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Quantum Information and Computing

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Modul No.04

Lecture No.20

Grover's Search Algorithm – II

In the last lecture I had introduced an algorithm called Grover's search algorithm and we said that this is basically searching a particular item out of an unstructured data base of N number of items and the our way of doing this was that we define a needle in the haystack function which has this property that excepting for the item that we are looking for the function can take in other arguments and it evaluates to 0 but for the particular item that we are interested in which I will be frequently referring to as the mark title.

The function evaluates to 1 the importance of the algorithm lies in the fact that we had seen that classically such a search requires of the order of N in fact typically a probabilistic algorithm would require N/2 searches if you are looking for one item out of N items now.

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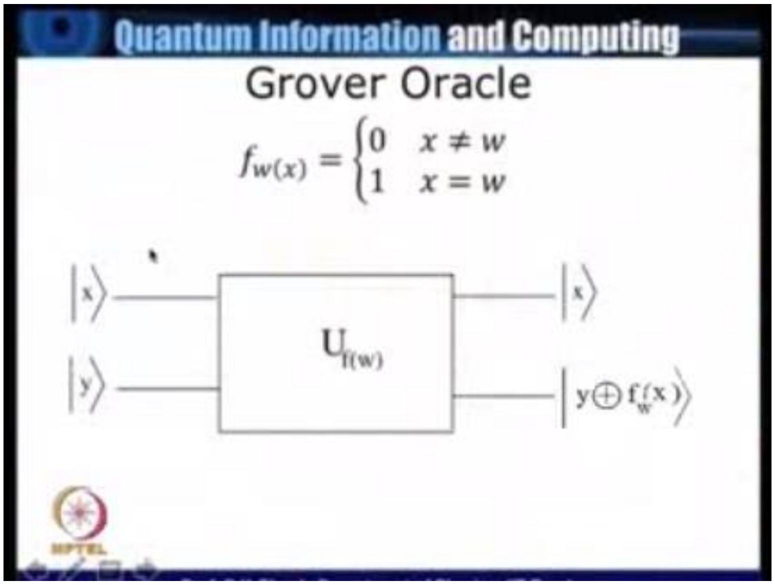

So they the Oracle that we define is like this that w is our string that I have been searching N qubit string and this evaluates to 0 if the input string x does not match with the string that I am looking for and it evaluates to 1 in the case where x is equal to w this is our normal representation of an Oracle, so here we have got the input x as we have seen several times that this input is a linear combination of a large number of states and this y is usually set a value depending upon what you are looking for.

And the black box or the Oracle it calculates this function corresponding to this linear combination of the input and the first line is still input without any change but these target bit which was y becomes y + the function that you have calculated, so that if y is the null string then you simply get f(w) and if y is 1 then you get the complement of that so basically the idea in this case was.

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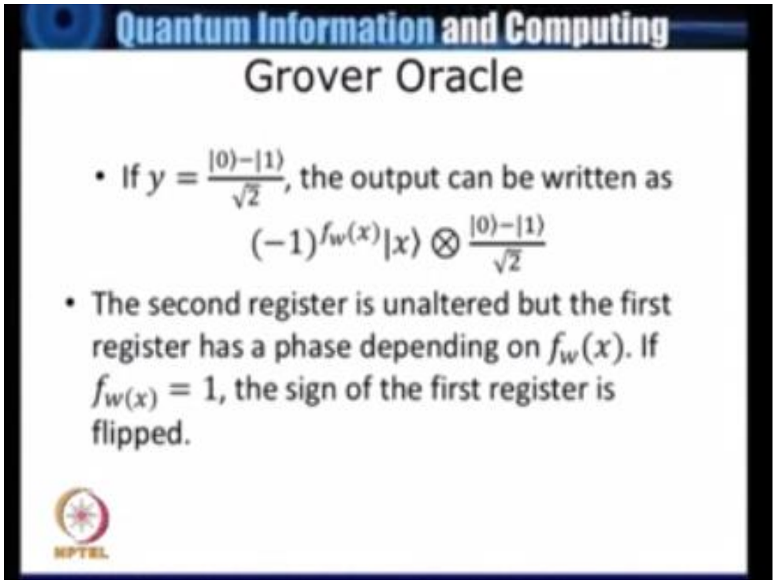

To set the target bit y equal to 0 – 1/ √ 2 so therefore the target bit has 2 components 1 0 and 1 1 and 0 obviously will not change the value of the function that you calculate which is output but one will give you it is complement and it is succinctly written like this that the output is written as -1 to the power this function that you calculate which if the function evaluates to 0 then it is -1^0 to the power 0 which is equal to 1 and of course x is the input which is left alone and the target bit is 0-1/√ 2.

So this is the phase that is associated with x you notice that this tells us that the second line or register is unaltered but the first register has a phase depending upon the value of fw(x) so this is what we talked about last time and we will today try to see how this is going to be actually implemented last time we also defined a couple of operators.

