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Quantum Information and Computing

Prof. D.K.Ghosh
Department of Physics IIT Bombay

Modul No.07

Lecture No.37

Von Neumann Entropy

In the last lecture we have been talking about Shannon entropy which is basically a classical information system which tells me what is the optimal compression that I can get.

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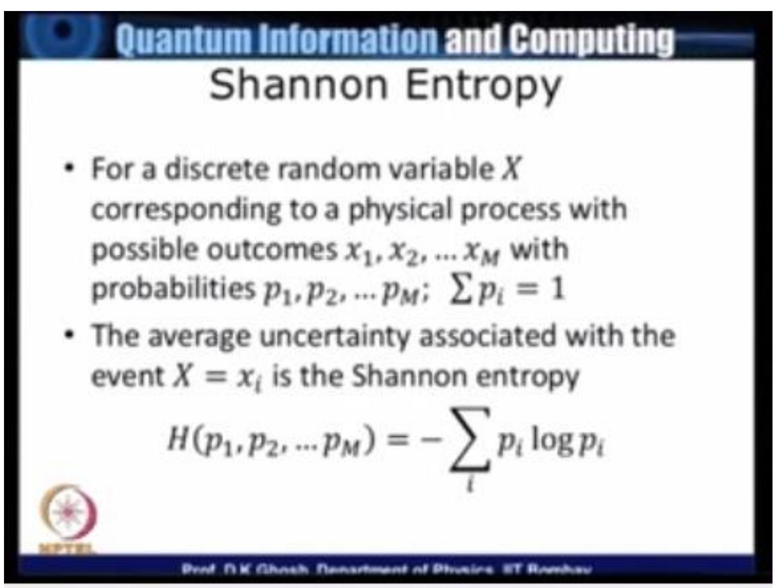

And if just to summaries what we said is supposing we have a random variable x corresponding to a physical process with has possible outcome which has possible outcomes x_1, x_2 up to x_M with respective probabilities P_1,P_2 up to P_M subject of course Σ P to 1 we said that the average uncertainty associated with the event is what is called as the Shannon entropy and has an expression which is $\Sigma_i P_i \log P_i$ we mentioned that traditionally it is conversational to take the base of the logarithm to be equal to 2.

The second point that we notice is that this definition does not depend up on the value that the random variable takes but rather or then on probability with which a particular event occur. We also said that Shannon entropy gives a lower bound on the uncertainty associated with an event and if the length of a uniquely describable prefix code happens to be n_i with the probability P_i then the length of the word should be ≥ the Shannon's probability function H.

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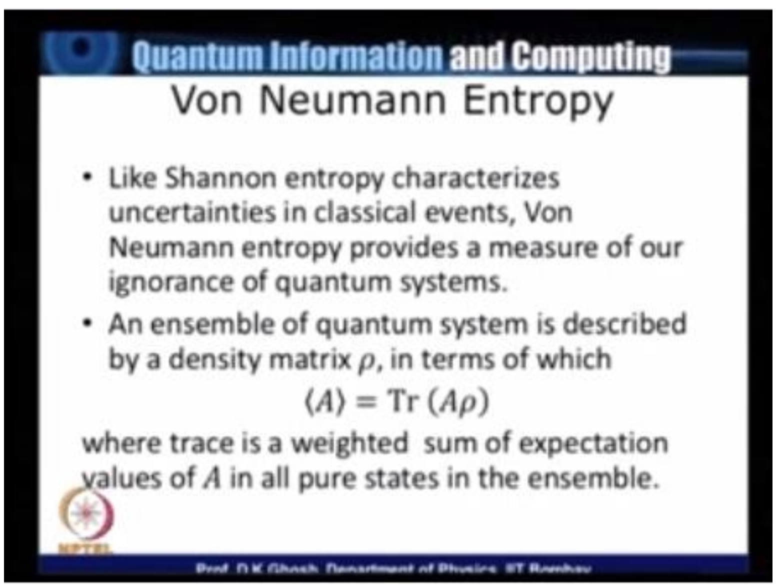

What Von Neumann did is to generalize this to the case of quantum systems and it did it by an argument which we will not go to but what happens is like Shannon entropy characterizes uncertainties of classical event, Von Neumann entropy provides a measure of our ignorance or uncertainties of the quantum systems like in Shannon case the situation that would look at is not that a pure system but of an ensemble described by a density matrix ρ we have had.

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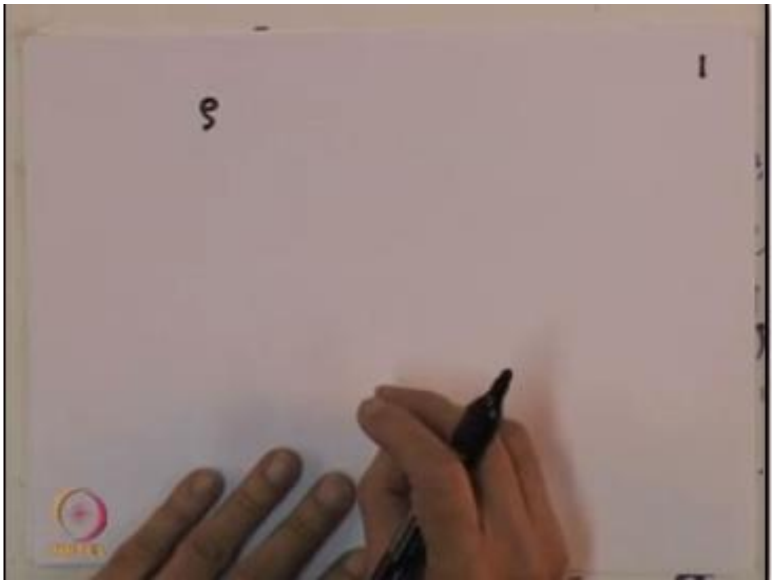

Quite a bit of a discussion on the property density matrix.

