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Quantum Information and Computing

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Modul No.03

Lecture No.13

Bloch Sphere and Density Matrix

In the last lecture we had introduced the concept of a density matrix and looked at it is property both for pure state and for mixed states. In particular we talked about certain properties of the density matrix namely respective of whether it is a pure state or not mixed state the trace of the density matrix is equal to 1 in both cases the expectation value of an operator ray is given by taking the trace of the product of the density matrix with the operator ray the consequence there of is that if you have a pure state then the square of the density matrix is the density matrix itself and that is because the density matrix has a simple structure that is cat's i with bra’s i so that when you take a square you get the same thing back.

Now that leads to trace of ρ^2 is also equal to 1 for a your state this is however not valid for a mixed state because one can show the trace of ρ^2 will be less than 1 now what we want to do today is to take you back a little bit and remember that for a pure state for a single Qubit pure state we had a geometrical representation on a block sphere and we will try to see whether there is an equivalent situation for the case of a density matrix and again I am talking about the Hilbert space in 2 dimensions that is the mixed system of single Hilbert states. So look at what the Bloch sphere actually represent so what the Bloch spheres did.

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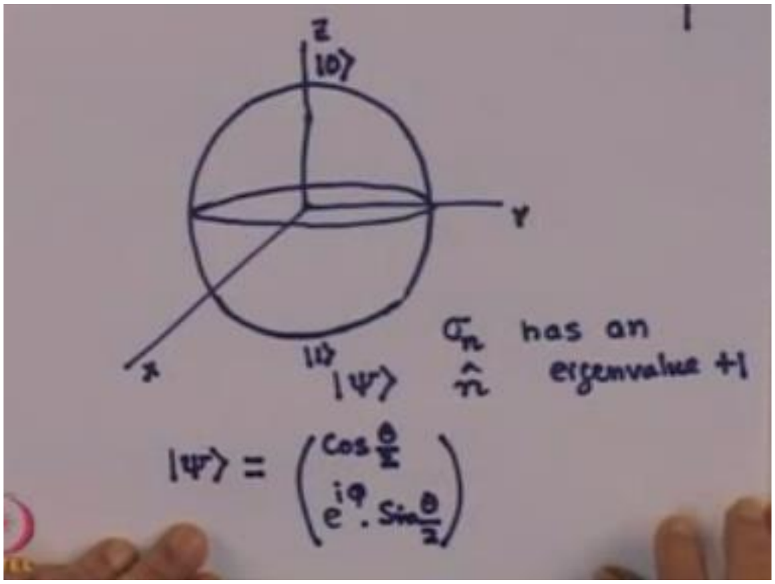

It was a unit sphere let me say that this is the equatorial circle so I take the this as the z axis this as the x-axis and the y-axis now what we had said is that every state has a position on this Bloch sphere every single Qubit state has a position on this Bloch sphere and where is this position now this is decided by that supposing you take the Pauli matrix along the direction n now the state corresponding to ψ is given by that ray in that direction n for which s σn has an eigen value + 1 that put our state on the North Pole to be 0 on the South Pole to be 1 and the point where the x axis intercepted the equator as 0 +1 / √2 etc…

Now this state ψ corresponding to the Eigen value + 1 of σn was shown to be given by cos θ /2 $e^{iØ}$ sin θ /2 so corresponding to this state we can calculate the density matrix.

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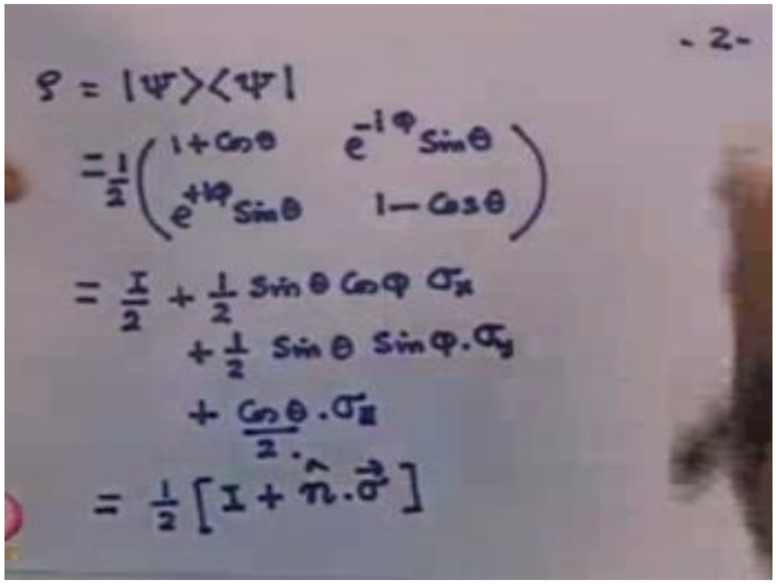

So that the density which is the product of ψ with ψ cat ψ followed by + ψ you can simply multiply the column vector with the corresponding row vector and show that this is given by 1 + cos θ /2 let me take the 1 / 2) $e^{-i\text{Ø}}$ sin θ $e^{+i\text{Ø}}$ θ and 1- cos θ this is nothing but a simple matrix multiplication matrix direct multiplication now as I told you any 2/2 matrix can be expressed as a linear combination of identity matrix and the three Pauli matrices namely Σx Σy and Σz.