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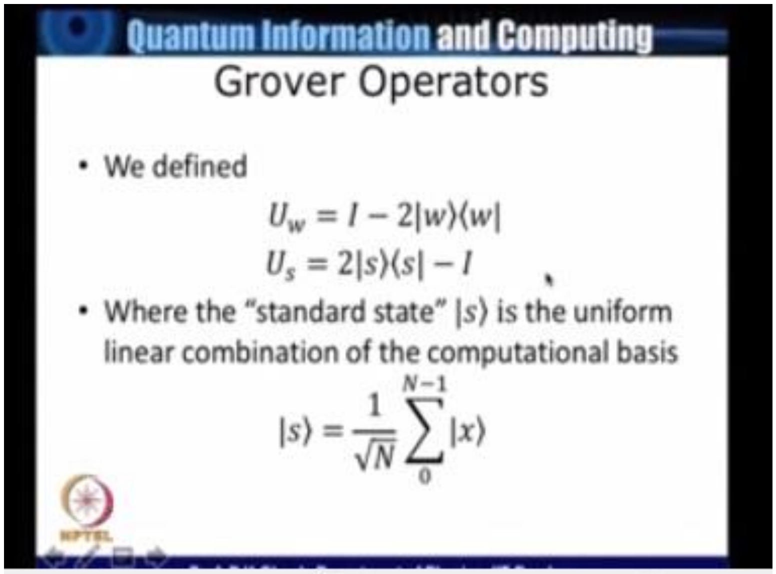

So first was 1 was a an operator which we called as the U_w and U_w is identity $-2|w\rangle\langle w|$ you notice what happens that if such an operator acts from the state w which is the mark state you get w – 2w which is equal to- w that is it simply flips the sign of the mark state a second operator which is U_s is 2ss- I notice the sense is essentially reversed.

So what happens is this that now s as I told you is a uniform linear combination of the computational basis so n qubit computational basis $1/\sqrt{n}\ \Sigma_x = 0^{n-1}$ of x now if this Us acts on this s notice this is a typical structure |S><S| is a an operator, so that I get s itself, in other words if I have an arbitrary vector and I resolve into a component parallel 2s and a component perpendicular to this, then the component of this arbitrary vector which is parallel to s will remain unchanged.

But the component which is perpendicular to s will change and we have seen a geometrical interpretation.

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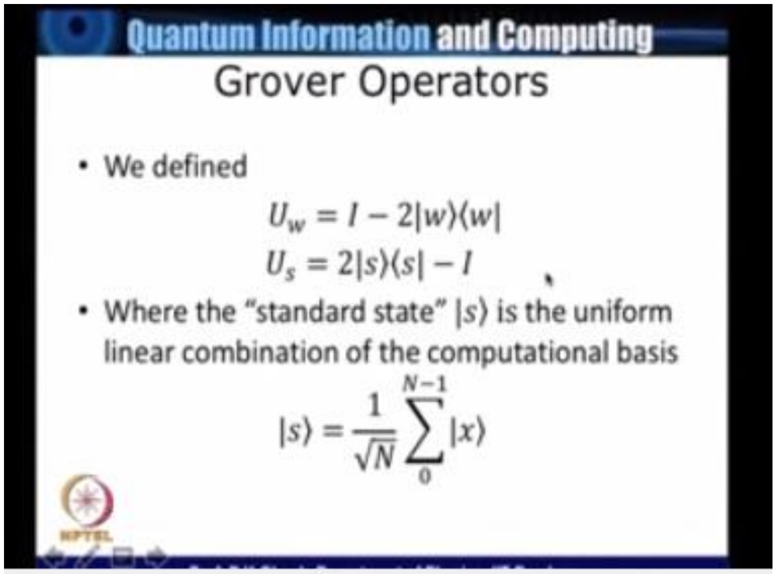

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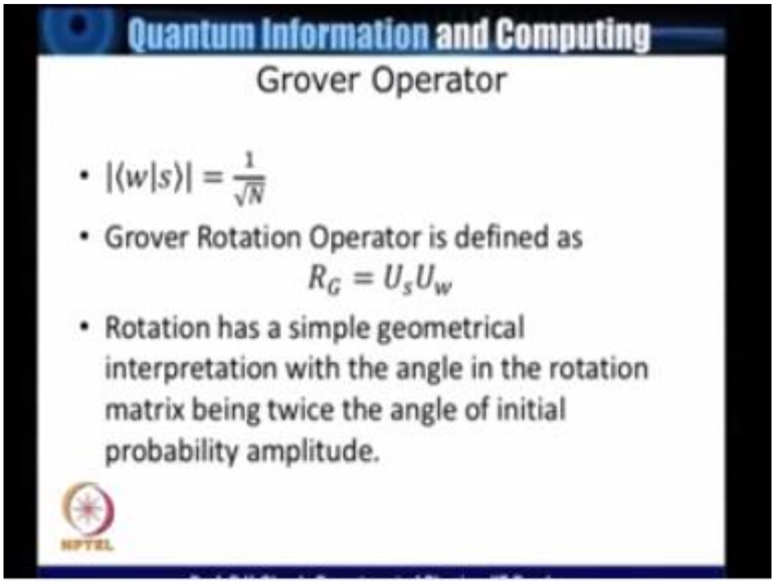

Of this which I will not repeat and basically what we said is this, that initially since I start with equal linear combination of the basis state I start with a w whose strength in the linear combination its amplitude is given by 1/√N this would be initial starting point, having done that we define a Grover rotation operator.

