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Quantum Information and Computing

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Modul No.03

Lecture No.11

Density Matrix - I

In the last lecture we had discussed super dense coding. We had pointed out that the coding or the dense coding or super dense coding arrows out of the desire by Alice of sending some classical information by sending a lesser number of quantum bits to Bob the in this lecture and the next one what we intend to do is to talk about the real systems which one actually considers see when we stated the laws of quantum mechanics or the postulates of quantum mechanics it was assumed that we are talking about a closed system.

A system which is there by itself does not interact with outside systems in other words our definition of the system was a self-contained system now generally what we consider our are study are much larger systems now what happens when we study much larger systems are the following one our first postulate was that the states are represented by raised in a Hilbert space now that postulate may not hold good.

And as I go along I will sort of explained by giving examples of why I am making that statement the second thing that we talked about, so far was that we talked about measurements in a computation all basis or even other basis but basically the measurements which are orthogonal projections on to your basis states, now that may not necessarily be applicable the third thing which we assumed is the big evolution is unitary now this postulate also may break down when we do not consider closed systems or what we will be talking as pure system. Now let us see this with an example.

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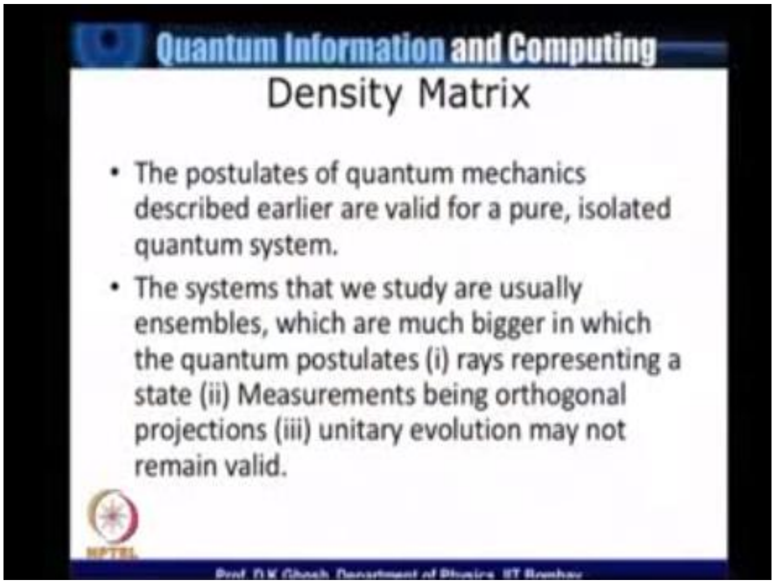

So this is what happens now just to give you an idea is the following.

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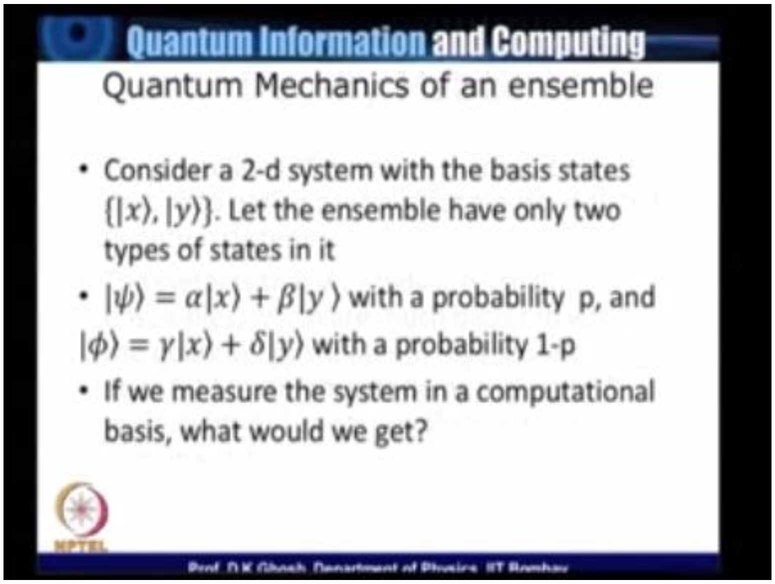

This open systems is essentially a collection of system, so let us just forgiving you a trivial idea consider it to diminutions one system whose basis states are xy it could have been 0 1 it does not matter and supposing I have an ensemble I am a collection of states but the states are not of one type so I have a state or type of state which have their state vector ψ as α times x + β times y you could have taken them α times 0 + β times 1 it does not matter you have another type of state there which is in the same basis written as γ times x + δ time y.

