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Quantum Infromation and Computing

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Modul No.05

Lecture No.25

Implementing QFT

In the previous lecture we had seen how quantum Fourier transform can be used to find out the period of the function actually what we did was to show that if you assume a given period then own there would be only certain states which will would be appearing in your first register in particular what we found is for the case of three qubits if you assume a function with a period of 2 the measurement of the first register will either give you a state 0 or give you a state 4.

They similarly if you had assumed a different period then the one can easily work out what would have been the content of the first register it is not a unique but on the other hand depending upon your value the periodicity of the function you can find out what are the possible results of measurement of the first register.

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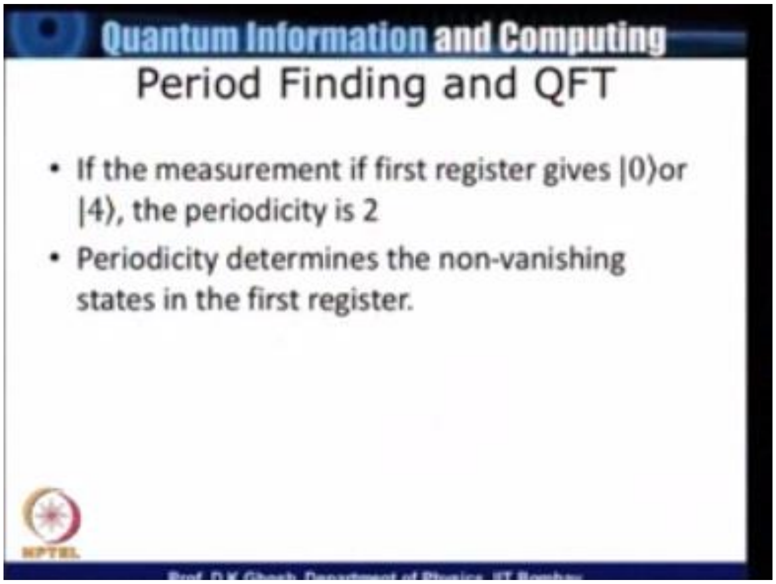

So therefore if in this example that we give you if the first register gives you a 0 or 4 one can conclude that the periodicity of the function is 2, so periodicity determines the non vanishing states in the first register now this is an important information.

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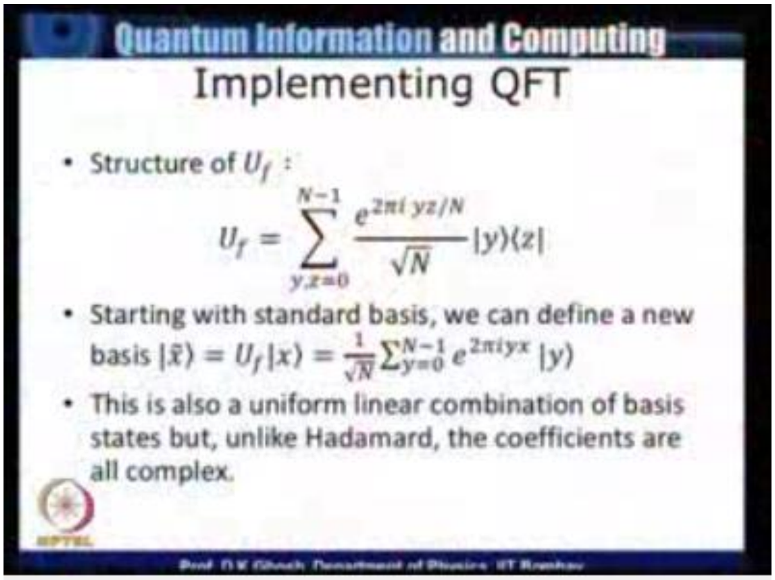

That will be utilizing later on the next question that we want to ask so we have seen that if we can take a Fourier transform this what will happen for a periodic function the natural question is there a unitary transformation which actually will do this job. So let me sort of explain what do I mean by this.

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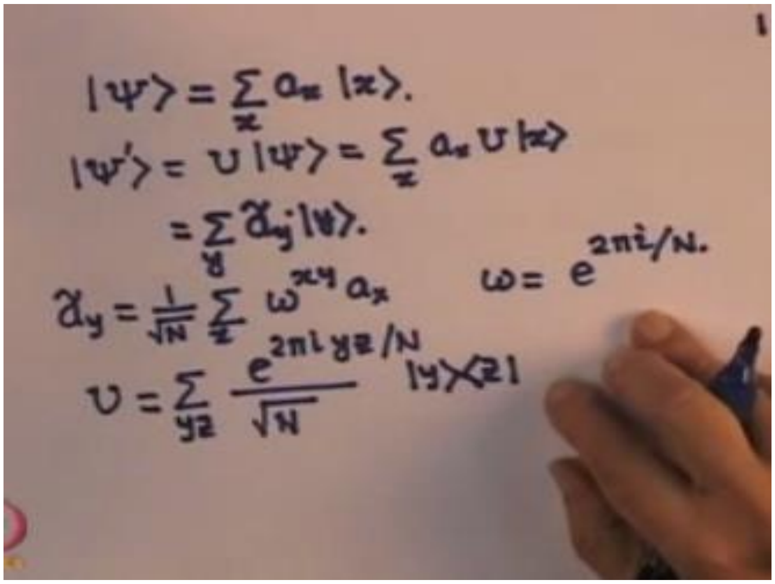

So I had an arbitrary state ψ in the computational basis Σ_x a_x| x sum and we said that my ψ′ is operator U acting on ψ which gives me by linearity Σ_x a_x U_x and that is equal to Σ_y $\tilde{a}_y$ | where you are $\tilde{a}_y$ is 1/ √N Σ_x ω^{xy} a_x where ω^N root of unity and as a result my U has a structure which is Σ_{yz} $e^{2\pi iyz/N}$ /√N and here y›‹ z you can see that this U acting on this here will because z must be than equal to x so therefore you will get the right factor that you want there so this is the type of operator that we are looking for.

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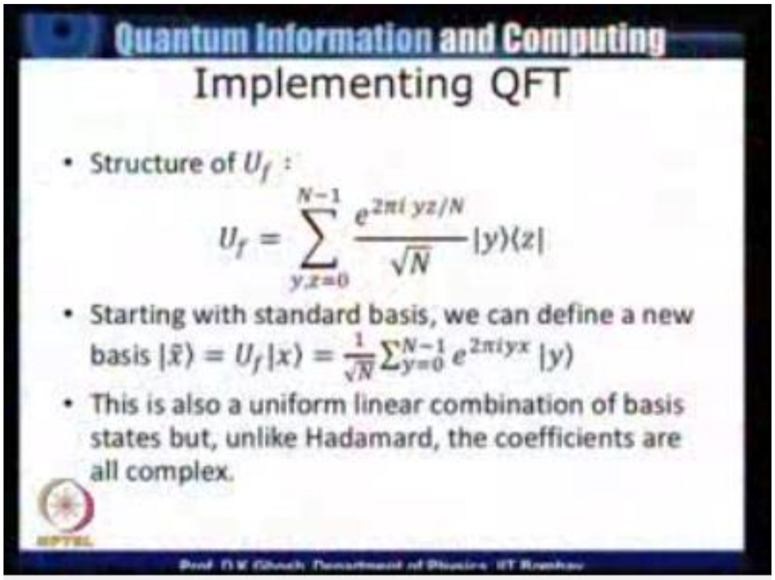

So once I have got a structure of X which is here.