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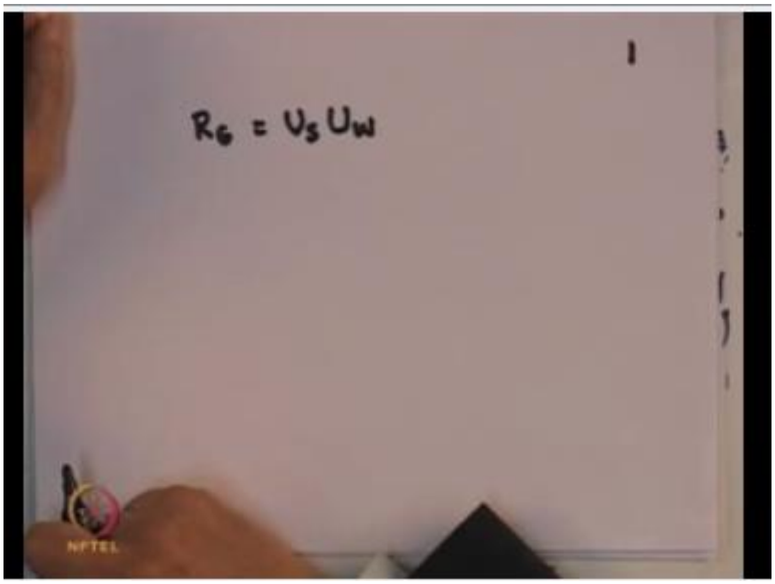

By $R_G = U_s U_w$ so U_w acts first and then U_s acts and we had last time seen that when this operator acts on an arbitrary state it has the effect of rotating that state by an angle which is the twice of the initial angle that it makes with the market state. So if you start with a particular value the every rotation my Grover operator $U_s U_w$ will make it closer by 2θ if initial angle between the marked state and the uniform distribution was S now was θ.

Now that makes the arbitrary state come closer and closer to the marked state but there is a problem that is we must a priori know how much or how many times I must apply this Grover operator because once since it is a rotation say once it has exceeded that marked place then of course the distance between them will go on increasing so this is what actually happens.

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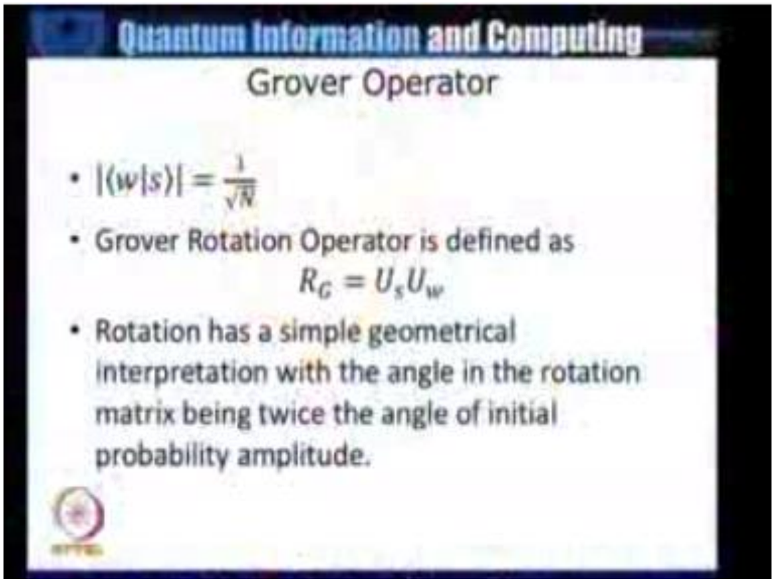

And so therefore this is what we did we gave the geometrical interpretation and today I will give a slightly different interpretation of the Grover rotation because while geometry is extremely attractive but it is not does not really take us very far in trying to repair an algorithm. So let us look at what is this new interpretation that we have.

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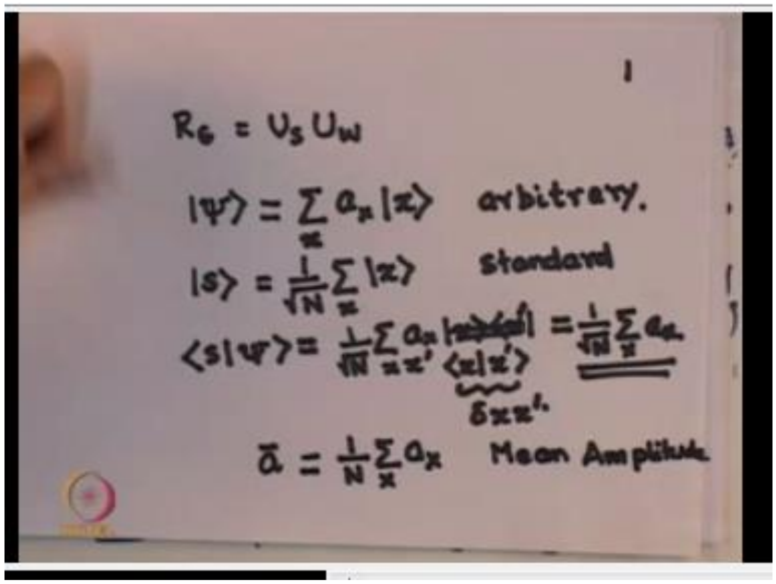

So we again start with an arbitrary State ψ which I will expand in the computational bases so computational basis I am writing as X and let us suppose that this is ax |x> so this is an arbitrary state and we have seen that my standard state S is simply an equal linear combination of these computational basis. So this is standard and this is arbitrary, so that what we have here supposing I take the scalar product of S with ψ.