The supposing I had n number of states in the system then n times p, p is the probability n times t is the number of states in the first category and n time 1 -p is the number of states of the second category another question that we ask now is that supposing we measure this system in a computational basis 0 and 1 or x and y in this case. What is the result that we would get?

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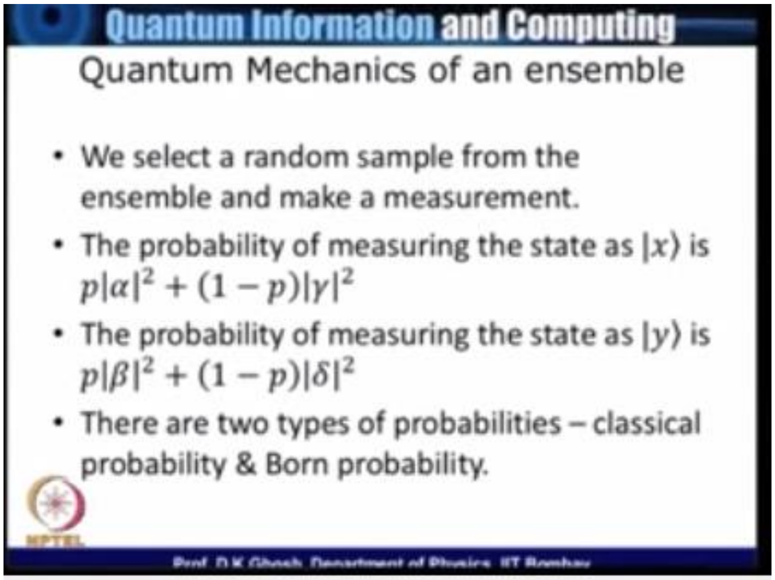

So the point here to notice the following that what is meant by a measurement in this case now when I make a measurement what I have to do is to pick up one of the state's belonging to this ensemble now when I pick it up whether it belongs to the state ψ or it has a state Ø that is determined by the probability with which I might mixture or the collection was better but that was a classical probability.

So I said p is the probability that in my collection the state that appears is ψ and 1 – p is the probability that the state that appears is Ø so when I make a measurement I need two things I pick up a state and I know with probability p to b ψ and with probability 1 –p to be Ø having done that I would make a measurement of the that picked up state in the computation basis so what would I get so let us return come back here.

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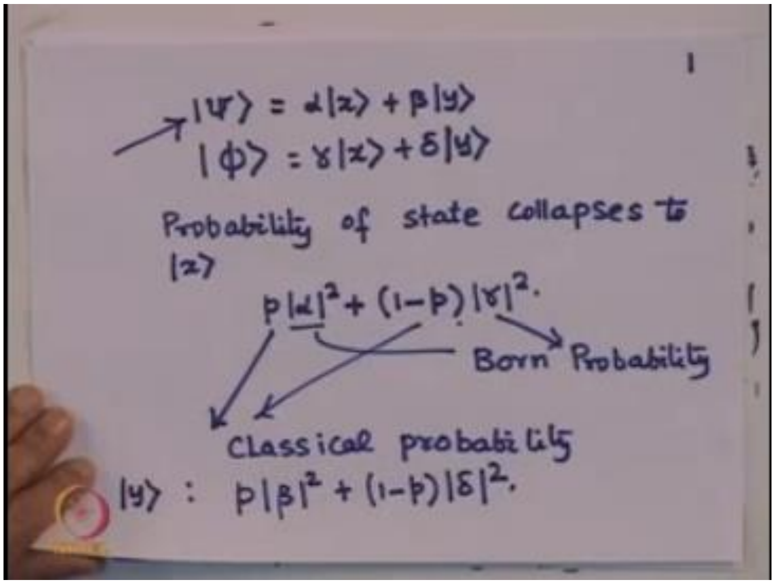

So let us suppose I have is a state ψ and that we have said is α x+ β y and it could have been a state Ø which is γ x + *δ* y now when I picked up at random if I happen to have got this state ψ I then my probability that my state collapses to let us say x now this is given by p times α^2 now this is provided I have picked up the state ψ but if I happen to have picked up the state Ø which is done with the probability 1-p and the probability of getting state x here is γ^2 so this is the product of two types of probability.