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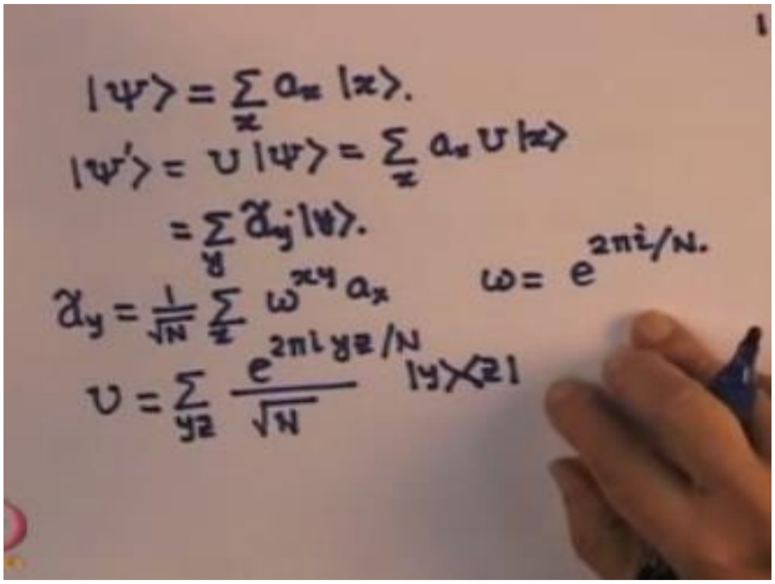

Now notice one thing it tells me that if I started with a standard basis so what happens if I let U.

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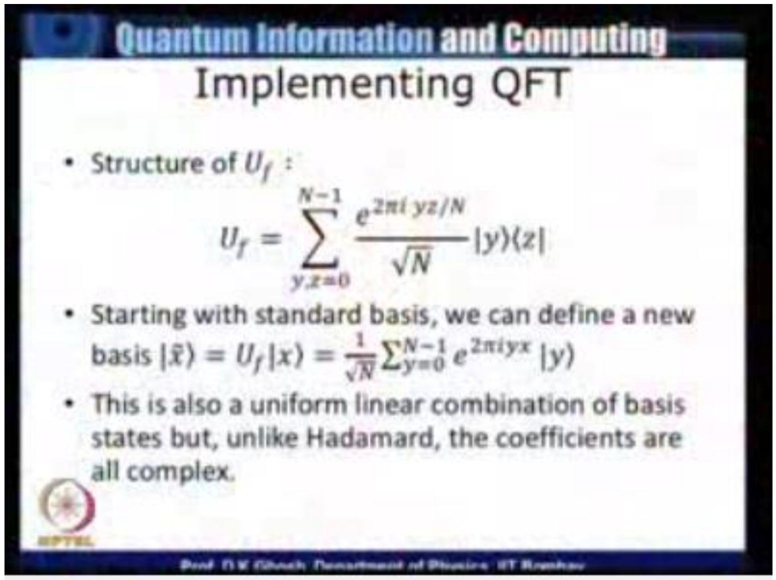

Act on my standard basis so I will get.

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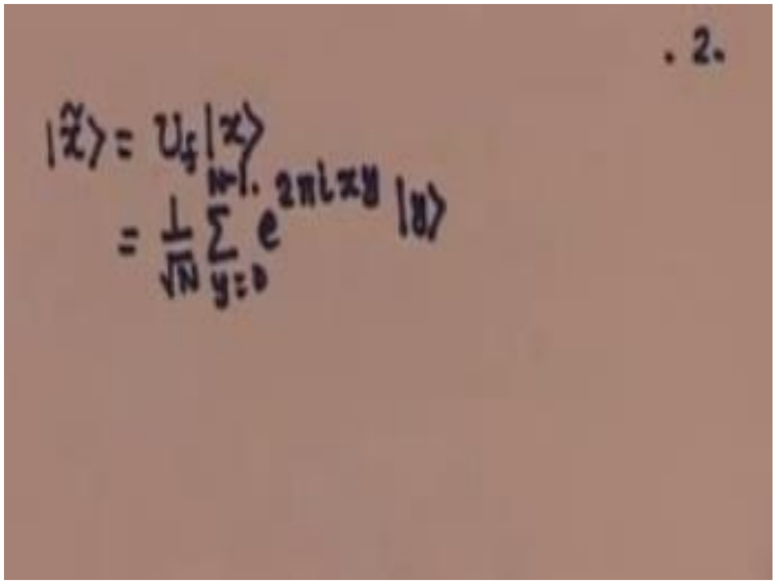

x~ which is my new basis is U_f acting on let us say one of the basis states is x so this will be equal to $1/\sqrt{N}\ \Sigma_{y=0}^{N-1}\ e^{2\pi ixy/}y$ now this is also a linear combination of the basis states but unlike they Hadamard transform this is not the coefficients here are not equivalent and they are not unity and in fact they are complex but these are all uni-modular because $e^{2\pi ixy}$ as has a uni modular the value of modulus is equal to 1.

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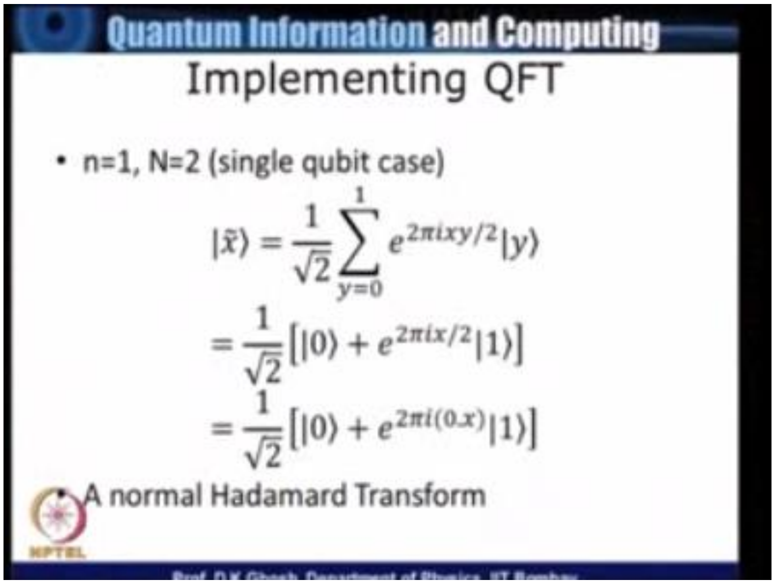

Now let me look at how does one implement the QFT the simplest way of doing it would be to look at systematically what happens for the single qubit case for the two qubit case and then they generalization will be obvious so let me take a single qubit case.