Now this has very simple structure so you can immediately see that this is i/ 2 which is this 1 1 and this, this + ½ sin θ cos Ø Σx + ½ sin θ sin Ø Σy recall that Σx had 0 1 1 0 Σy had 0 –i i 0 so therefore if you expand this $e-^{i\text{Ø}}$ as cos Ø – iØ this is fairly straightforward and of course you have cosine θ times Σz of course 1 / 2 again so this thing can be written as 1/2 you have an identity matrix there + n the unit vector n which has component sin θ cos Ø sin θ sin Ø and cos θ dotted with the Pauli vector Σ.

So this is the you know expression for ϱ which is obtained fairly straightforward and you look at the components of n which gives you θ n Ø choose a particular θ Ø go over on the Bloch sphere and you can find out a position on the Bloch sphere corresponding to any state Ø that is now what we can do is we can now look at what happens to point this remember we said it is unit sphere and we talked only about the points on the surface of the unit sphere now what we want to do is this that we would see that the points within the Block ball Bloch sphere blog ball both interchangeably used terms the point inside the Bloch sphere correspond to mix states,.

Now as we have already said the mix states do not have is a state representation, mixed states do not correspond to it a particular vector but you can write down the density matrix corresponding to it, but the reverse of course is not true if I have a state it corresponds to a density matrix, so therefore all points on the surface of the sphere also represent density matrices.

But corresponding to the pure state, so what we are going to show now is that the points inside the Bloch sphere they represent a mixed state. So let us look at how does it work, okay.

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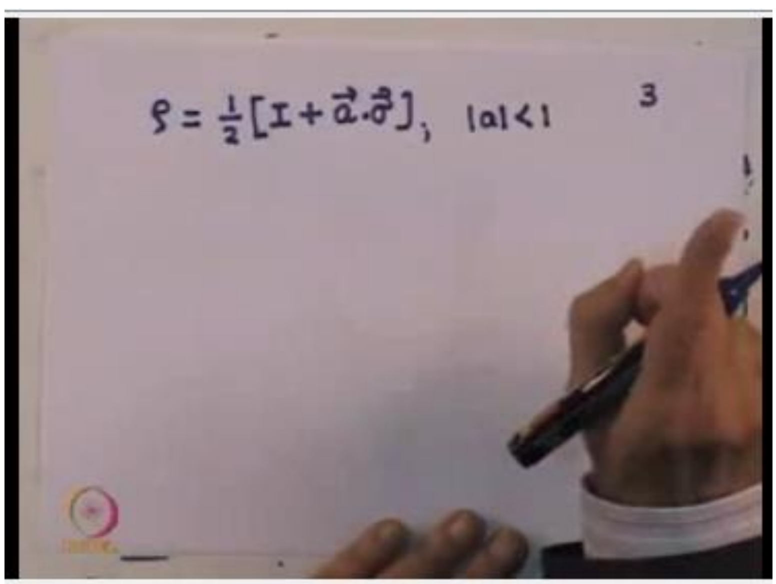

So let us look at a representation of Rho which is ½ [I + now remember I had a unit vector n dotted with sigma but I am using a vector ‘a’ dotted with sigma, define this. Now if you do this definition with |a| < 1, what you can find is that this represents a state which is a density matrix corresponding to a mix state.

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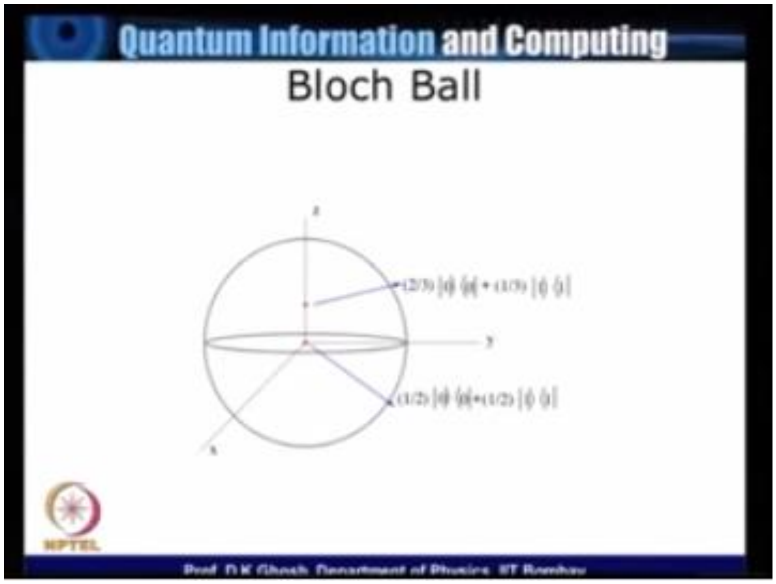

Now let us look at this picture, so I have said that it is representing.