This is a classical probability p and (1-p) and this is the Born probability and these this p and (1-p) they are classical, obviously the probability that it collapses to a state |y> is given by P times |ß$|^2$ + (1-p) | δ|2**.** So when you consider a general system which is not which does not contain only one type of states this is the extra considerations that comes in. Consider on the other hand a composite system now supposing I have a composite system.

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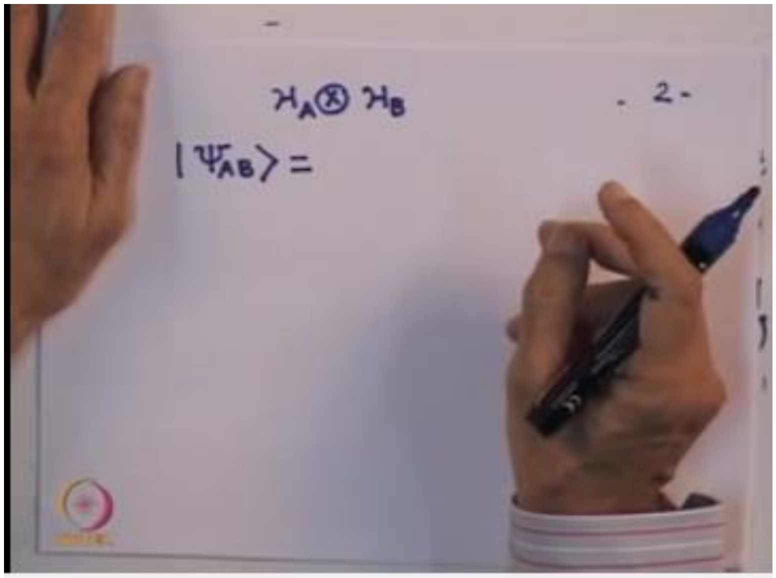

And let us suppose the composite system I will call as A x B and the corresponding Hilbert space is H_A x H_B, now suppose I have Ψ_{AB} as a state in this Hilbert space, now if it so happens that Ψ_{AB} factorizable, remember we are using the word entanglement we are saying if this be if the states are not entangled.

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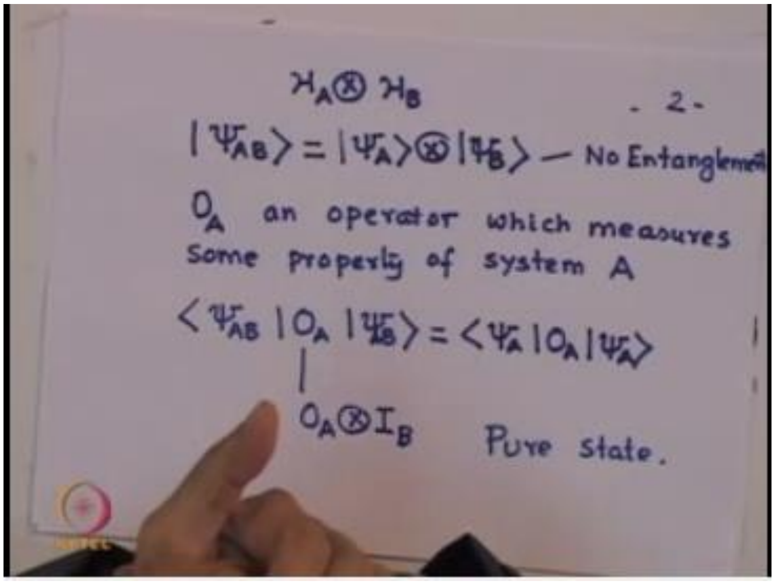

Then I can write Ψ_{AB} as product of $|\Psi_A >$ with $| \Psi_B >$ now when such a thing happens if I consider an operator, supposing O is an operator OA is an operator quit measure some property of system A, then what I am trying to do is Ψ_{AB} $| O_A$ Ψ_{AB} > now since this is this acts only on A and this is factorizer, I can write this OA as product of OA direct product with I_B where I is an identity operator on B.

And then this will give me $<\Psi_A| O_A |\Psi_A >$ multiplied by Ψ_B I_B Ψ_B but that is equal to 1 by normalize sum of the Ψ_B, so this is what I would get. So in this case the system behaves like a pure state, so the there is no entanglement so no entanglement the composite system behaves like a pure state, now notice if the system does not allow such factorization the usual nomenclature uses.

Supposing A is my system of interest and B is an environment, by the word environment we mean any system with which the system A interacts. It need not be a big system it can be as small as the system A itself, so if the interaction is such that the wave function does not factorize.