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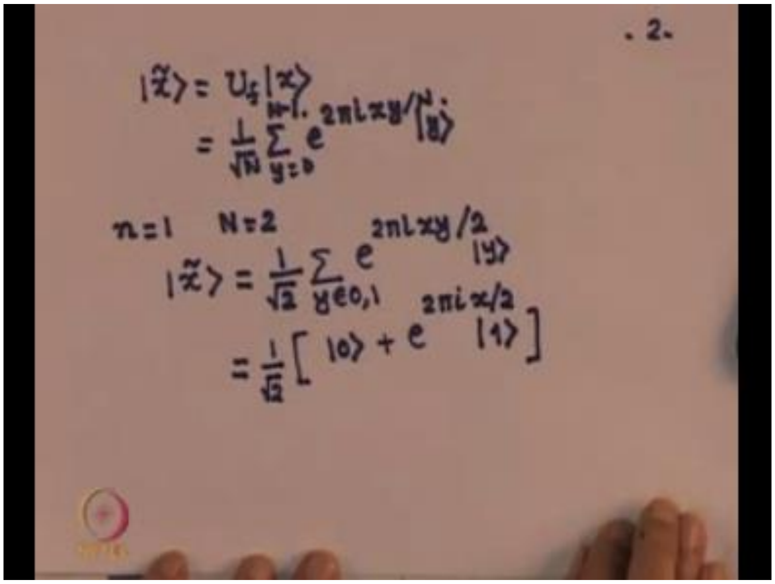

N is equal to one corresponding to N is equal to two now what is the x~ so x~ is 1/√2 Sum over Y is equal to 0 to1 $a^{2\pi ixy/2}$ |y> does n missing there. So since Y takes value 0 and 1 this is 1/√2 |0> the coefficient becomes equal to 1, $e^{2\pi i}$ now y is equal to 1, so I left with an X so I get x/2|1> Now I am going to write it in a slightly different notation and the notation is this, that if you recall.

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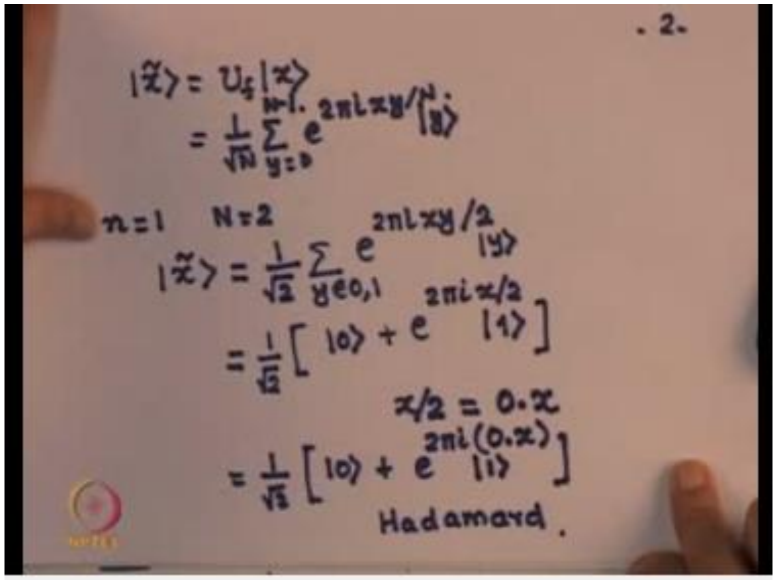

Since my x, y etc in this case this way of writing takes value 0 and 1, the x/2 is binary expansion is simply 0.x, let me explain what I mean by this, see in a typical decimal number when you write for example 0.4 what do you mean? When you write 0.4 it means 4/10, if you write 0.03 it means $3/10^2$ that is 3/100, in this case my basis is two, so therefore if I write something divided by 2 the binary point or binary decimal is a little bit of an oxymoron statement.

But binary points representation is 0.x, so I will write this as one 1/ √2 [|0> + e $^{2\pi i}$ times instead of writing x/2 I will write it with 0.x |1> Now what is this? Now notice this is nothing but a normal Hadamard transform, why? Because if this X happens to be zero so that I have got $e^0 = 1$, I get 0+ 1/√2 on the other hand if this x happens to be equal to 1, now remember 0.1 in binary point representation is ½.

So that I get $e^{2\pi i/2}$ which is $e^{\pi i}$ which is equal -1, so I get 0-1/√2 which is as it odd to be in case of a Hadamard gate. Now this is of course trivial, so therefore in this particular case the QFT for n=2 is the implemented just by having a Hadamard gate, now that we have seen that for n = 1.

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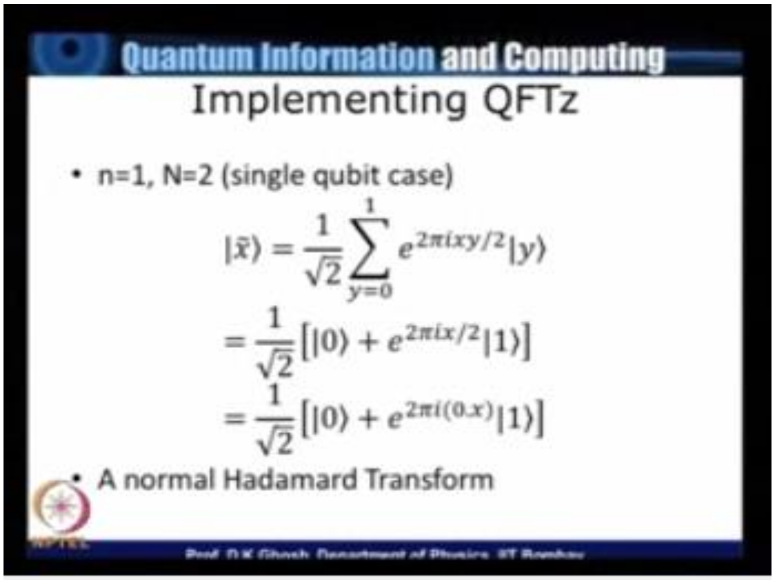

That N= 2 the Fourier transform may be implemented by application of a normal Hadamard transform, we will continue and let me take two qubit case n = 2.