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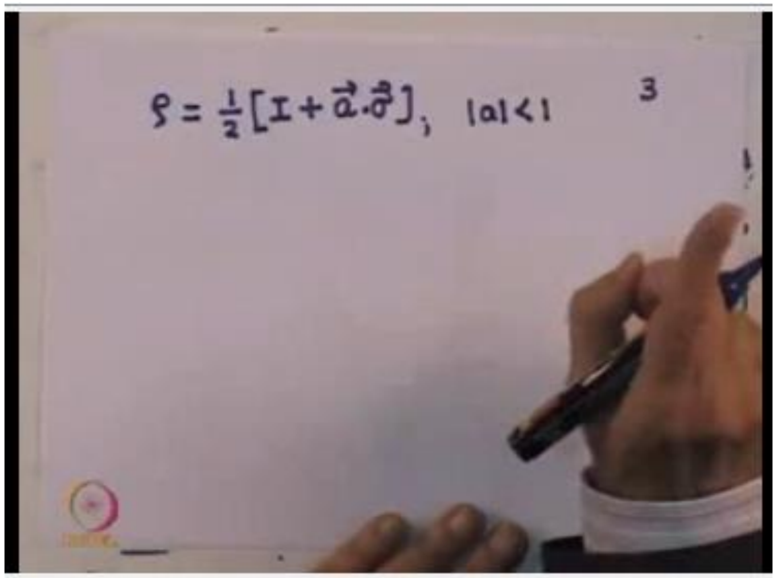

½ [I + a. σ] with |a|<1.

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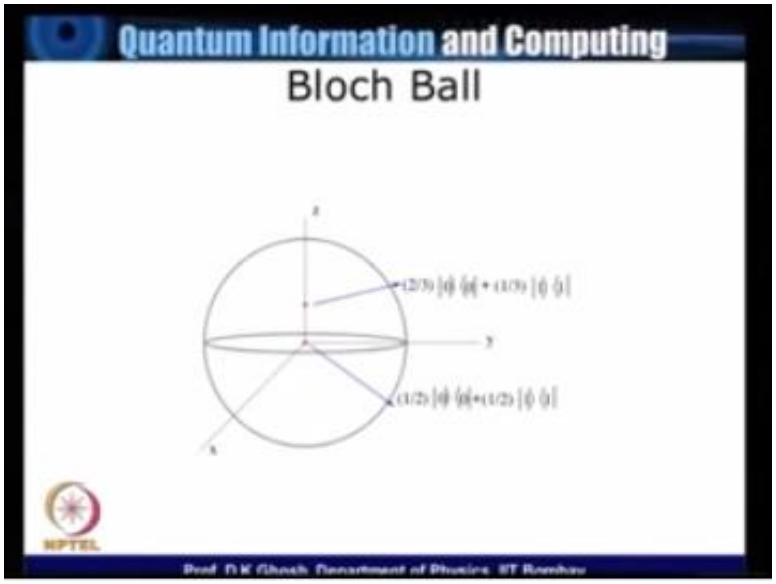

Look at the slide here now supposing I am looking inside the Bloch sphere, a point which is here which is one-third way up, see this a unit sphere one-third way up along the z-axis. Now we will show just now that this represents this state. 2/3 00 + 1/3 10, the point here the center it corresponds to ½ 00 + ½ 1, Now let us see how does it work out, so let me take Z component to be equal to a/3.

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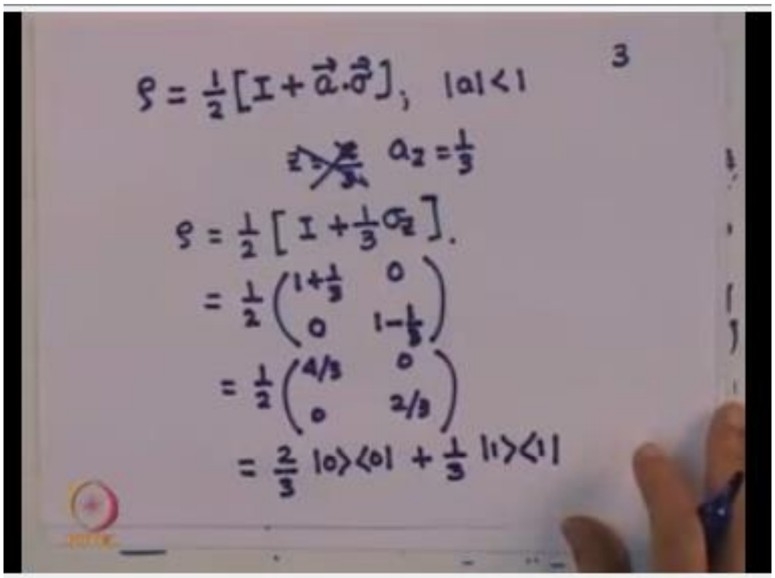