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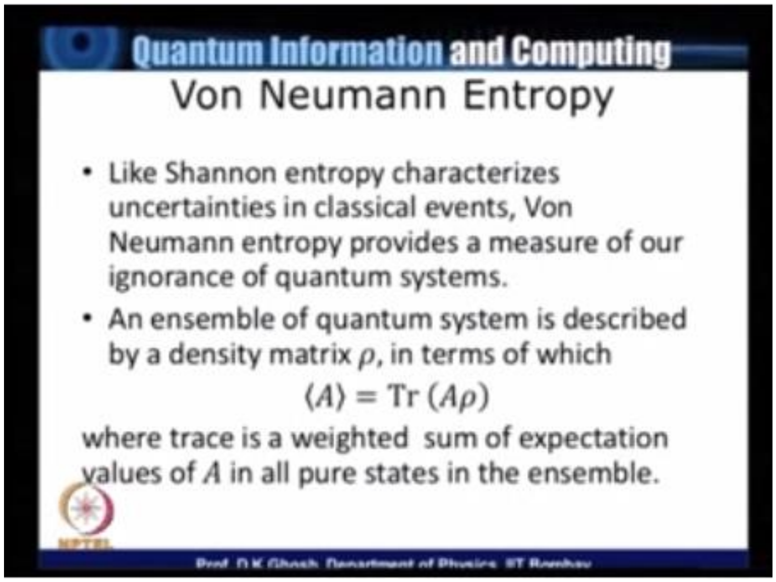

And we have said if I have an observable A.

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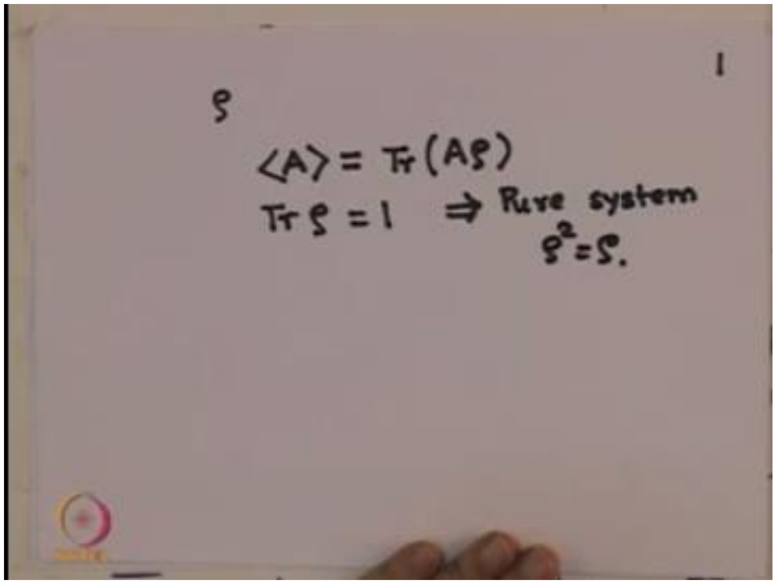

The expectation value of the operator A is given any Tr of A with ϱ where ϱ is the density matrix and that the density matrix as a property that Tr ϱ =1 for a pure system we had seen that ϱ^2 must be also = ϱ.

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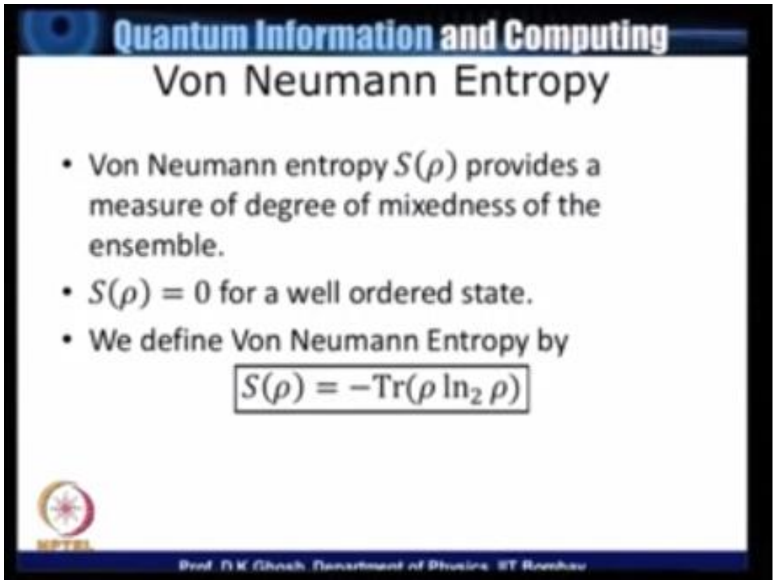

So Von Neumann entropy S(ϱ) provides a measure of the degree of mixedness in a ensemble so firstly I require for a pure system like in case of Shannon entropy for a pure system.