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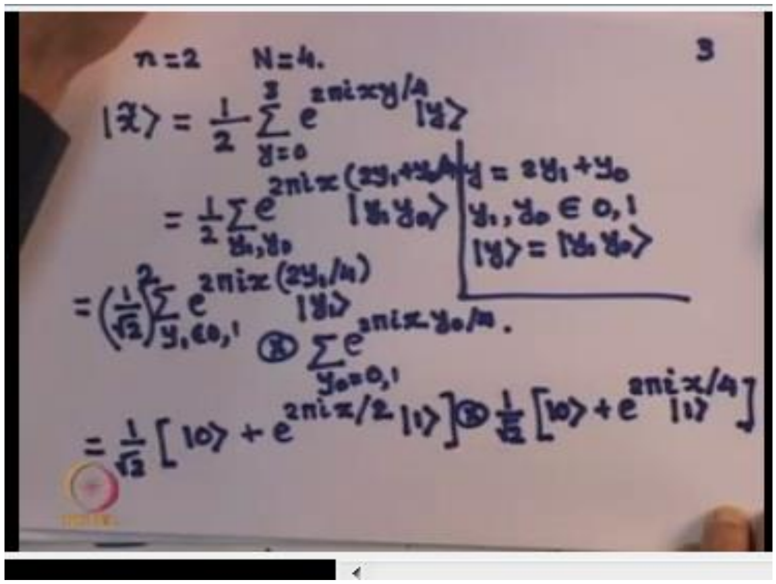

So that N = 4, now let me write down the Fourier transform so this is 1/√n square because there are two of them, y = 0 to 3 $e^{2\pi ixy/4}$ and y of course. Now remember that in this way of writing x and y are two qubit situation, so therefore let me express expand this by writing y = $2y_1 + y_0$ I will for the moment keep x the same, where y1 as well as y0 they take value 0 and 1 and my state y I will write as y1, y0 there is a direct product essentially.

So let me write this as equal to ½ sum over y1 and y0, keeping x the way it is 2y1+ y0, y4 and y1, y0 let me break up this term into product of two and distribute this 1/2 into 1/√2, so the first term I will write it as, some over y1 $e^{2\pi ix}$ (2y1/4) and |y1> product with some over y0 equal to 0 1 here also it is 01 $e^{2\pi ix}$ and you have y0/ 4. And there is a 1/√2 which is coming along with this, so let me since I do not have space there let me put it like this.

Now we are going to see what are these terms like, realizing that my y1 takes value 0 and 1, let me first split it up. So I have 1/√2 |0> because if y1= 0 this e^0 = 1 + $e^{2\pi ix}$/ 2 because this is 2/4 and y1=1, so I have got 1 here and multiplied with 1/√2 first term is |0> as before + $e^{2\pi ix}$ y_0=1 so it is x/4 |1>, so this is, this is what i get as the Fourier transform of the 2 qubit situation. Let me now expand this x.

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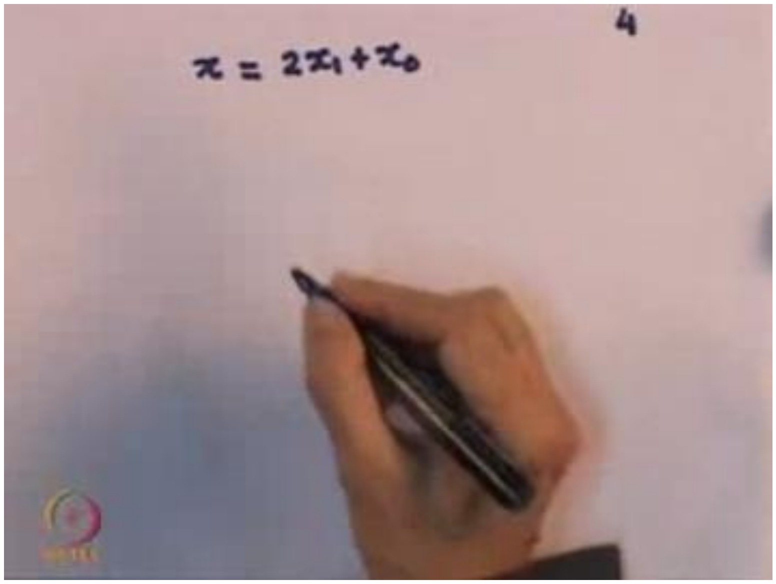

And realize that $x=2x_1+x_0$, now if that happens if you look at this term here.

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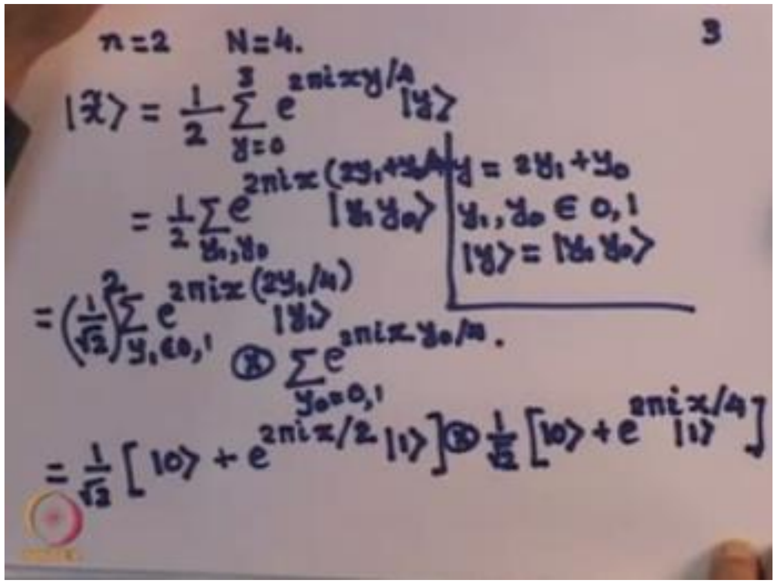

I have got 2πix/2.