So in that case my row in corresponding to this is ½ [I + 1/3] so z component is a/3 I have said and my ‘a’ is of course the sorry let me take Z component to be equal to az to be equal to 1/3. So this is σz/3, so this is equal to ½ [1+1/3] of course off diagonal elements are 0 and since σz has -1, so this is 1-1/3 which is equal to ½(4/3, 0, 0, 2/3) and this you can trivially show is equal to 2/3 |0> < 0| + 1/3 |1> <1|. So this is what that point corresponds to, suppose you have to calculate.

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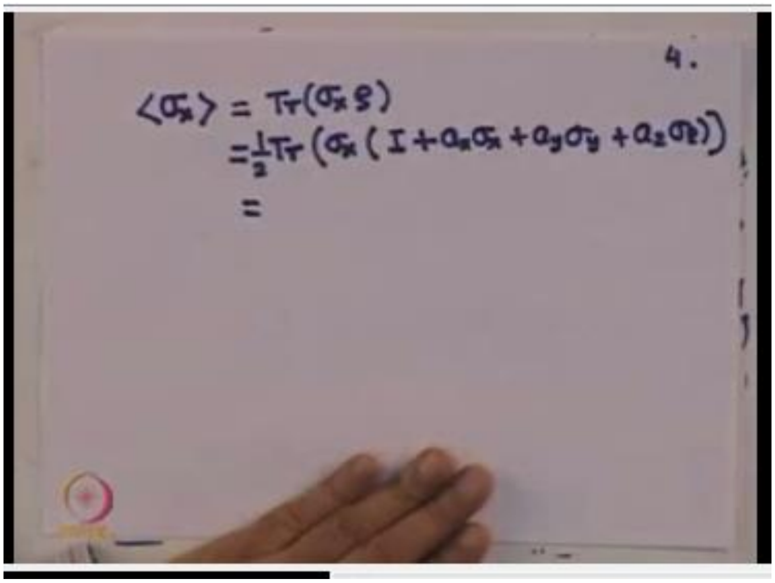

The expectation value of let us say X component of σ this is given by Tr (σ_x ϱ) by definition, so this we can expand it now Tr (σ_x) remember we said that the ϱ can be written as I + well there was one more two there, a_x σ_x + a_y σ_y + a_z σ_z, remember that σ_x ^{2} is an identity matrix but σ_x, σ_y product is i times σ_z etc and the trace of each of the Pauley matrix is equal to 0.

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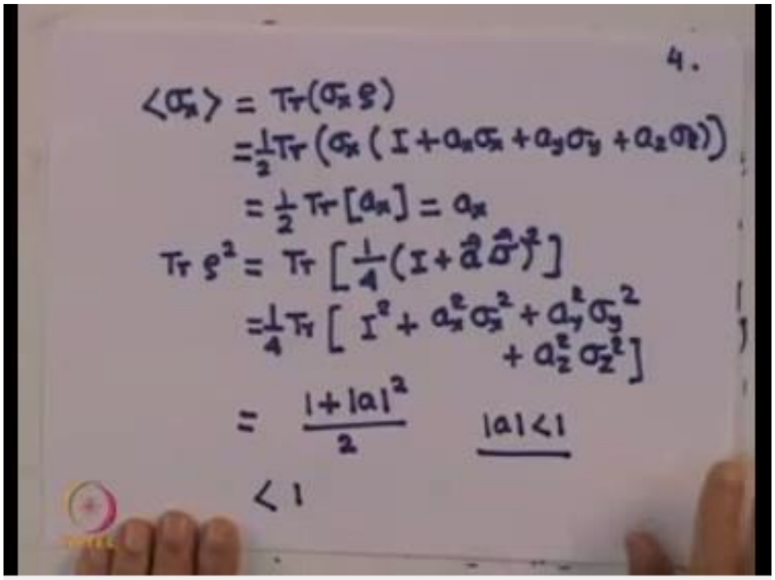

So therefore if you carry on the multiplication through you get this term will give you 0 because it is trace of σ_z but there is a term here which is ax times $\sigma_x{}^2$ so therefore I get ½ Tr ($\sigma_x{}^2$ times ax) which is equal to ax, so these are two by two matrices so therefore this is nothing but ax itself and likewise you can show σ_y average is ay, az etc, you calculate Tr (ϱ^2) the trace of ϱ^2 is place of $(1/2)^2$ which is ¼ + I + a. σ^2 expand this out again.