Then this is 1/ √N by normalizing this will be some over X ax and X, X of course, so which is nothing but 1/ √N times some over X ax actually I should have written it like this, X X’ so that by this is this should be just a scalar product of X with X’ so that this δ x x’ and this is what I get. Now look at what is this, if you look at this what you find that this is equal to this ax is the amplitude of X in the state ψ.

So I can write this as if I define a - bar is defined as 1/ N sum over x ax which is nothing but the mean amplitude, then I can write this, I can write this as.

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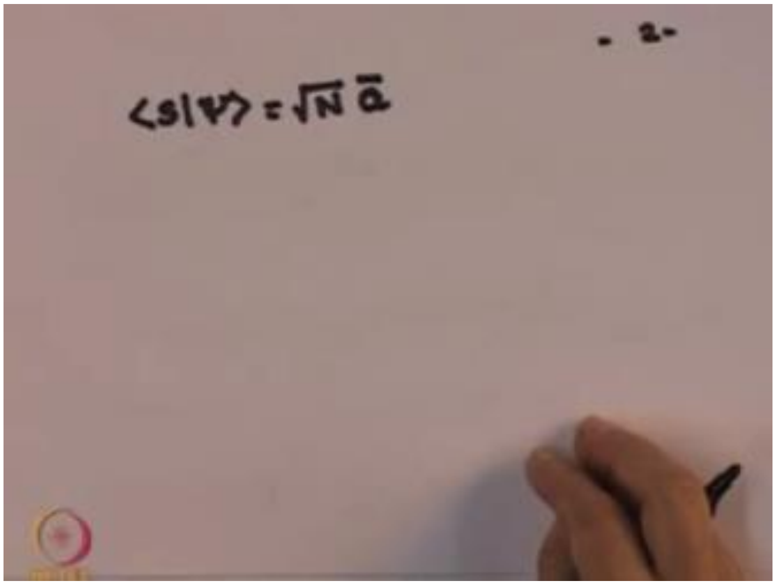

$<s|\psi> = \sqrt{N}$ times the mean amplitude a bar, where a bar is simply the arithmetic average of the amplitudes of ψ in a computational basis. Now let us look at what, what does U_s acting on ψ will be.

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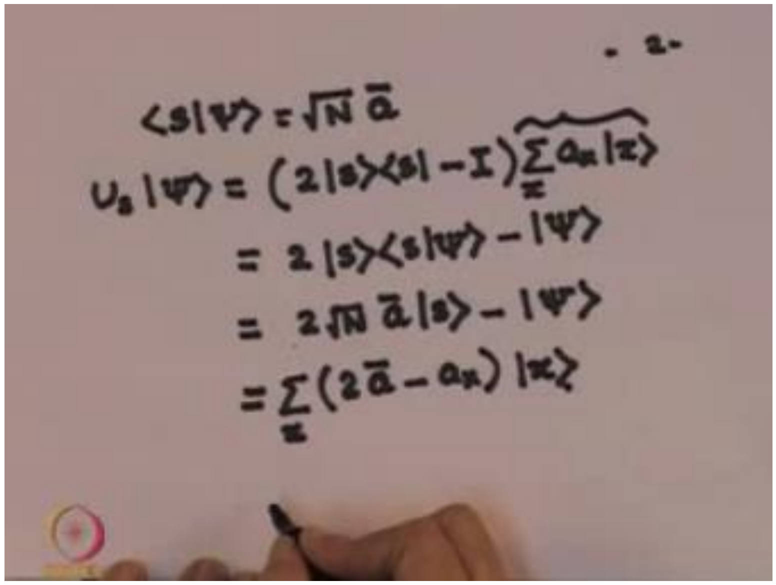

Our definition of U_s was it U (2|s><s|-I) and this is acting on ψ which is ∑x a_x|x> so this is equal to 2 times notice I have here the |s><s| with ψ because this is, this is actually ψ and -ψ of course, now this we have just now seen is given by √N times a bar.

So therefore, this quantity is 2√N times a bar this is, this quantity and of course I have got the vector s there –ψ so what I can do is to write this as equal to ∑x(2 a bar –ax) |x> now this is just the expression for ψ and I have expressed now s in terms of 1/√N ∑x |x> so that this is becomes an identity. So if you notice what is the interpretation of the application of U_s on an arbitrary state ψ. So what is happened is this, that before application of U_s my amplitude of x in my states a_x.