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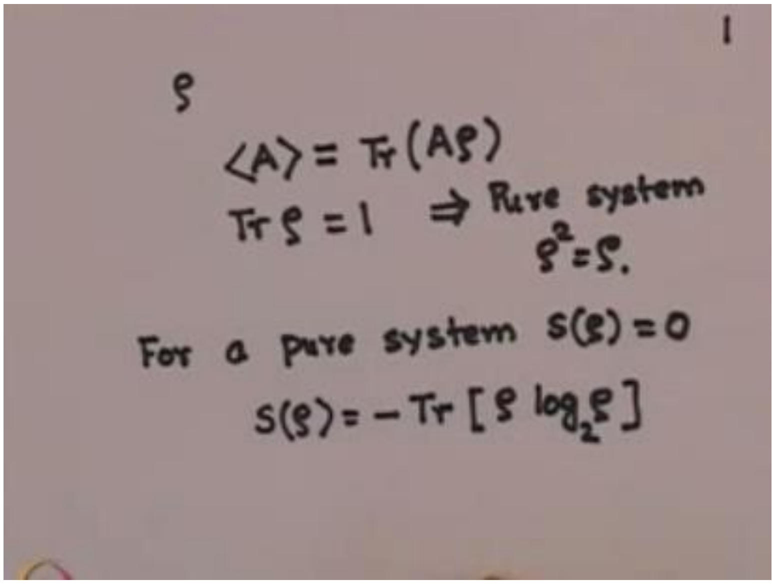

$S(\varrho)=0$ Von Neumann took his use of Shannon discussion of entropy and he defines the entropy if it is now carries his name as equal to minus remember it was $-\Sigma_i$ Ti log P_i log P_i so this is –Tr [ϱ log ϱ] and as before I can take the base of logarithm what I want but they as I said the traditionally you will take the base is 2.

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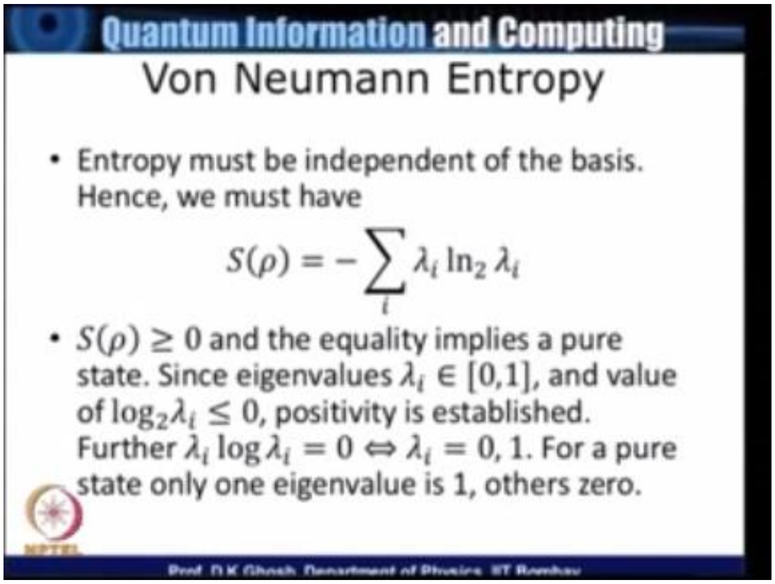

So let us look at what are the properties of this system like this. First thing is since entropy must be independent of the bases in which you are calculating because is that Tr so therefore I could go over to a representation in which ϱ is diagonal.

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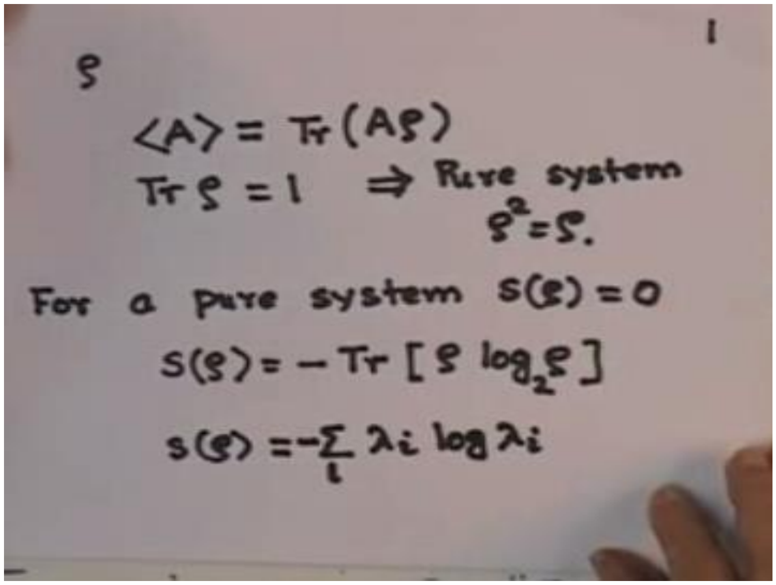

And then my S(ϱ) would then be given by $-\Sigma_i \lambda_i \log \lambda$ there are few properties of this which we will discuss. So of course from the definition it becomes clear that your S(ϱ) is equal to 0 for a pure system and the reason is the following if you take.

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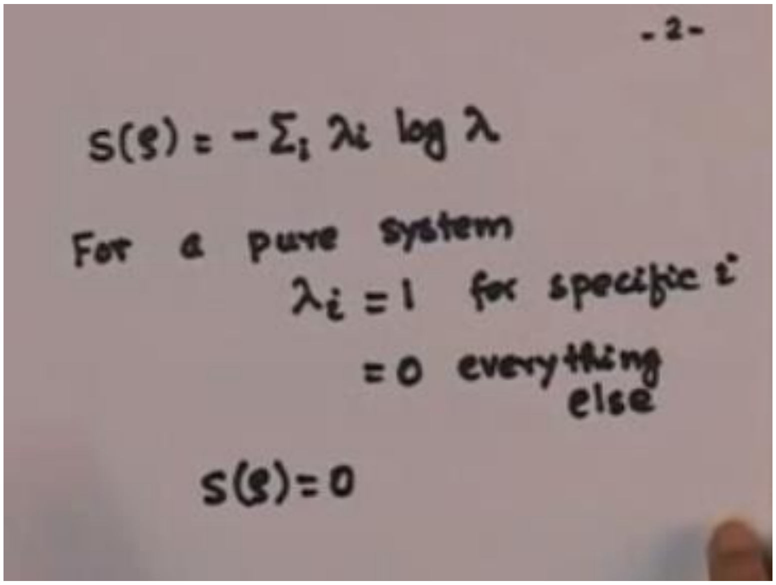