¼ taken out, now when I expand this, by first writing a. σ as ax σ_x + ay σ_y + az σ_z so this is a sum of four terms when you take a square you would say there are too many terms but the problem is not that difficult because these will contain the cross terms which have product of σ_x, σ_y which are nothing but I times σ_z and trace of that will be equal to zero, the only terms which will not be equal to zero.

Are those for which I get $\sigma_x{}^2$ $\sigma_y{}^2$ $\sigma_z{}^2$ in other words I will be simply left with ¼ [I^2 + ax^2 $\sigma_x{}^2$ + ay^2 $\sigma_y{}^2$ + az^2 $\sigma_z{}^2$ Tr ($\sigma_x{}^2$) is identity, so therefore I am left with ax2 + ay2 + az2 which is nothing but $|a|^2$ so therefore this quantity is 1+ $|a|^2$ /2, the factor of 2 in the numerator comes because of the trace of identity matrix is equal to 2, so you notice again that since we have said |a|<1, so Tr(ϱ 2) for this case as expected is indeed less than 1. So therefore the point satisfies all the properties that you require for a mixed state density matrix, so therefore the position of a point inside the Bloch sphere tells you.

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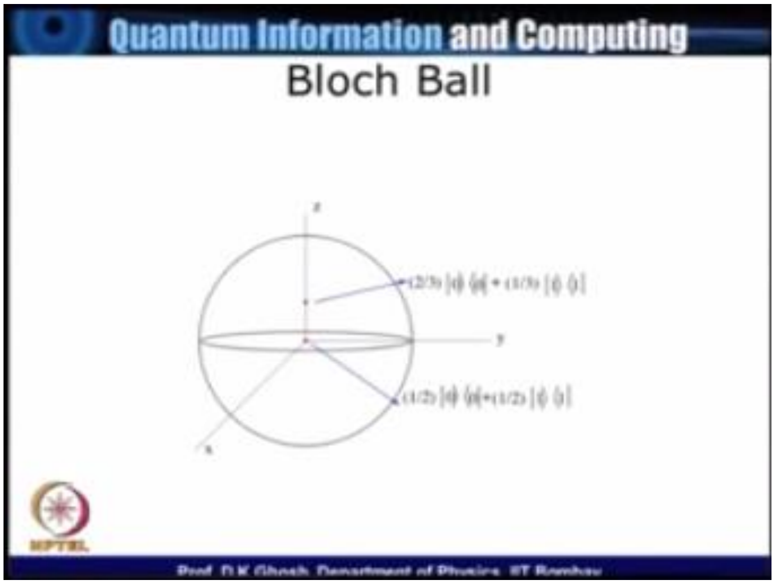

That what is its location of the mixed state, now having done that let us go over to another point of interest. Now as we have pointed out several times our systems are never closed systems, they usually interact with environment surroundings, the surroundings by definition need not be a very big system. For example, I could have a one qubit system interacting with another qubit which is of not interest of interest to me.

So I talk about a system of interest which I say is A and anything which interact with it which is of not of my interest I call it the environment and let us designate it as system B, so in other words i have of necessity to deal with a composite system consisting of A and B. But my interest is not on this composite system, but to get or extract out of it the properties of the system A which is the system of interest will, now how do I do it.

Now this is done by a prescription which is taking reduced density matrix. What reduced density matrix does is to in some sense average out the environment and the definition is shown here on the slide.

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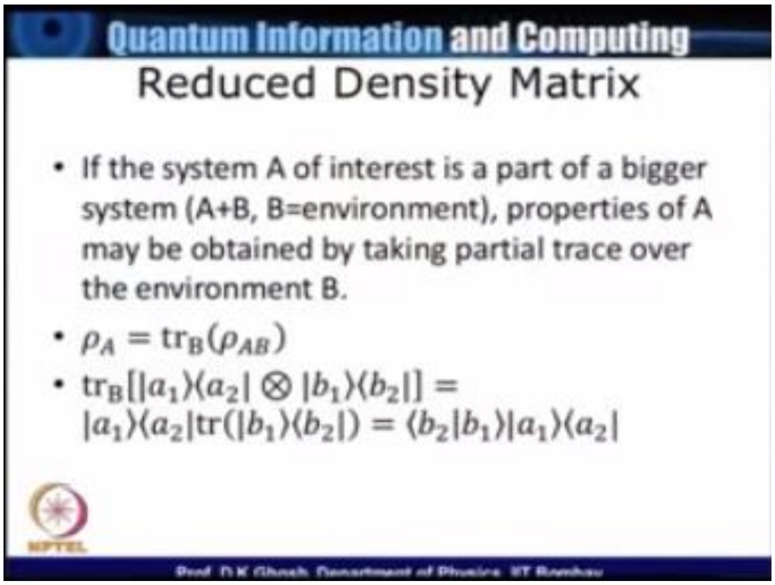

What I am trying to say is if you take partial trace of the environment, so my density matrix of interest ρ_A is obtained by taking the trace over the environment which I have called as B of the density matrix of the composite system. Now you can see immediately how, why it happens supposing you take let me work it out here.