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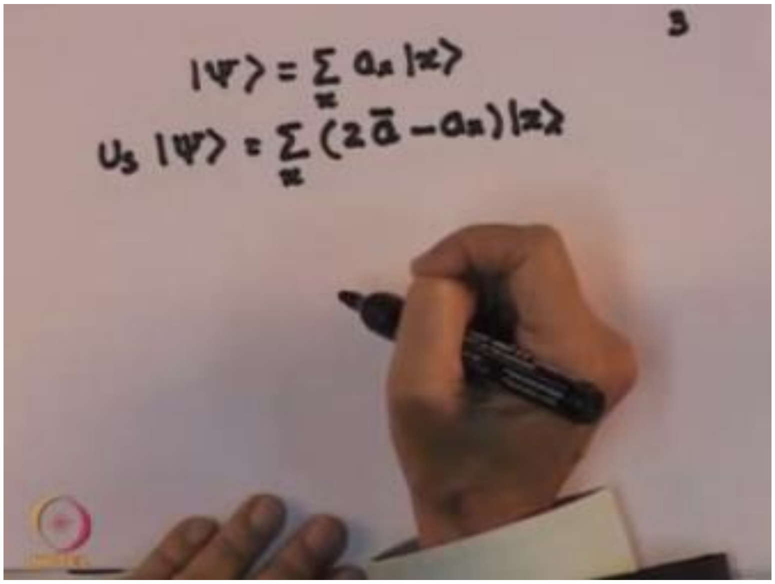

Recall my $|\psi>$ $\sum x a_x |x>$ so this was before reverse is applied and we have just now said that U_s of $|\psi>$ $=\sum x$ 2 times the mean amplitude -ax acting on $|x>$. Now suppose I measure the amplitude with respect to the mean then my initial amplitude.

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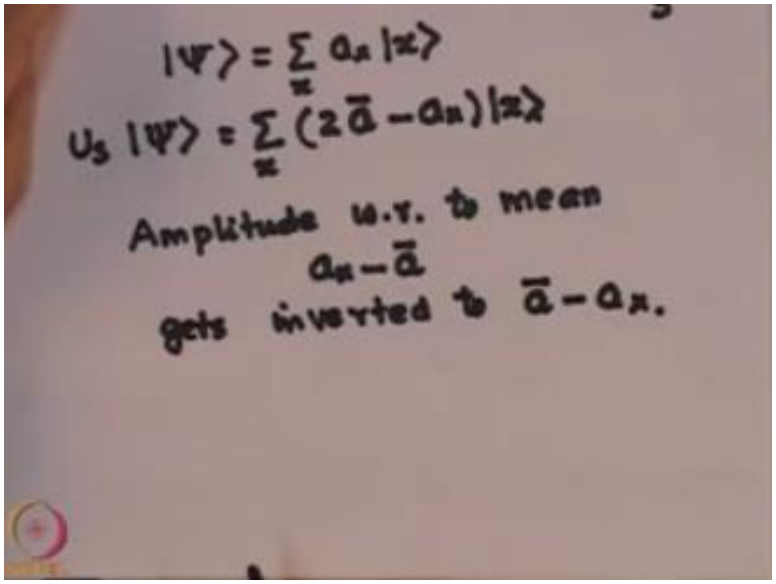

Amplitude with respect to the mean, my initial amplitude was a_x - a bar and since my final amplitude is 2a bar -minus a_x, so if I compute it with respect to a bar this is becoming a bar -a_x so this gets inverted. Now this is a nice observation to make that the amplitude with respect to the mean of an arbitrary state on application of U_s gets flipped or inverted. Now we will see the meaning or what happens as a result of this, let me take as a specific example.

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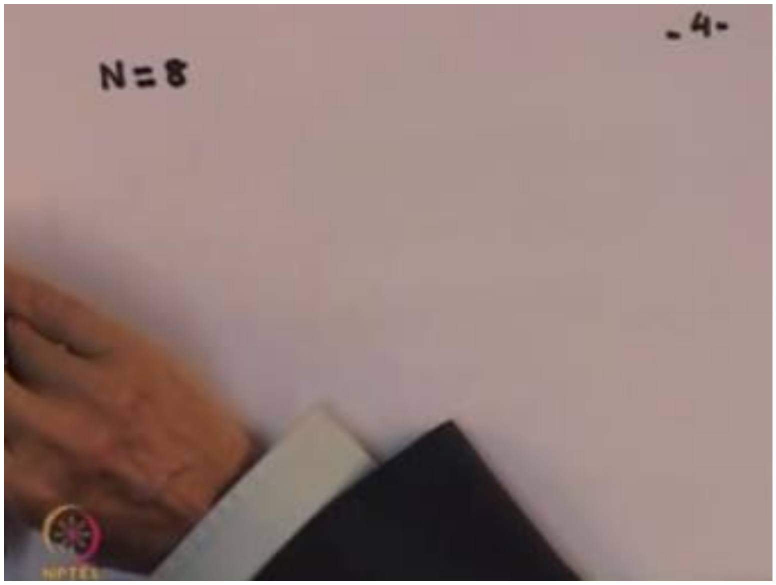

Number of items =8 we had seen that N=4 gives us with a single Grover rotation they marked state, now let us suppose I have N=8 I do not actually care which one is the marked state because the Oracle knows it my calculation of the function f w of x so therefore I will assume that the oracle know did lose it and my analysis of it is going to backtrack with the knowledge that track with that knowledge the oracle has that what actually has been happening so let us look at that suppose n = 8 and let us suppose W = 4.