$S(\varrho)$ to be equal to $-\Sigma_i \lambda_i \log \lambda_i$ recall that for a pure system one of my λ_i is must equal to one and all other is 0. So therefore, I get -1 log 1 -0 log 0 but 0 log 0 is to be understood in as a limit which is equal to 0 so therefore, for a pure system may be a particular λ_i =1 for specific i and is equal to 0 for everything else. So therefore, $S(\varrho)$ =0 that is one property which we wanted and that is fine.

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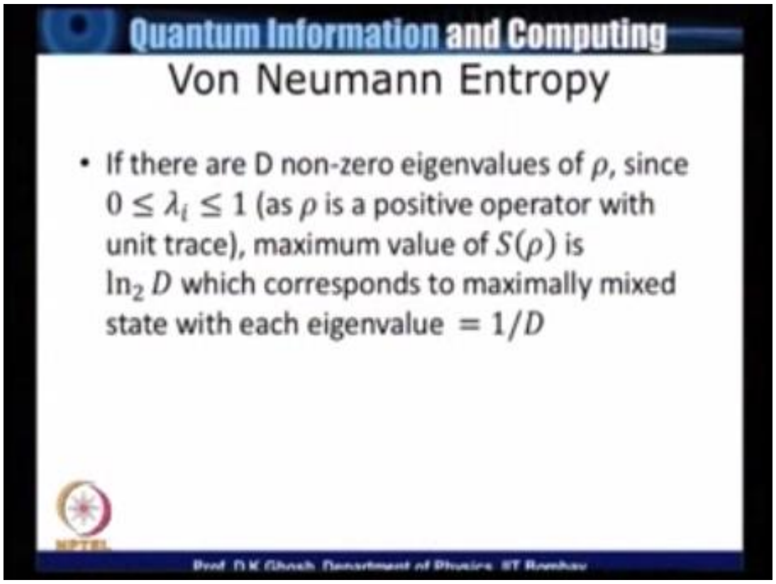

Now what is the maximum value of the entire this requires a detailed mathematical proof I will not go through it, but since I said I will give an intuitively which is very simple thing to understand since we said that entropy is the measure of the degree of mixedness of the system supposing I am D dimension they have a D dimension then if there are D number of non-zero Eigen values and obviously the maximum mixedness comes for the case where each of the D Eigen values is equal to 1/D.

And that is because the sum of the Eigen value of the density matrix must be equal to 1 because Tr of ϱ is equal to 1 so therefore I must have the S(ϱ).

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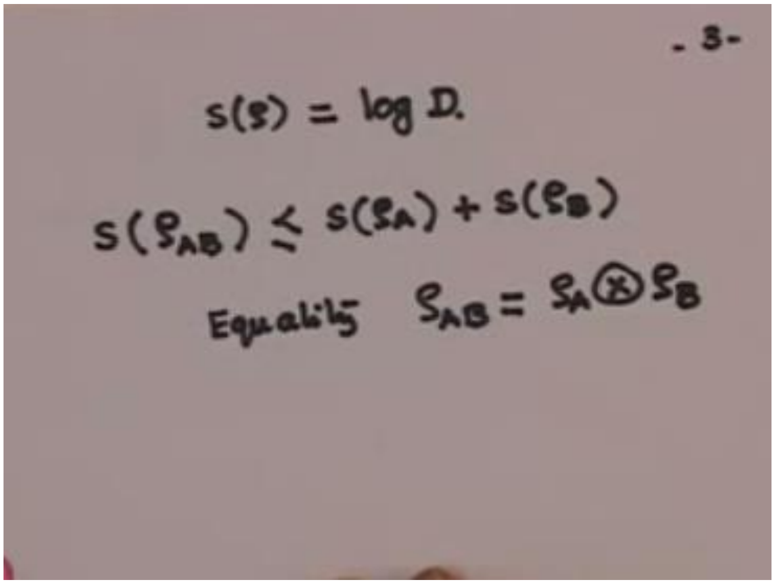

Maximum log of D.

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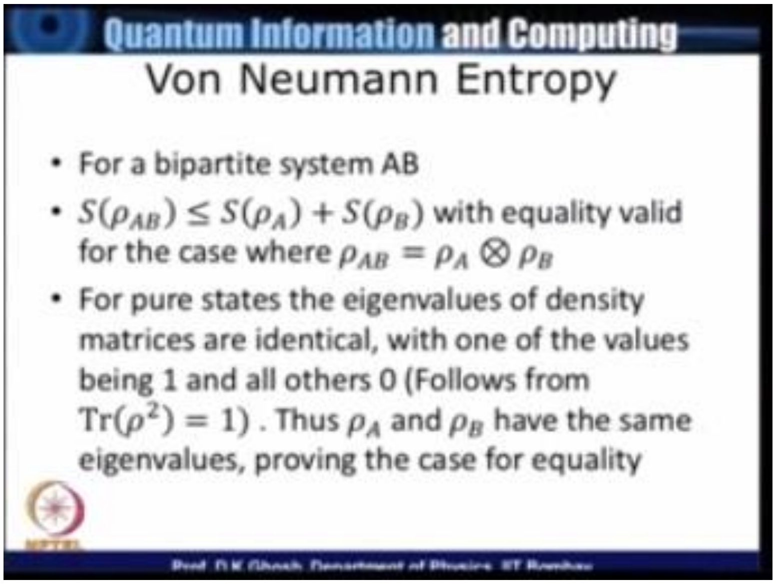

There is another property. Which is valid for a bipartite system multi qubit systems we have discussed and so what we are saying is this.