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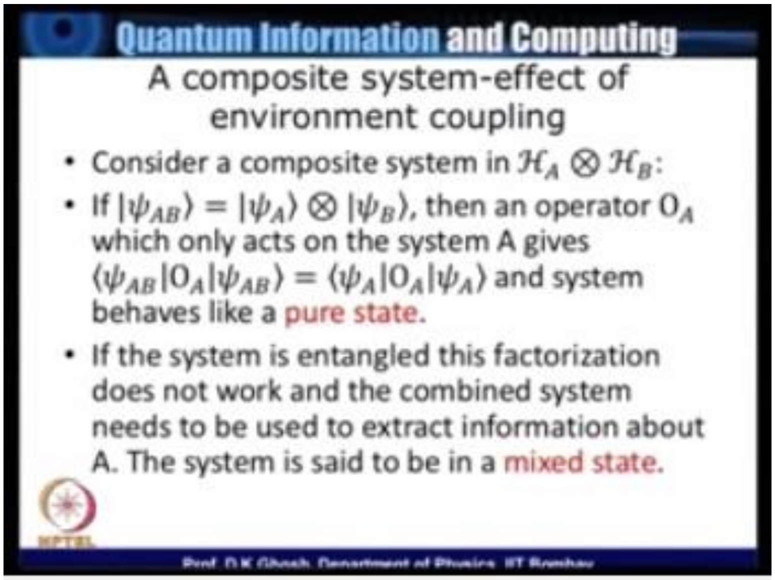

Then the above argument does not work and we need to extract information about properties of system A by a different idea, such systems are known as mixed system, mixed state. Now what we are going to do, in this lecture and the next is to talk about the postulates of quantum mechanics, as they need to be modified for such mixed system.

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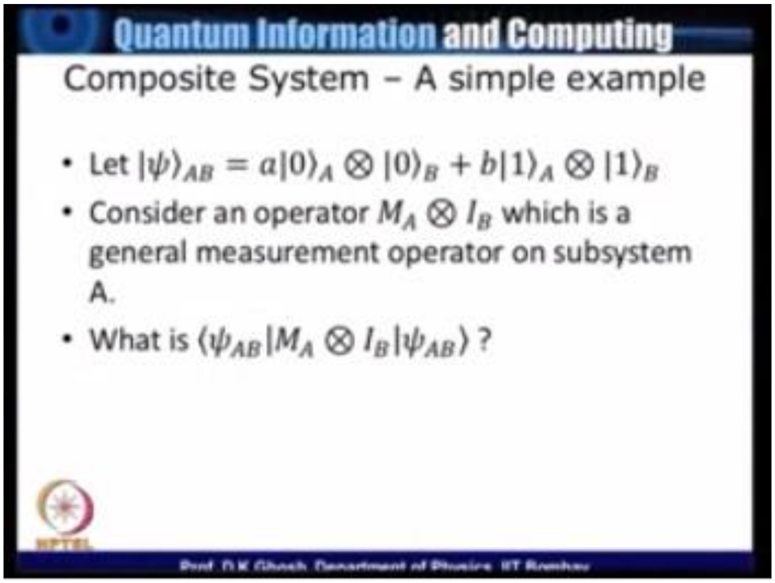

Continuing with our example, let us consider a very trivial case of a composite system which is entangled. So I am what I have done is to take Ψ_{AB}.

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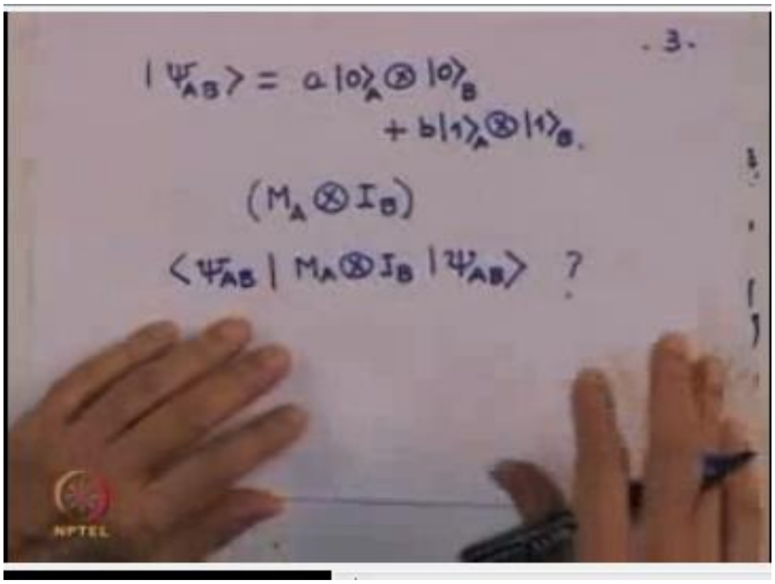