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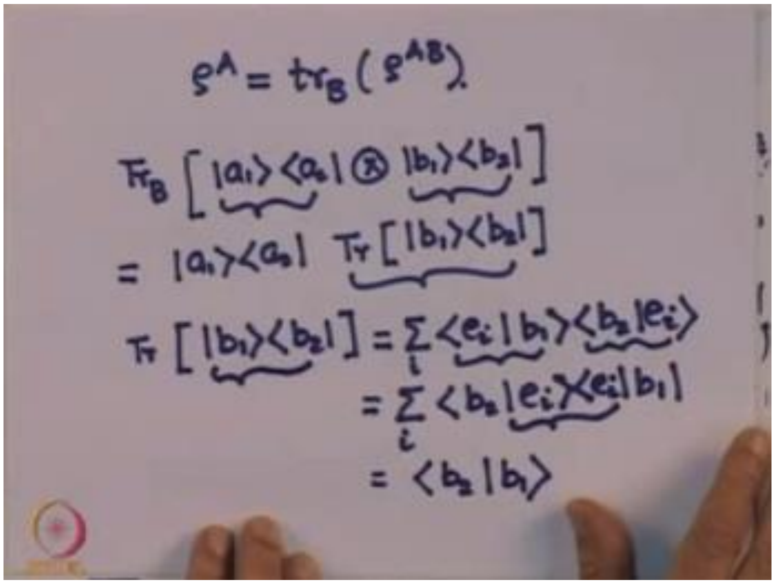

Supposing you take, let us look at how what does this trace with partial Tr_B supposing I have a trace of a thing like this, |a1>< a2|| b1>< b2| so arbitrary a1 b2 it a1, b1 extra. But, but these belongs to the helmet space of my interest and this is your environment. So when I say that take the trace partial trace with respect to this, this will give me |a1>< a2| times trace of this quantity |b1><b2|, now this is a number and you can see what is this.

Supposing I have a basis e_i in this space then what is my trace of [|b1>< b2|] by definition of trace it is sum of the diagonal elements, sum over $\sum_i$ $<e_i|b1><b2|e_i>$ diagonal elements. Now I can do the following, since these are scalar products I can interchange the order, so i will write it as sum over $\sum i<b2|e_i><e_i|b1|$ but notice that this is the density matrix corresponding to this is state e_i and if you take a sum over ∑I, because of the completeness that gives me an identity.

So therefore, this is nothing but $<b_2b_1>$ scalar product. So in other words when you take that trace of things like this there is a simple prescription that you simply change the order and take the scalar product, so that is what we get.

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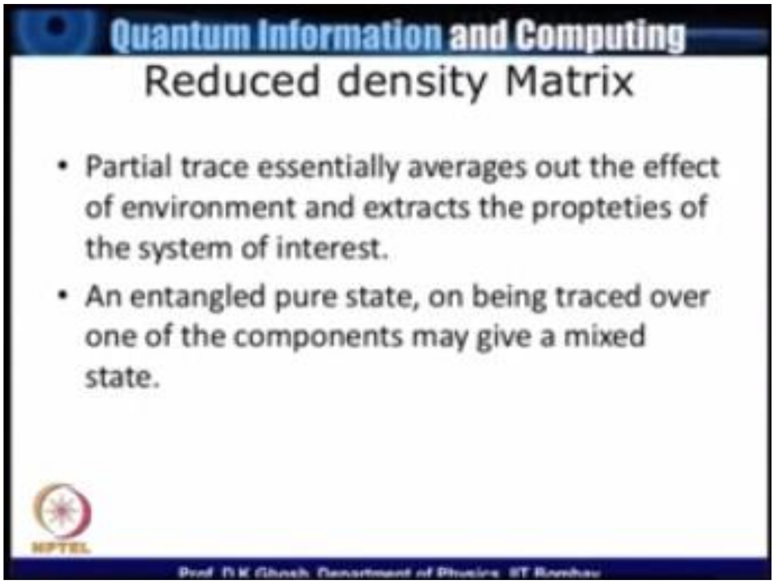

So the question that you would ask is, why do I take a partial trace we have already physically pointed it out that taking a partial trace is required because we want to extract information about system of interest and we would like the environment in some way to be averaged out. So let us suppose M is an observable.