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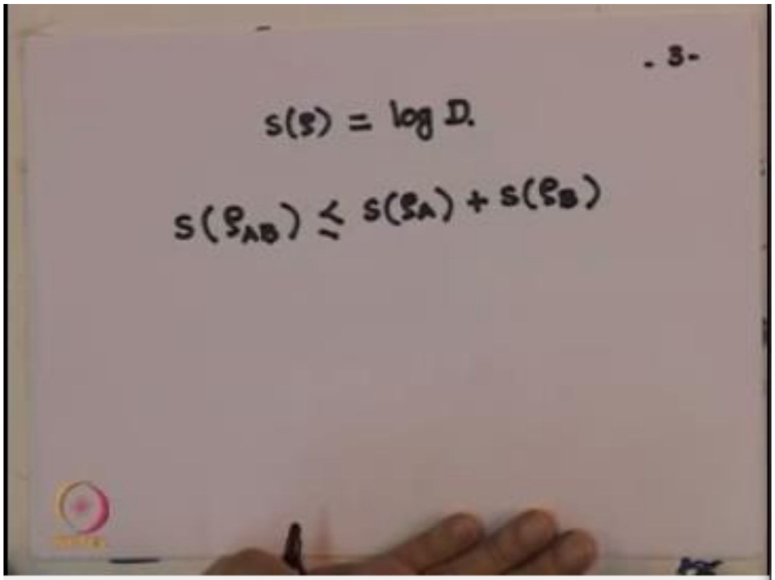

That entropy for a composite system $\varrho_{AB} \leq S(\varrho_A)+ s(\varrho_B)$ once again I will not be proving these, but I will illustrate them there validity vise on example.

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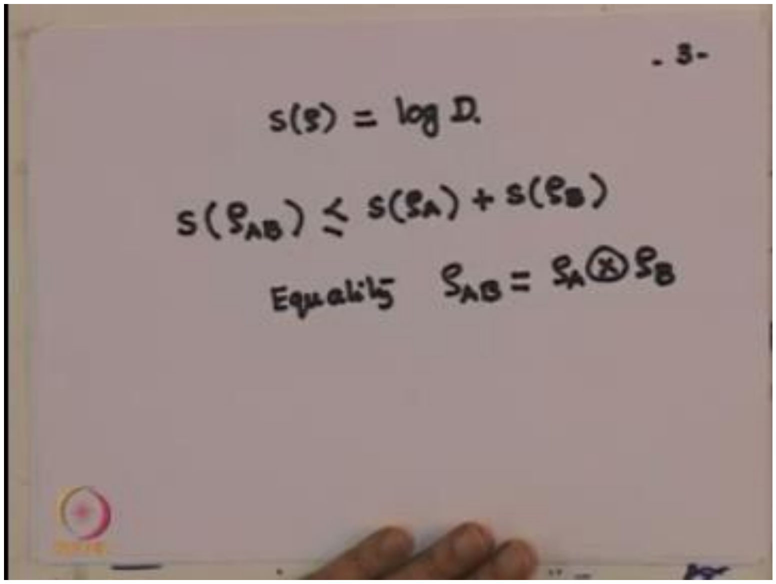

The equality holds if my ϱ_{AB} for the multi qubit case or two qubit situation in this case is ϱ_A ϱ_B factorize. Now you can see why this at least should be true.

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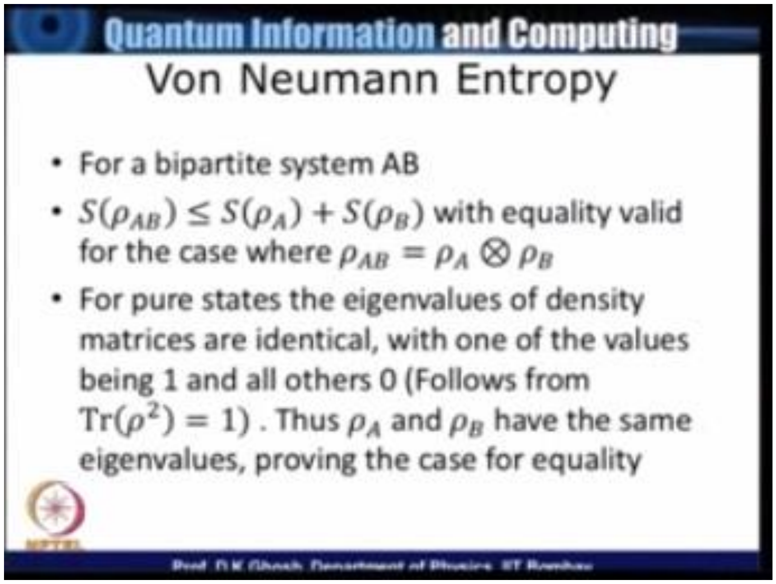

So you know that for pure states the Eigen values of the density matrices is an identical with the one of the values being equal to 1 and all others being equal to 0.