Is equal to suppose I write it as some constant times $0_{A,}$ $0_{B,}$ so I am taking a single qubit state to be both A and B and so the base is 0 and1 in both, but this is a composite system, so I write this as this plus let us say 1a 1b, this is obviously not factorizing. So the question is this that, how do I calculate? Supposing I have an operator in the state space of A and B is an identity, now this is the operator which I am interested in calculating the expectation value.

So how does one what is in other words I am interested in finding out, what is Ψ_{AB}, M_A x I_B $\Psi_{AB.}$ So let us do a bit of a an algebra with this same example and look at it the following way.

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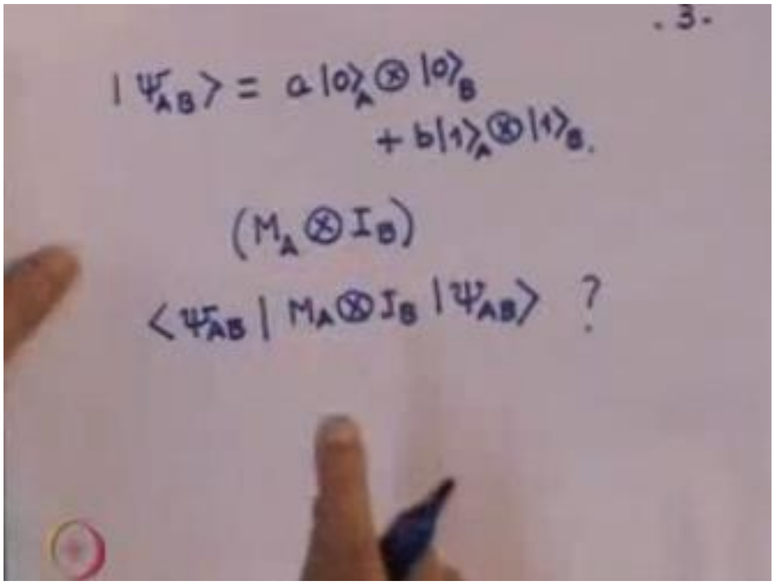

So well I have written this ψ_{AB} like this and we are going to consider what is the expectation value of this operator. I will do a bit of a formal arithmetic here.

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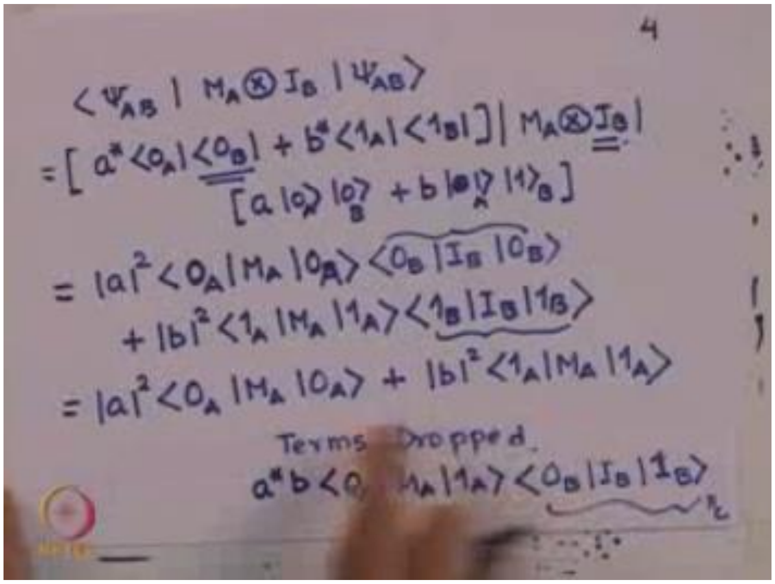

So we have $<\psi_{AB}|M_A xI_B|\psi_{AB}>$ so let us expand this out, so we have said already this is A00+ B 11, so this is the [so therefore I get a* $<0_A|0_B+b*<1_A|1_B|>$ then I have my operator stand which between bracket and then I have got $[a|0_A> |0_B>+ b|01_A>|1>_B]$ I can just multiply it through and get four terms there. Let me write down the non vanishing terms first, so I have got a*xa which is $|a|^2$ I have got $<O_A\ 0_A\ |M_A|\ 0_B>$ now if you look at it I have an I_B there, so I have got $0_B\ I_B\ 0_B$ let us write it for formality $<0_B\ |I_B|\ 0_B>$ this is should be $0_A+\ |b|^2$ similar term $<1_A|M_A|1_A>$ multiplied with $<1_B|I_B|1_B>$ there if two modems.

