case  $S^A$  so my  $S^A$  is the Eigen values since it is already in a diagonal form Eigen value are  $\cos^2 \theta$   $\sin^2 \theta$  so I get  $S^A = \cos^2 \theta - \text{sign in front of it logarithm}(\cos^2 \theta) - \sin^2 \theta \text{ logarithm}(\sin^2 \theta)$  which is equal to.

So recall what is it that I am doing here this is nothing but  $2 \log(\cos \theta)$  so I get  $-2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$ . Now I will obviously get a similar expression, identical expression where I consider  $Q_B$  which is obtained from  $Q_{AB}$  by tracing out A because the expressions you have absolutely symmetrical there is no reason to expect that my  $S^B$  would be any different.

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7

$$\rho_A = \begin{pmatrix} \cos^2 \theta & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$
$$S^A = -\cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)$$
$$= -2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$$
$$S^B = -2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$$
$$S^A + S^B = -4 [\cos^2 \theta \log(\cos \theta) + \sin^2 \theta \log(\sin \theta)]$$
$$> 0$$

So my  $S^B$  will also be  $-2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$ . And so therefore my  $S^A + S^B$  when I just add them up I get  $-4[\cos^2 \theta \log(\cos \theta) + \sin^2 \theta \log(\sin \theta)]$  remember the  $\cos \theta$  and  $\sin \theta$  are less than 1 so the logarithm turns out to be negative but there is a minus sign in front of it which will make it greater than 0, but the pure state  $\rho_{AB}$  the entropy was 0, so that sort of tells you that.

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$$S_A = \begin{pmatrix} \cos^2 \theta & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$
$$S^A = -\cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)$$
$$= -2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$$
$$S^B = -2 \cos^2 \theta \log(\cos \theta) - 2 \sin^2 \theta \log(\sin \theta)$$
$$S^A + S^B = -4 [\cos^2 \theta \log(\cos \theta) + \sin^2 \theta \log(\sin \theta)]$$
$$> 0$$

The entropy of the mixed state is less than the sum of the entropies of the two component pure state. Having done this we would proceed to the quantum communication I would not have time to discuss the reason why Von Neumann entropy become significant for quantum communication because a theorem very similar to the Shannon entropy case for noiseless coding is applicable for the compression of quantum colloquia.

There are many theorems parallel theorems for quantum systems but that would take us outside the scope of this course and in the next lecture I will be talking about different protocols for quantum communication.

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