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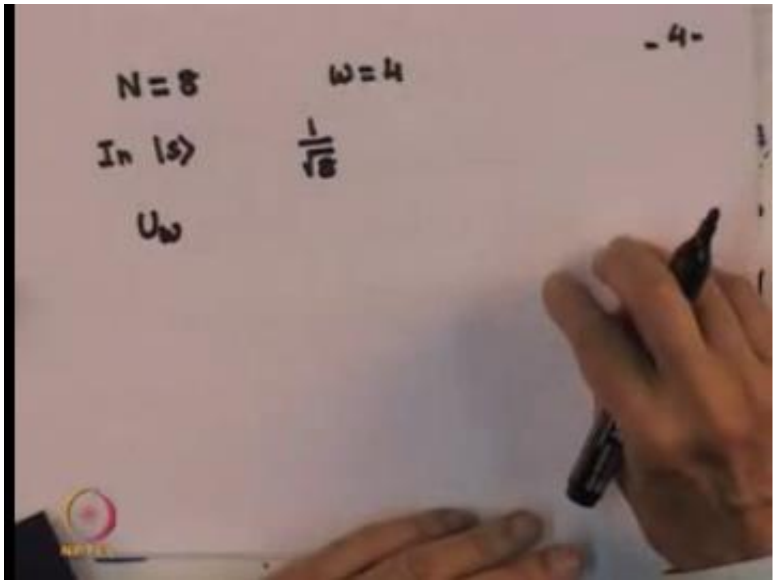

Is the illustration you could take w could be anything else because all the states I have the same status it would not make any difference to our calculation so in s each basis state has a strength 1√8 this is the amplitude of each of the computational basis in the standard state now what does uw do this is not u_s this u_w now what we have seen is u_w will invert the amplitude of W only u_w inverts.

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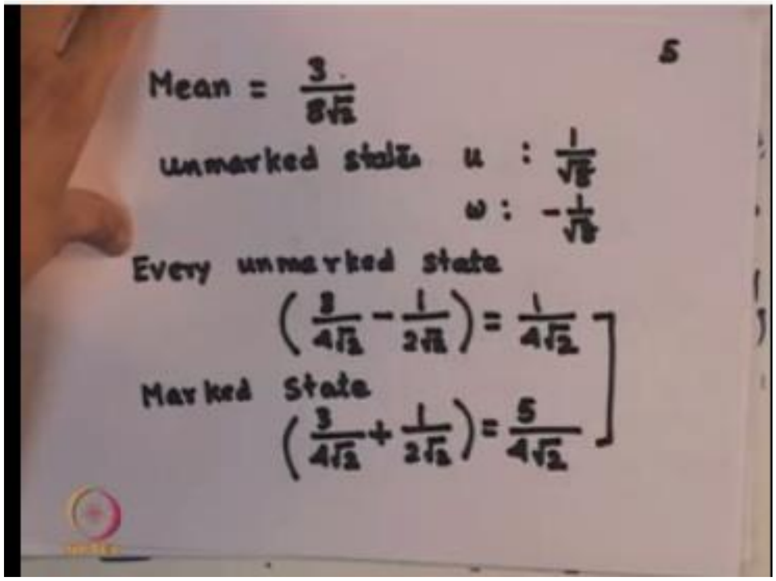

The amplitude of w alone so since my starting point has also been 1√8 after application of u_w the amplitude of W becomes - 1√8 of it all others still remain one hour square root of it so let us will calculate the mean now my mean then is there are 8 items of 1 over 8 7of them still has 1√8 has the amplitude now mark state has -1 √8 of the amplitude so this is equal to 6/ √8 and that is nothing but 3 /8√2 so now I look at what does application of u_s given nowadays we said that as a result of us.

The new amplitude will become 2a - x so let us look at what does it mean let us rewrite mean is equal to 3 by a true to the unmarked state let us call it you unmarked state there are seven of them the amplitude is whatever √8 the marked state which is we have been calling it as w is - 1 √8 and that is how I got these means so let us look at what is happening by means of that so every unmarked state whenever I have amplitude two times a bar the a bar I am already written down.

So 2 times a bar is 3 / 4 √2 - 1 √8 which is 1 /2 √2 and = 1 /4 √the marked state which is w remember it had a sign difference so therefore this will be 3 / 4 √2 + 1 / 2 √2 which is nothing but 5/ 4 √2 compare these on compare none of these two you find that the marked states amplitude has been amplified so that it is five times as much as they are not state amplitude which means the corresponding probability density has become 25 times now this is what we are calling as the Selective amplification.

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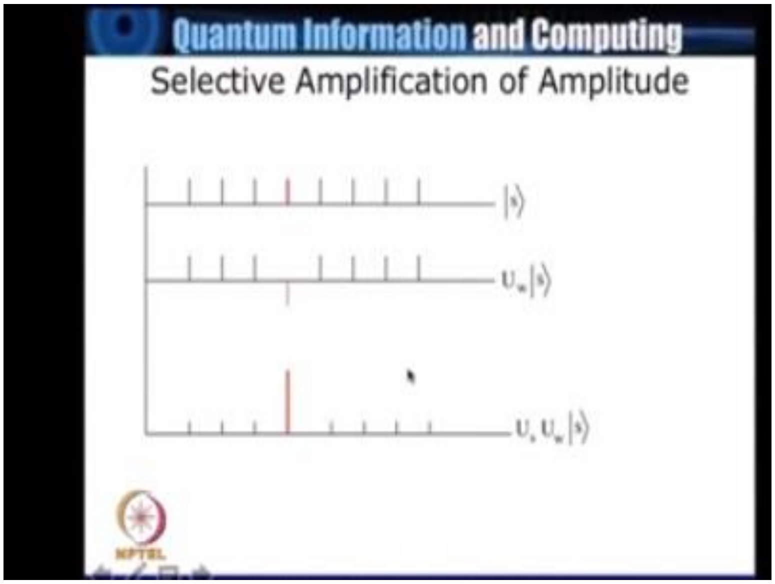

And look at the picture here so initially I have this state as it is not visible here but there is a red color there and, and that is my marked state the fourth one the application of u_w will invert that leave all others the same when you apply u_s after that all these become much smaller and this becomes amplified so that the amplitude is five times as much so the probability becomes 25.