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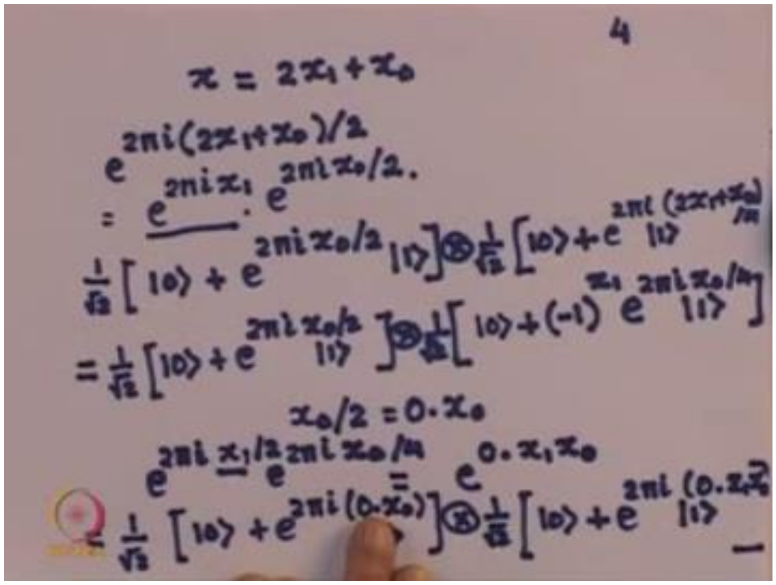

So $e^{2\pi ix/2}$ x is $(2x_1+x_0)/2$ and that is equal to $e^{2\pi i}$ this 2/2 so it is x_1x $e^{2\pi ix0/2}$ so notice one thing for both x_1= 0 as well as 1 this term becomes equal to 1, because e^0=1 and $e^{2\pi i}$ is also equal to 1. So therefore this term that I had written down here can be rewritten as 1/√2[|0>+ $e^{2\pi ix0/2}$ is coming from here because this is 1times vector 1 multiplied with 1/√2 again [|0>+ but this time I have 2πix/4 so let me write it as $e^{2\pi ix}$ and x is $2x_1+x_0/4$ |1> I do not change the first term let it keep the way it was $e^{2\pi ix0/2}$ |1> multiplied by 1/√2, let us look at this term, this is |0>+.

Now look at this term here, it is $e^{2\pi i(2x1)/4}$ which is nothing but $e^{\pi ix1}$ so $e^{\pi i}$ being equal to -1, so this term is $(-1)^{x1}$ and the phase term is still there which is $e^{2\pi ix0/4}$. Now we need to rewrite these in a particular way, I had already told you that this thing which is $x_0/2$ now since it is a single qubit $x_0/2$ can be written as $0.x_0$ in the binary point representation. And let us look at what do I get here, here I have got $e^{2\pi ix0/4}$and here there was a -1 to the power x_1 which I had actually written from here.

So let me bring back the phase notation so it is 2πi so 2 it so x_1 is what I had, so this is equal to $e^{0.}$ see here I have an x_1 and I have here an $x_0/4$, so this is $0.x_1x_0$ sorry here that should have been a 2, because this is $2x_1/4$, $x_1/2$ is $0.x_1$, $x_0/4$ is $0.x_0$, so finally if I collect them all together and getting 0 + e2лi 0. x0 1 over √2 0 + e2лi 0. x1 x 0 1the thing that I want to point out is the following that these terms actual this term is not particularly important because here i get - 1 to the power x0 so which is nothing but an ordinary Hadamard transform but supposing I transform the first q bit Hadamard then this term.

Where there is a phase rotation depending upon what the value of x0 is then i will have a problem and because of that i need to change you notice here we have got a2лi0.x1 which is nothing but - $1^{x\ 1}$ this is what you expect from an ordinary Hadamard transform but the phase factor is $e^{2\text{лi}}$ x 0 x 4 and that phase is there only when x0 = 1 but if i had changed the value of x0 here instead of keeping the old value then this would mess up things not, not the way I want it so what is the important is to remember that I want certain action to be taken in a particular way.

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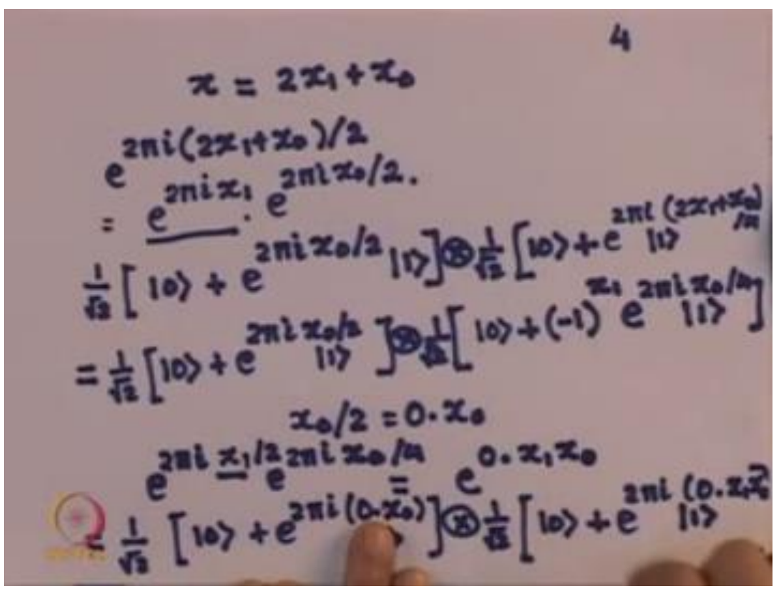

And since I need the x0 to provide a control I cannot apply the gate on x0 before I have done this and that is the reason why we will see in the slide.

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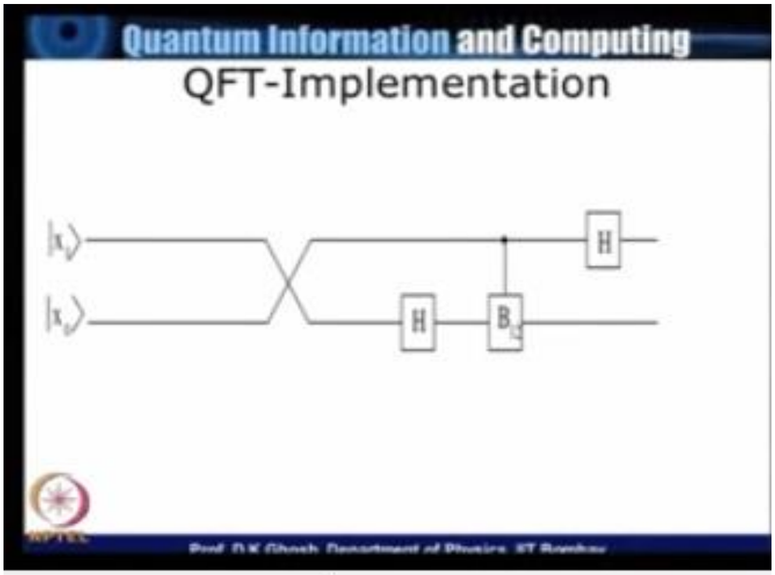

That our circuit requires a reversal of x0and x1 before the various gates are given now how does one implement such a the way to implement this is this we define what is known as a controlled B_{jk} can get.