So notice that these terms are one because I_B does not do anything just keeps O_B and this is normalized and this term is also 1. So what I get here is I will come back to the two terms which I have dropped $<0_A|M_A|0_A>+\ |b|^2<1_A|M_A|1_A>$ now what about the terms I have dropped, the terms that I have dropped are the cross terms, terms dropped so the cross terms for example are $a*b<0_A|M_A|1_A>$ multiplied by <0B this is one of the terms $|I_B|\ 1_B>$, so this is clearly 0, because I_B acting on 1_B does nothing and then by orthogonality this is 0.

The other cross term also becomes 0, so this is the result I have got. So what is this thing compactly.

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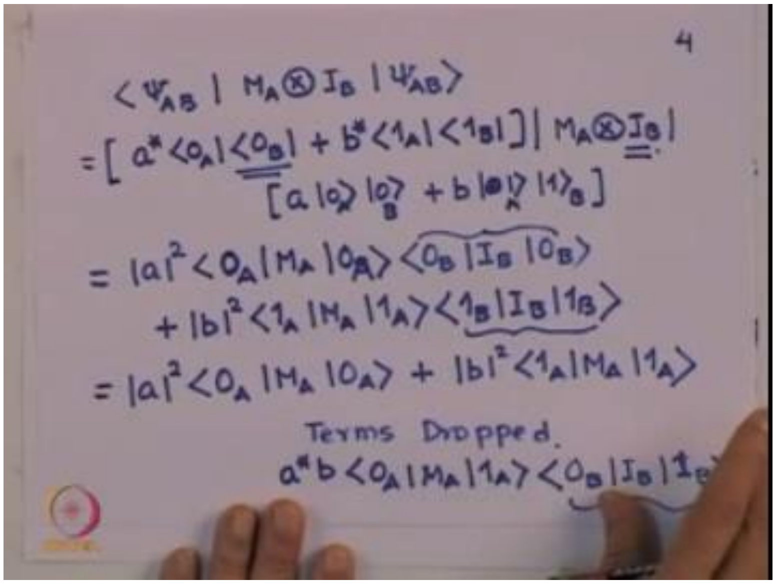

So this is nothing but a^2 times the diagonal matrix element of M_A + b^2 times the diagonal matrix element of M_A in state 1. So if I define, if I define ρ_A

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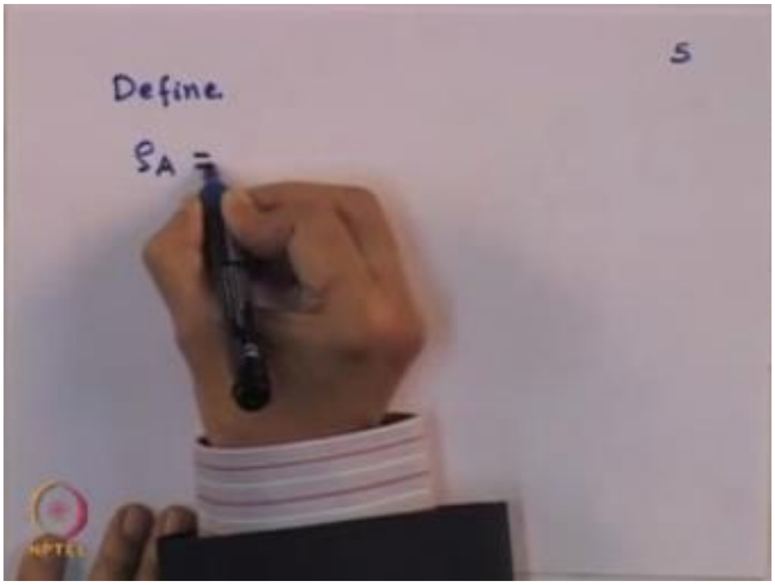