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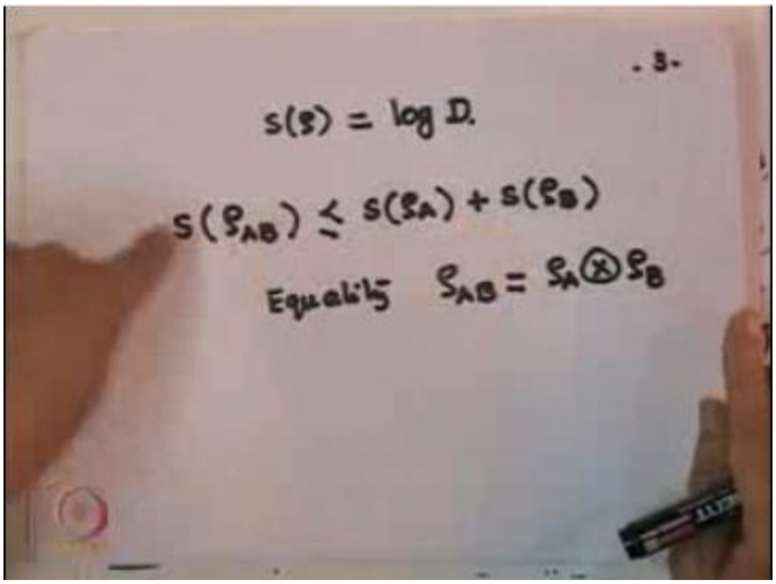

Just ϱ_A ϱ_B must have their same Eigen values so therefore equal this is the standard mixing theorem that is when you mix this system SA and SB the S(ϱ_A ϱ_B) becomes less than equal to this. As I said there will be very few regress proof I will get for the Von Neumann entropy let me give you a few examples for this consider a pure state ϱ.

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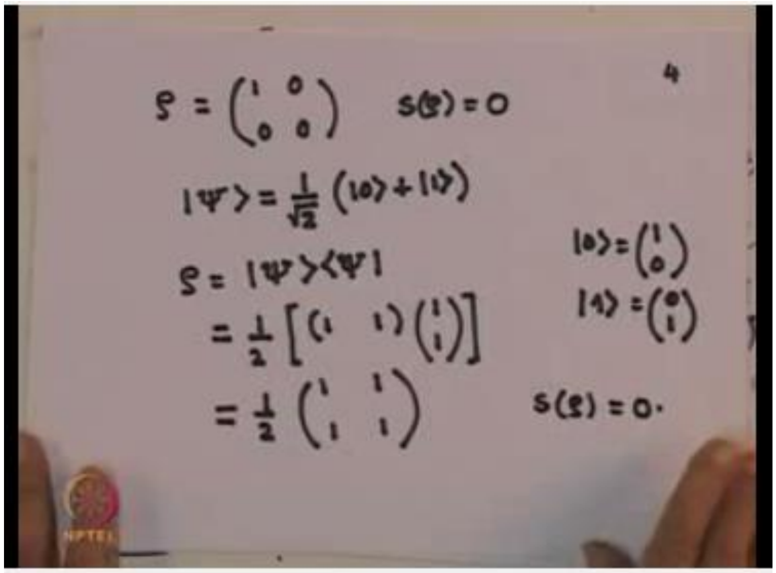

So I know that the Eigen values are 1 and 0 so supposing I go we are representation in which design so obviously this we so I know of course $S(\varrho) = 0$. Now consider a state $|\psi\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle)$ remember the diagonal states that we had talked about. So my rho here so this is the pure state is $|\psi\rangle\langle\psi|$ and this is equal to if I go over to the matrix representation I have $1/\sqrt{2}$ twice at the herein so it is ½ remember that $|0\rangle$ is 1, 0 and $|1\rangle$ is 0, 1.

Now the corresponding bra's are 1, 0 row etc. So I get 1, 0 + 0, 1 which is nothing but 1, 1 and this is of course 1,1 because this is just the diagonal bases. Which then is ½ if we multiply these two you get 1, 1, 1, 1. So you can see immediately that the Eigen value of this matrix will be 1 and 0 so once again $s(\varrho) = 0$. But consider a next cube, for example the maximum Linux state which we have discuss several times has supposing.

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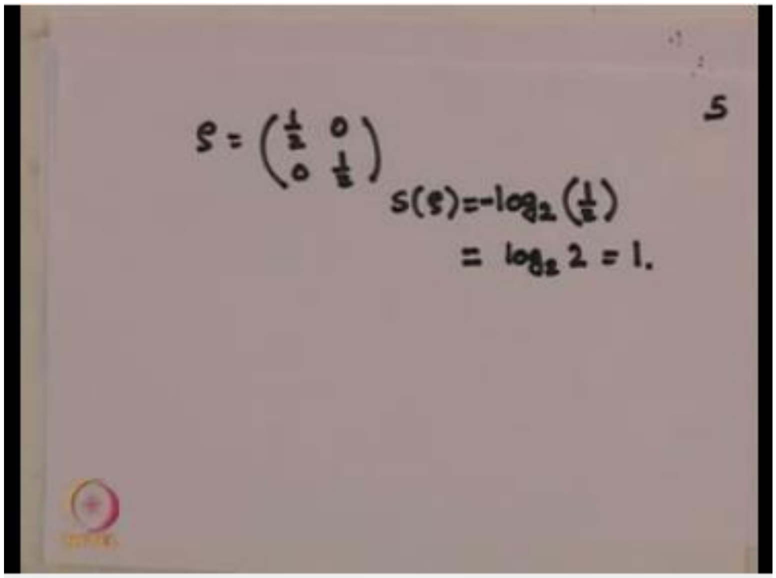

ϱ = ½, 0, 0, ½ s(ϱ) you can easily immediately $\log_2(1/2)$ – ½ (log 2) ½ so that adds up to one, so this is equal to simply $\log_2 2 = 1$. Now let me give you an illustration, of that inequality that I talked about. Let us consider the following situation.