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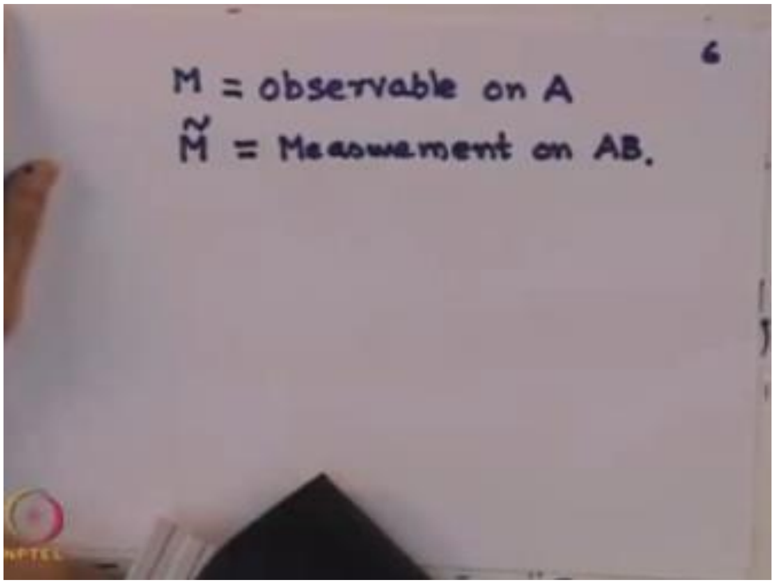

On our system A, let me say M~ this is a measurement that I do on the composite system AB, now obviously I am interested in making finding out the result of measurement on the system of interest. But however I am constrained to make the measurement on the composite system. Now suppose I have the basis an Eigen basis for the system A, then my composite system can be written as.

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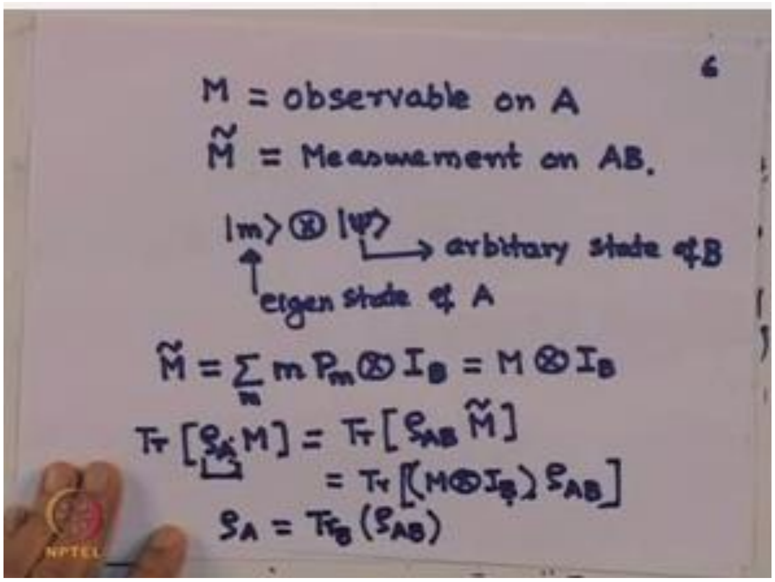

Let us say an Eigen state M so this is an Eigen state of the system A direct product with let us say |ψ> with represents an arbitrary state of system B. Now then since this is the Eigen basis I can write down my M ~ as sum over m using the spectral composition projection operator P_m. So in other words I am interested in making a measurement on P_m in the next lecture we will be talking about the how the measurements are taken.

And so this thing is nothing but my M x I_B where m is operated corresponding to observable on A. So if I take now, trace of ρ_a with m and I want this to be trace of ρ_{AB} with M ~ I need this because I have to make a measurement on the composite system in AB and M is an operator on a so this supposing I had a pure system this is what my result would have been, but since I have a composite system this is what I have.

So therefore this I can rewrite as equal to trace of M x I_B which is your M~ expression ρ_{AB} when I am taking a trace it is in my tree on which order I write it, like this relation is going to be equal to that relation, you can see that this is simply obtained if you decide that this ρ_A is given by taking the trace over B of $\rho_{AB.}$ So this is this is the story of the reduced density matrix. I will close this session with A an example supposing I take the entangled two qubit state the Bell state

00 +11/√2. Now you can immediately calculate the density matrix corresponding to this which is half 0, 0, 0, 0, + 0, 0, 1, 1, + 1, 1, 0, 0, and of course 1,1,1,1. Now remember that this is the pure state your stated the Bell state.