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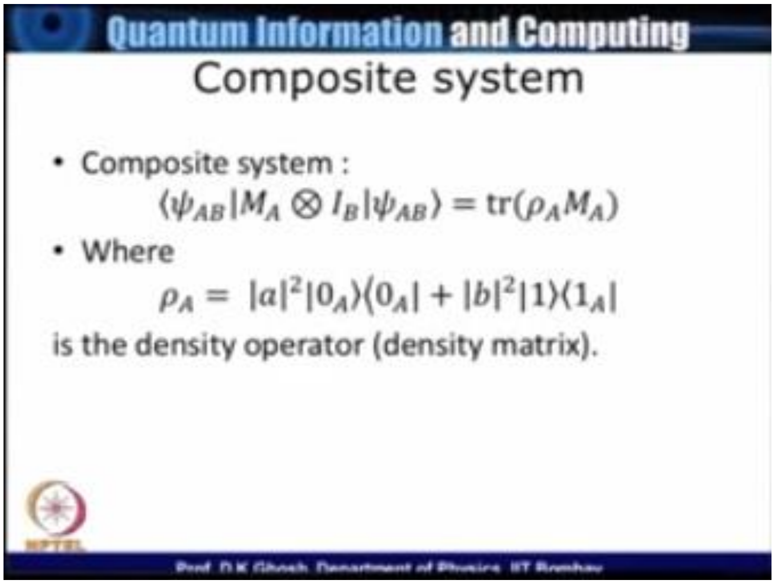

You can look at the slide which is also there, so I define ρ_A as.

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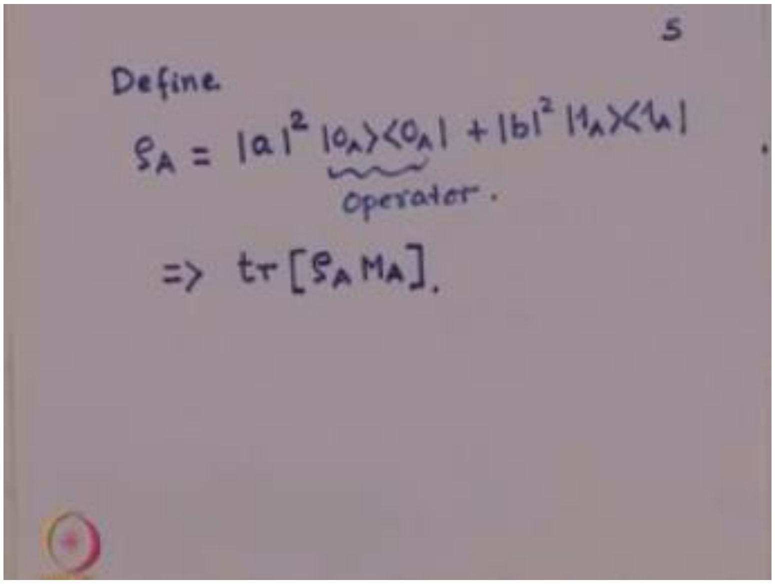

$|a|^{2|}0_A> 0_A|+|b|^2|1_A><1_A|$ is an operator. Remember, that a kit followed by a bracket is an operator this is an operator. So if I define this way, then what I have got here is to write this expectation value as trace, trace is nothing but the sum of the diagonal elements. So trace of ρ_A with M_A with this definition.

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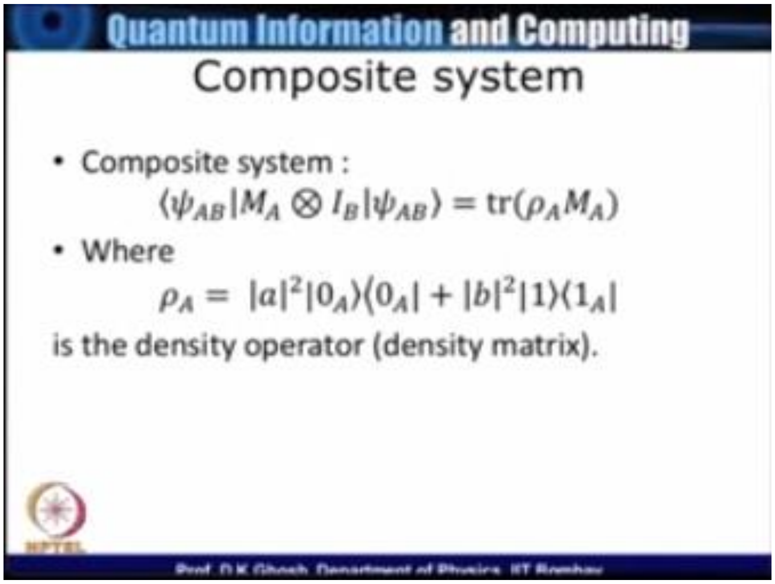

This is what it works on, you can simply plug in this ρ_A into my previous expression and get this.