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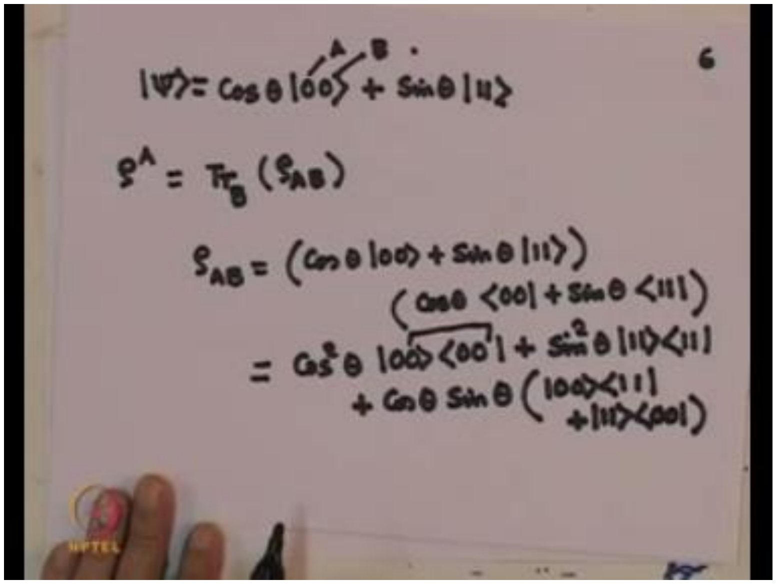

Consider a state which is Cos θ |00| + sin θ |11| 2 qubit state, well since this is the pure state I can immediately calculate what is the entropy of these two states and find out that the entropy = 0. So let us look at what happens to my ϱ here. So in this case look at what is ϱ^A okay like this is my bit A this is my bit B, remember how I calculate ϱ, I calculate ϱ^A by taking a partial trace over B of ϱ_{AB}.

And what is ϱ_{AB} ? ϱ_{AB} is simply this get multiplied by the corresponding value. So let us write ϱ_{AB} = cos θ |00| + sin θ |11|multiplied by cos θ <00|+sin θ<11|. So I can write down this as = $\cos^2$ θ |00><00| + $\sin^2$ θ |11><11| + cos θ sin θ (|00><11| + |11><00|). Now if I take a trace of let us say B remember what we said, so B is this second bit, we said that when you take a trace of a ket followed by a bra all that you get is the product of the bar with the cketat, do you know product of bra with a ket. So if I am taking a trace.

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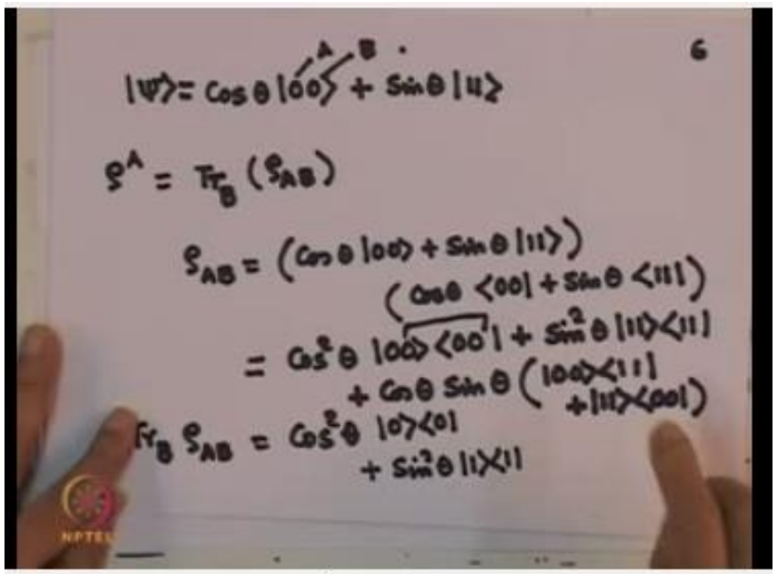

I am going to get equal to $\cos^2 \theta$ |0><0| the B that is spaced out but I have 0 here and a 0 here so that I get 1+ $\sin^2 \theta$ |1><1| these terms will give a zero because when I am taking a trace over B I have trace out 0, 1 which is the scalar product of 1 with 0 or in this case is 0 with 1 both of them become equal to 0. So if I now, now that I have got a one qubit situation I can express ϱ_A.

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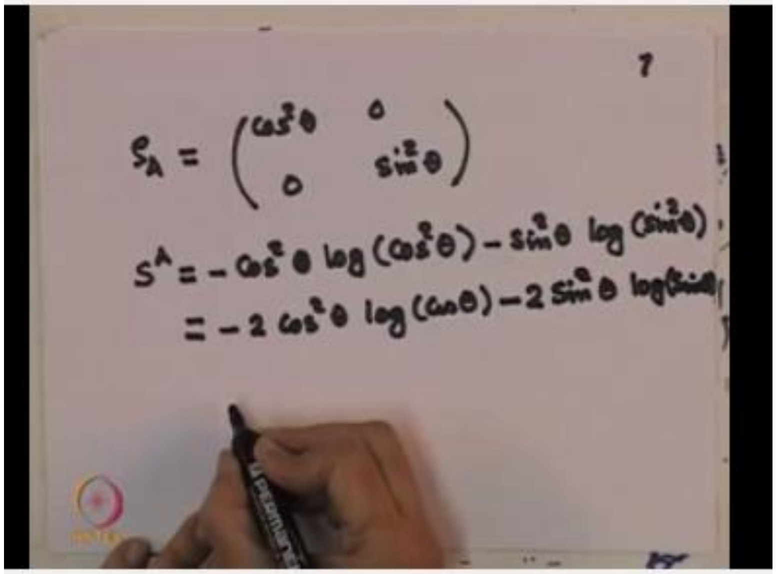