Now suppose I am interested in calculating item of interest is first qubit so what do you want to do is to trace take the trace over the second qubit of this rho because this row was on the composite system. Now what do I get I have already stated that if you are taking a trace over the second qubit, so what would I get from here for instance this will give you there is a half outside let us keep it like that this will give you 0, 0 trace of 0, 0 this is corresponding to the first qubit and this corresponds to the second qubit.

But we have just now shown that this is nothing but the scalar product of this with this which is equal to 1 and likewise you can write down say for example this one will be 01 and scalar product of one with zero which is zero. So this term is not there likewise this stuff will not be there but this term will be there so I will be left with half of 0, 0 plus 1, 1 what is the interesting thing this is the mixed state, so I started with the pure state.

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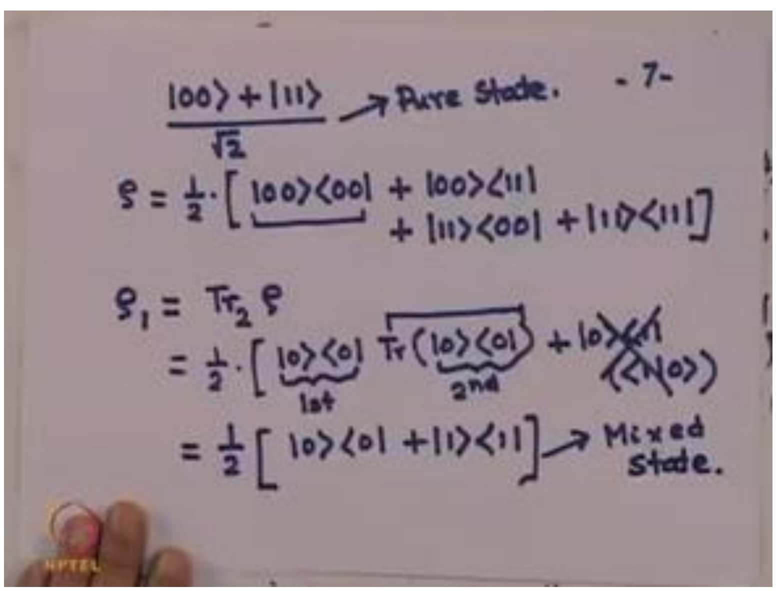

Which were entangled states when I averaged over the second cubit the state that I got was a mix state density matrix that gets the density matrix you can check immediately?

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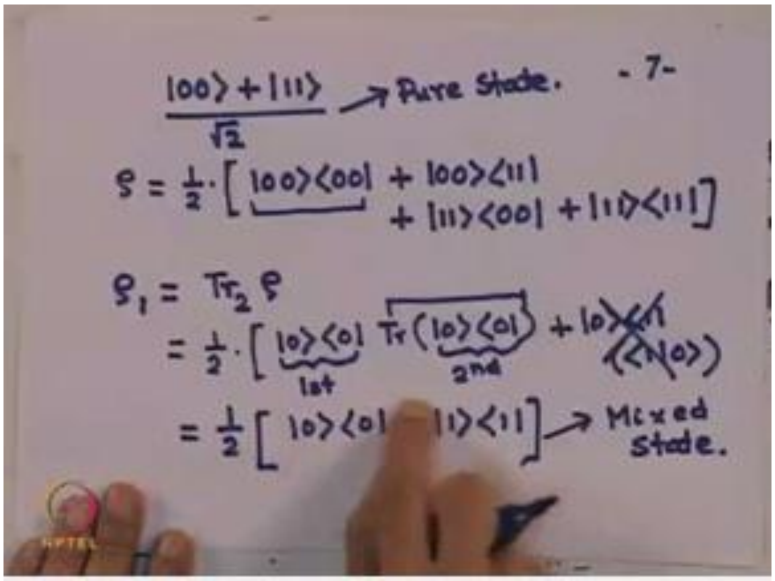

Because by identity resolution this quantity is nothing but I / 2, so the before I leave this topic of density matrix which we have taken last two or three lectures it is probably appropriate to summarize what we have got what we said is frequently we need to talk about systems which are not closed, systems which are parts of an ensemble, and when that happens the original postulate of quantum mechanics saying that a quantum state is represented by array in the Hilbert space that does not hold good anymore.

To take care of such systems we need to define a density matrix we obtain the properties of density matrix properties like trace of $\rho = 1$ it is a hermitian positive operator etc., and we related it to the expectation value or we related the expectation value of physical observables to the density matrix by saying for instance, that if you want an expectation value of an operator A it is given by trace ρ_A.

We also talked about how to distinguish a pure state from a mix state because they are trace though trace of the density matrix is equal to 1 for a pure state square of the density matrix is equal to the density matrix itself, so these were the properties of the density matrix which we considered in the last two or three lectures.

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