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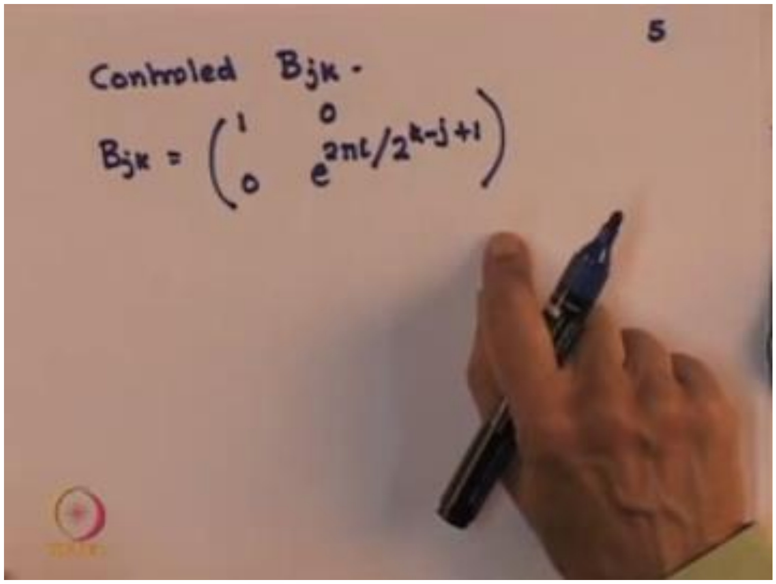

Define the matrix B_{jk} by the following100 e2лi / $2^{k - j +1}$ the real k_j etc. Occur here we will see when we extend our logic for n = 1and n = 2 to the general case but, but this is what a controlled B_{jk} gate does so let us look at.

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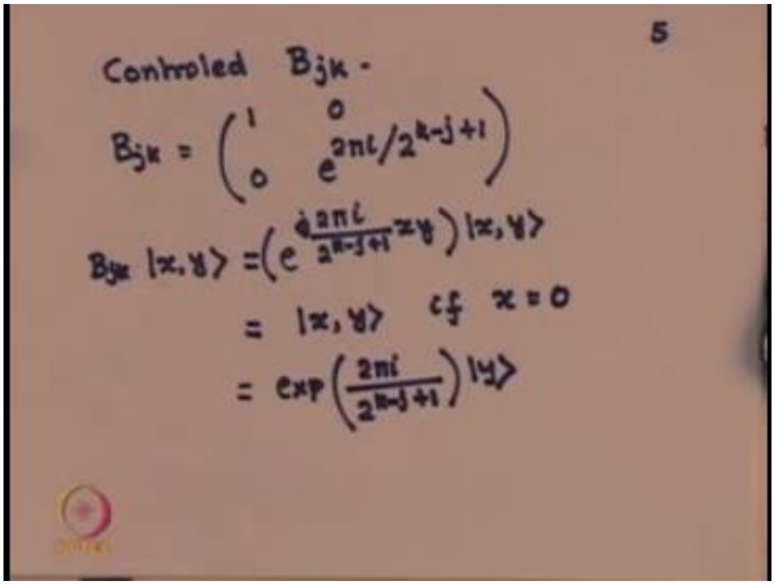

What is the effect of controlled B_{jk} on let us say state xy the controlled B_{jk} gate acting on this gives me e^{i} okay let me write it 2лi /2 $^{k - J +1}$ xy that is a phase factor xy now what does it give me it tells me if x = 0 this is nothing but xy on the other hand if x = 1 it picks up a phase that is exponential of 2лi / 2^{k-j+1} y now with this let us return back to.

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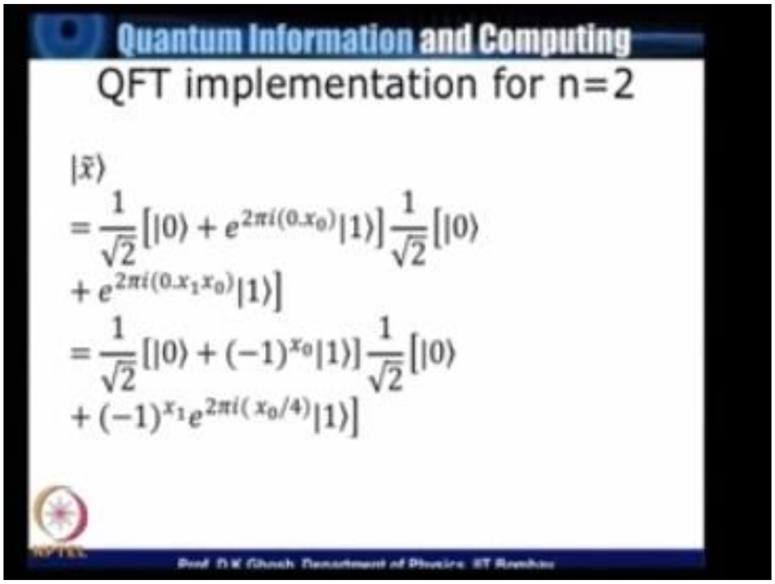

Our previous slide so we said my X~ which was the Fourier transform was given by the slide.

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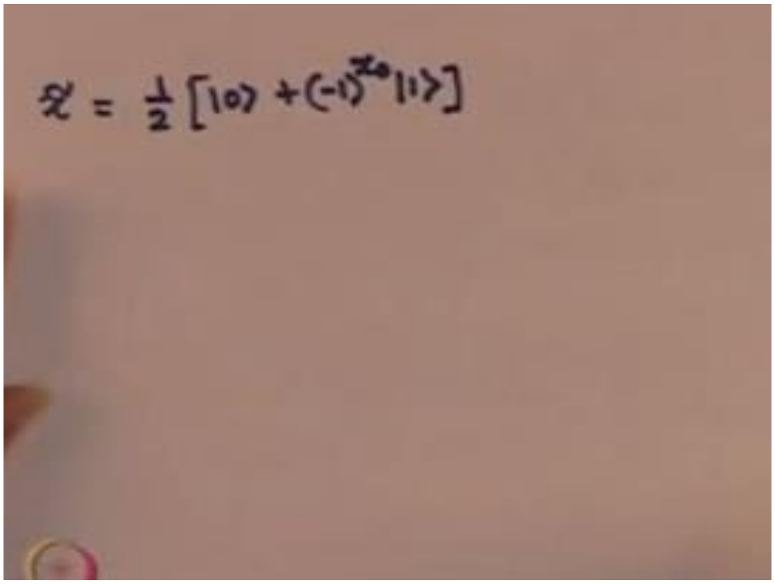

So X~ then becomes equal to first one as we said is just Hadamard it 1/2 0 + -1^x 0 1 nothing to be done there.