Which is by this, so this is nothing but $\cos^2\theta$ and $\sin^2\theta$ which is of course a proper density matrix because you can see that the ρ_A adds up to 1. So what happens to my entropy in this case S^A so my S^A is the Eigen values since it is already in a diagonal form Eigen value are $\cos^2\theta$ $\sin^2\theta$ so I get $S^A = \cos^2\theta$ - sign in front of it logarithm ($\cos^2\theta$) - $\sin^2\theta$ logarithm ($\sin^2\theta$) which is equal to.

So recall what is it that I am doing here this is nothing but 2 log ($\cos\theta$) so I get $-2\cos^2\theta \log(\cos\theta) - 2\sin^2\theta \log(\sin\theta)$. Now I will obviously get a similar expression, identical expression where I consider ρ_B which is obtained from ρ_{AB} by tracing out A because the expressions you have absolutely symmetrical there is no reason to expect that my S^B would be any different.

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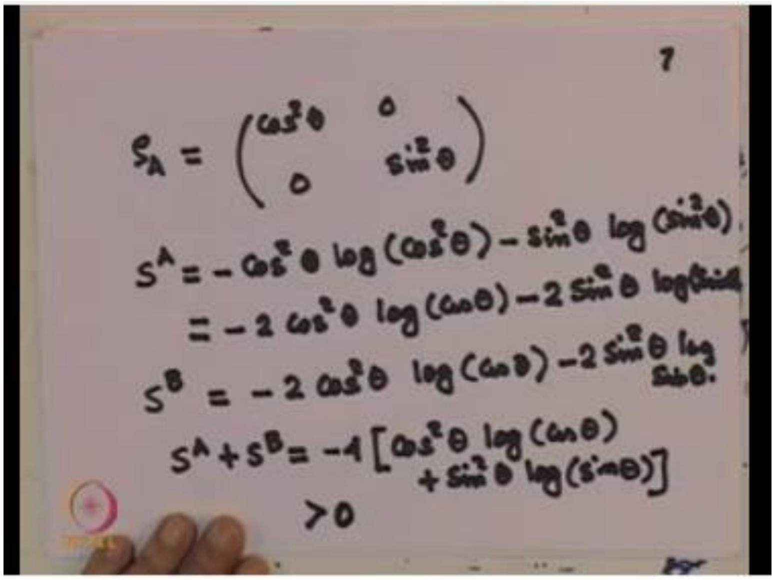

So my S^B will also be $-2 \cos^2 \theta \log (\cos \theta) - 2 \sin^2 \theta \log (\sin \theta)$. And so therefore my $S^A + S^B$ when I just add them up I get $-4[\cos^2 \theta \log (\cos \theta) + \sin^2 \theta \log (\sin \theta)]$ remember the $\cos \theta$ and $\sin \theta$ are less than 1 so the logarithm turns of the negative but there is a minus sign in-front of a which will make it greater than 0, but the pure state ϱ_{AB} the entropy was 0, so the that sort of tells you that.

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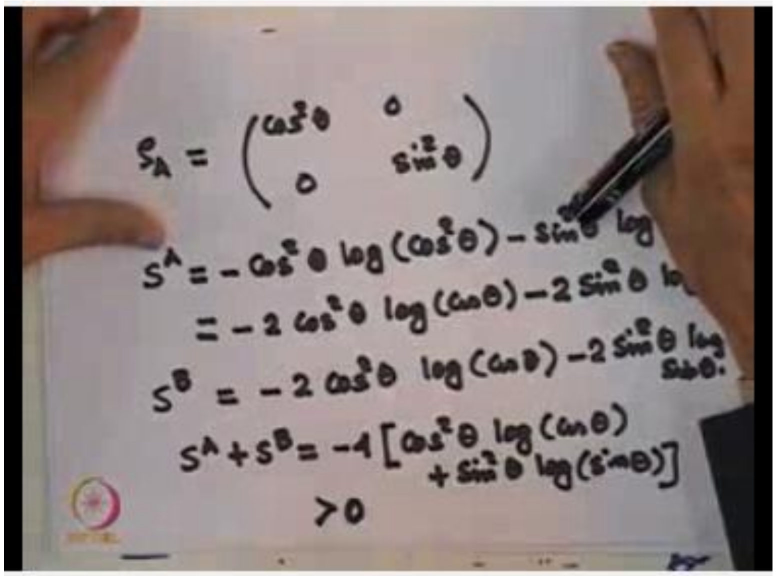

The entropy of the mixed state is less than the sum of the entropies of the two component pure state. Having done this we would proceed to the quantum communication I would not have time to discuss the reason why Von Neumann entropy become significant for quantum communication because a theorem very similar to the Shannon entropy case for noiseless coding is applicable for the compression of quantum colloquia.

There are many theorems parallel theorems for quantum systems but that would take us outside the scope of this course and in the next lecture I will be talking about different protocols for quantum communication.

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IIT Bombay

Prof. R.K. Shevgaonkar

Head CDEEP

Prof. V.M. Gadre

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Tushar Deshpande

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