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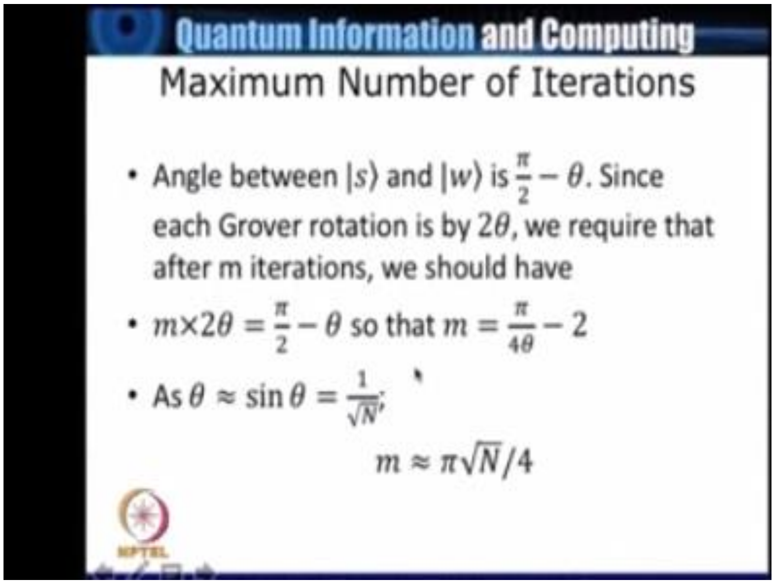

Times next question that we want to ask is we have already pointed out that we must a priori know the number of iterations because if you do not then since every rotation is making it closer to the market state by an angle to θ there would be a time when you will actually exceed it of course it could still lie closer to the mark state than the previous one big on the other side but very soon it is going to be different.

So therefore we must be in a position to control how many times I want to apply the Grover, Grover rotate that is very easy to calculate look at the slide.

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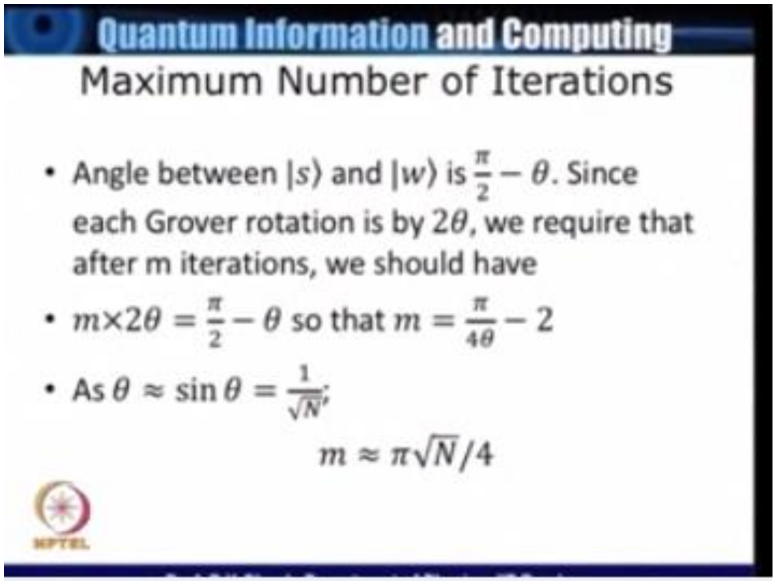

That gives you the structure I will work it out only on the table the angle between S and W in our diagram that I showed you last time was π/2 – θ, π/2 - θ because I had taken sine θ = 1/√n, now since each go over rotation rotates it by an angle to θ if you have m number of rotations the angle of rotation is M times 2θ, now this M times 2 θ should be as close to π/2 – θ as possible and in the ideal case it should just be equal to π/2 – θ. Which we saw was possible in case of n = 4.