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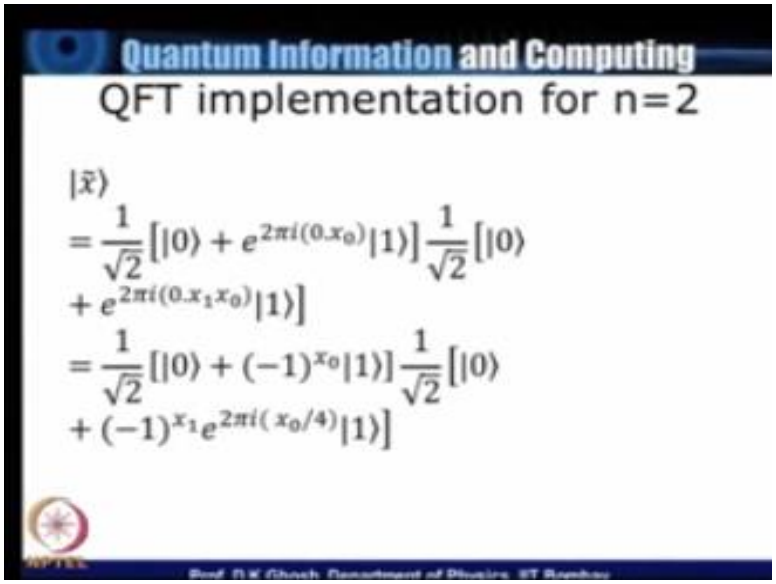

The second one if you refer to the slide again it was this term here 0 $+e^{2\pi i 0.x10}$ which we have seen can be written like this, but you see this is the control phase rotation.

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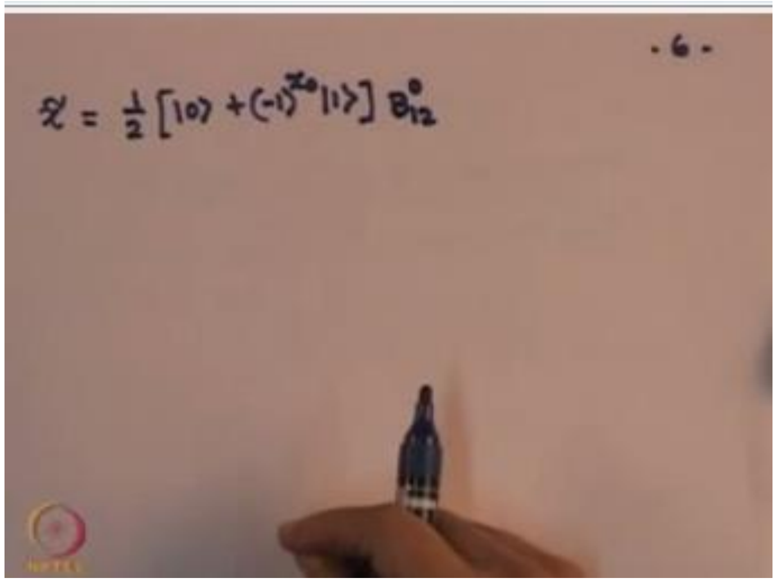

So what we will say is this, this is equal to B 1,2,0, why 1,2 because I had defined.

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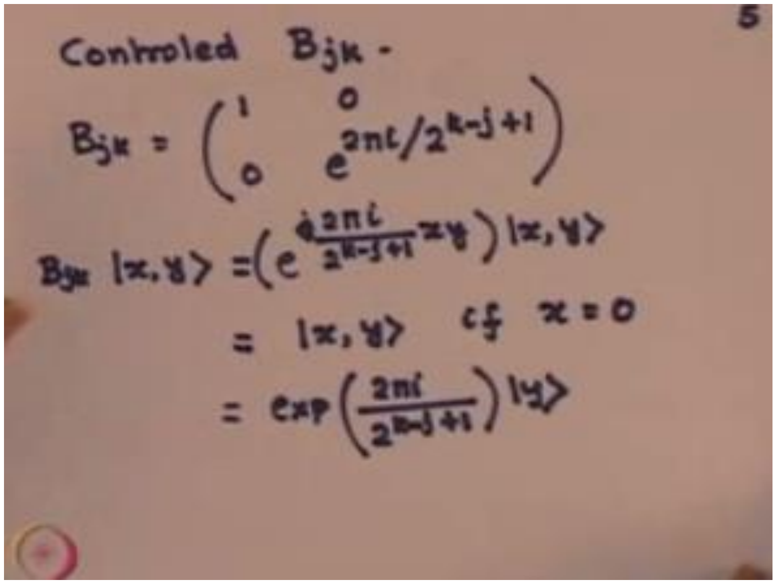

BJK gate as given by this so if J = 1 K = 2i get 2-1+1, so it is 2 square there $e^{2\pi i/2}$ now this is precisely what we had there in the slide.

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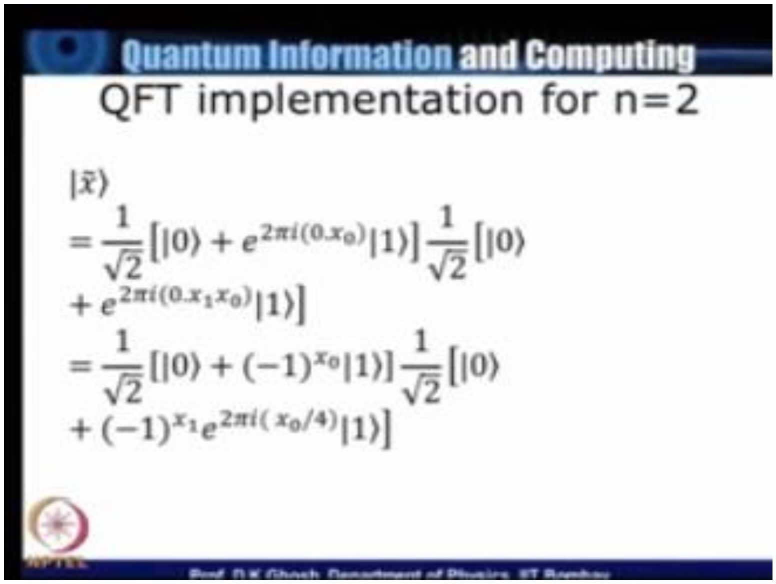

So that this thing we should be able to write as B 1, 2, 0.