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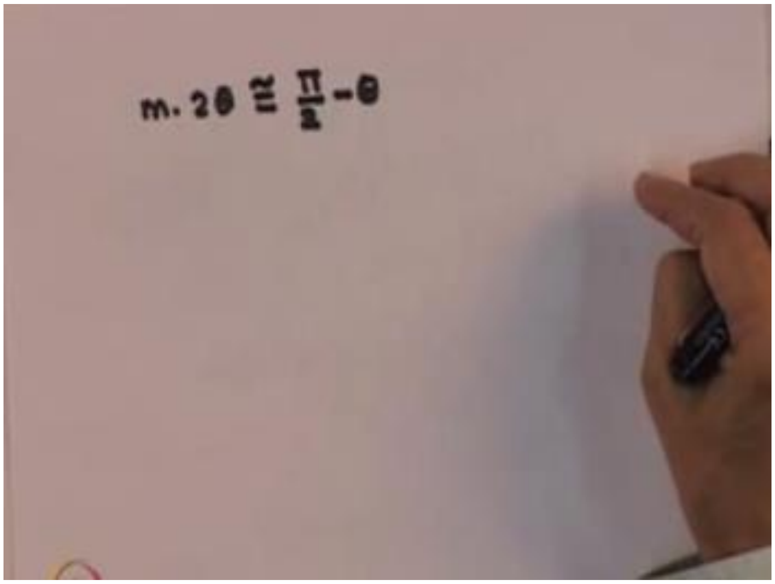

In case of n = 4 if you recall my θ was 30 degrees because sine of 1/√4, so if sin that was thick our 30 degrees 2 θ is 60degrees and π/2 - θ is also 60 degrees so therefore n = 1was the original now if you have solved this in general then you notice m must be equal to approximately equal to but π/4 θ – 1 / 2, now since we have said θ = sin θ = 1/√N for large n I can ignore this term and I can find that m = π √N /4 this is the quadratic acceleration that I was talking to you that is the number of iterations.

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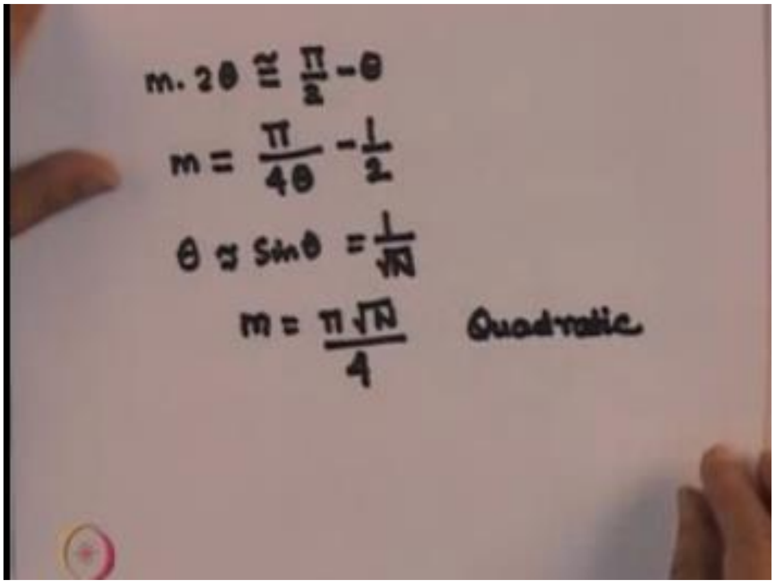

After which the standard state on which when I applied the global iteration will come as close to the mark state as possible with very large amplitude is of the order of square right now what I have done which you can very easily do in your leisure time I am taken a very large number but typically a database search will have much more. So try to work this out with a very trivial program that you read.

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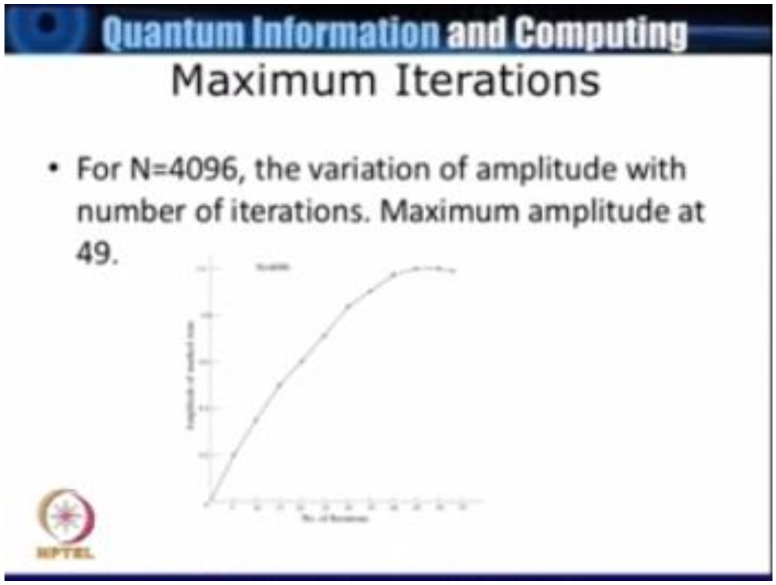

This N is equal to 4096 I have taken 4096 because it just happens to be square of 64, so that initially my amplitudes are 1/64 what have plotted here is variation of the amplitude that I am got of the marked state as a function of number of iterations, now what is found here is that when the number of iteration is 49 you will find that if you take that formula that I gave you it comes very close to 50.

Then the amplitude of the marked state almost becomes equal to 1and after that however the amplitude starts decreasing with respect to this 49 and then it will sort of complete cycles. So what we have done in this lecture is to look at a an interpretation of the Grover rotation in terms of the fact that it inverts the amplitude around the mean and we have seen that as a result of that the amplitude of the marked state gets selectively amplifier and with a very large degree of probability if the number of iterations is of the order of $\sqrt{N}$ I would get the standard state with which I started coming close to the mark state which I am looking for.

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