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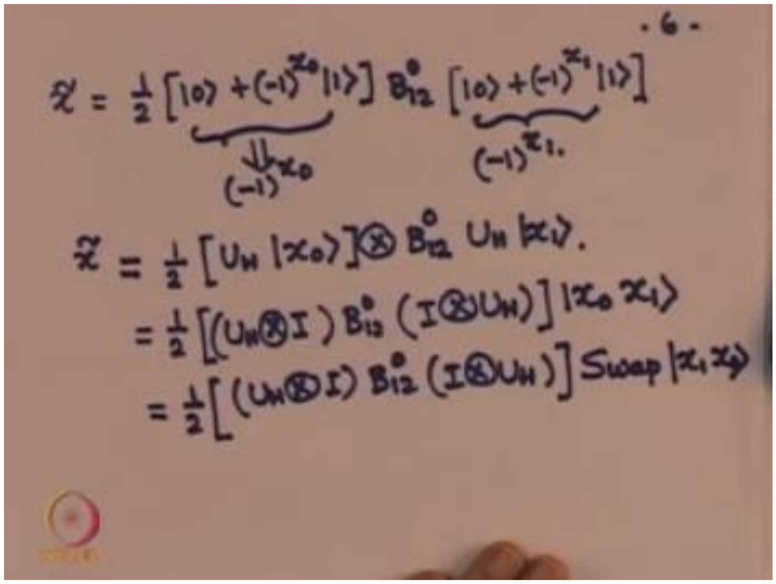

Acting on 0 +$(-1)^{x1}$ on 1, so this states are entangled because you notice the first one which refers to $x_1$has a $(-1)^{x0}$ but the second one has $(-1)^{x1}$. So in other words the output is in reverse order. Now so what do I do the way to mend this situation is to do the following, X~ is ½ the first term is U_h X_0 second one is B_{12}^0 U_h acting on X_1which is equal to half of U_h x I B_{12}^0 I x U_h the different order because one of them acts from the X_0 the other one acts on Y, but it acts on X_0 X_1 this is what we want.

But you realize that what we had was a state which had X_1 X_0 so therefore I would write this as 12 U_h I B_{12}^0 also swap X_1 X_0. So what it means is this before applying these operations. We must swap the order in which X_1 and X_0 are appearing.

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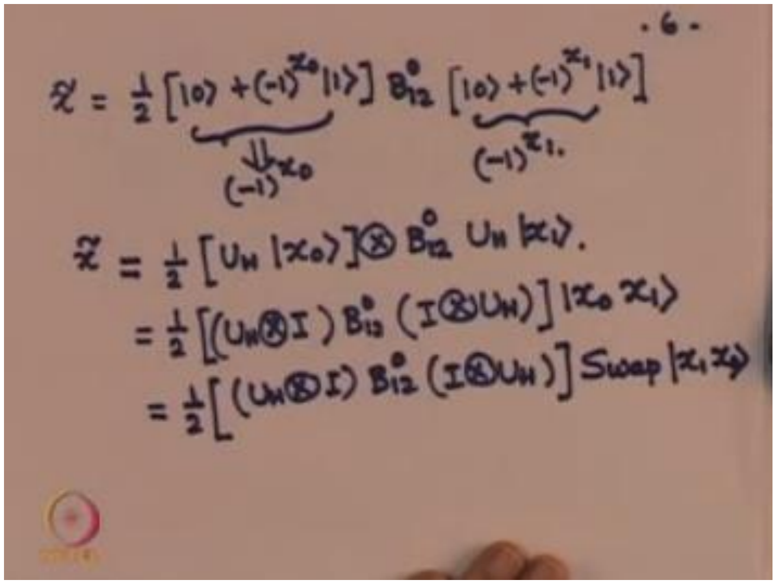

And then carry on this thing so that my final state comes up in the current order because what we noticed is my X_1 and X_0 were getting swapped.

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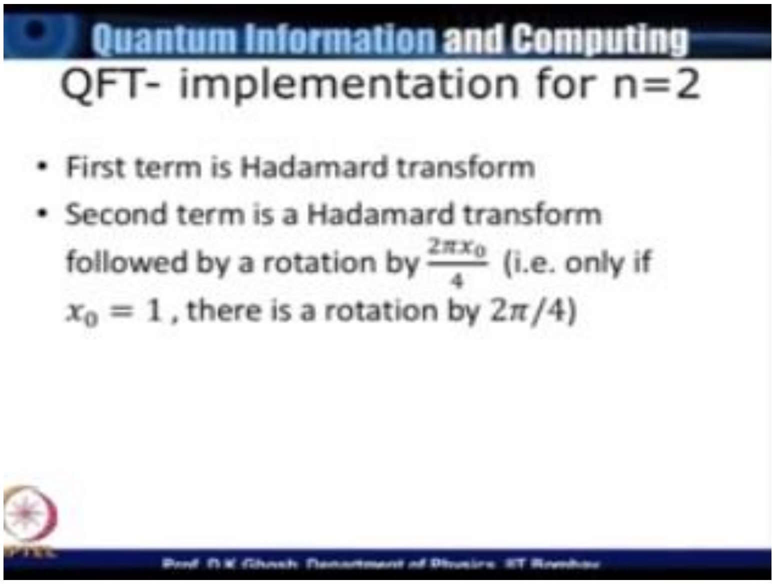

Now look at a picture so this is this is what we said here that there is a rotation selective rotation depending upon what is the value of X_0 in the second one.

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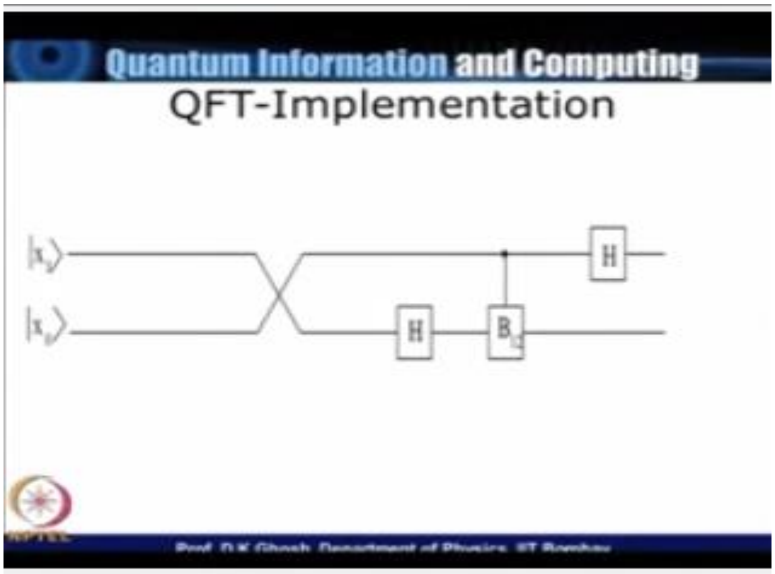

And a simple circuit will tell you this that supposing you had X_1 and X_0 now you do not want these operations to just take place on X_1 and X_0. So what I do is I swap this question we swap it and then as we have seen that on the first line which is not affected by this control I put a Hadamard gate there and then on the second line I put a Hadamard and be wondering. Now if I do that then the operations would have been done in the order that we want.

Because the way my states came they were reversed. So we have done couple of things in this and the previous lecture one was to point out that quantum Fourier transform can be used to infer the period of a function which is periodic with a discrete periodicity, and the second one we saw by specific examples of 1 and 2 qubit cases how to implement Fourier transforms in a quantum circuits and for that we had two additional things namely a controlled BJK gate and a swapping of the bits before we apply this transforms.

We will see later when we extend this idea to a general n qubit case the, what we have to do is to permute the bits first and then apply the operations the BJK gate the Hadamard gate etc.

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