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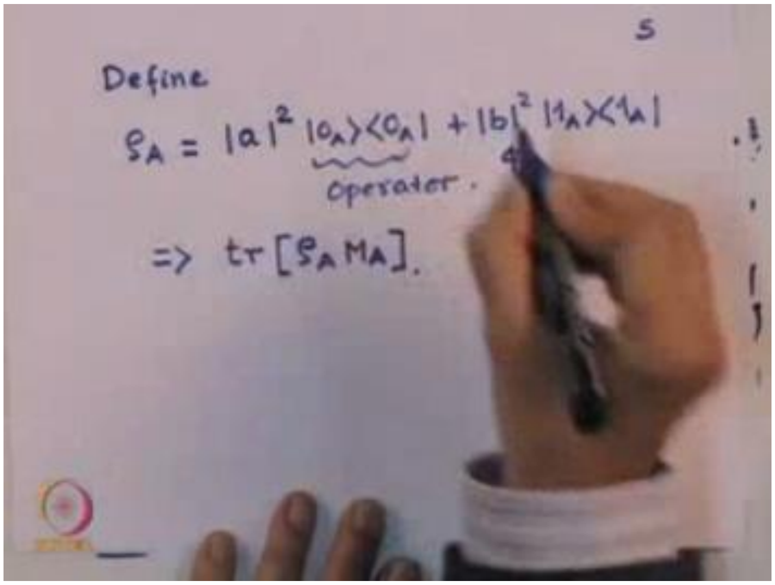

So this operator that I have defined here, this is called the density operator this rho a this is my density operator for the subsystem a also known as density matrix these are interchangeably used, we have already pointed out several times that any operator can be expressed in a matrix representation, so these are known as the density operator. Now to look at what is the difference between the two consider the following things.

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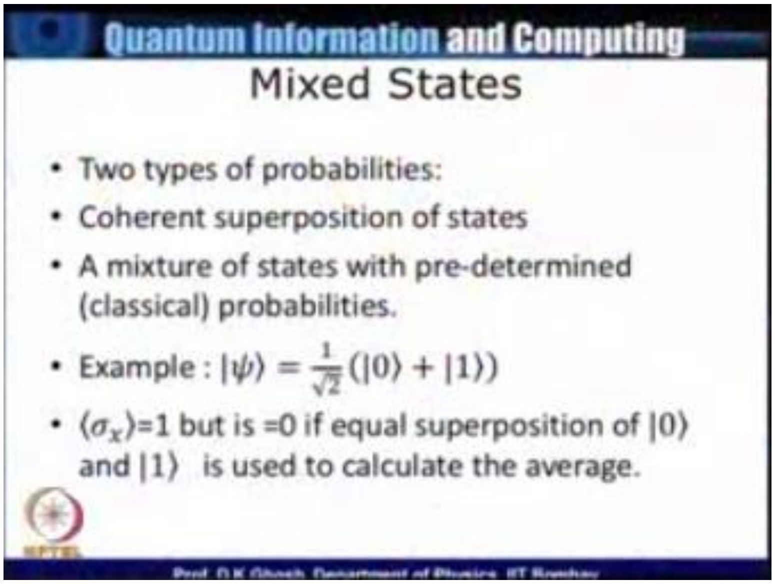

Let me first give you an example, so of a pure state, a pure state is a coherent superposition of the basis States for them. So let me take ψ to be given by 1/√2, 0 + 1 this is what we have been talking about several times. Now supposing I take this state and calculate what is the expectation value of for instance σ_x operator, now you can immediately do this because this 1/2 because a brand they get each one will have a so I have √2.

So I have this that, I have 0+1operator σ_x acting on the state 0 +1 we know that σ_x changes a 0 to 1and 1 to 0, so therefore this becomes 1 and this becomes 0 just multiply them this, this term this into this becomes 0 this into this is 1 into this is 1, so that is 1 plus 1 is 2 by 2 that is equal to 1. Now however supposing instead of this coherent superposition I had a equal superposition equal mixture if you like off state 0 and 1.

Now what is the difference, the difference is that this is the pure state quadrant superposition here what we have to do is to look at it I have a collection in which I have states 0 and state one in equal mixture.

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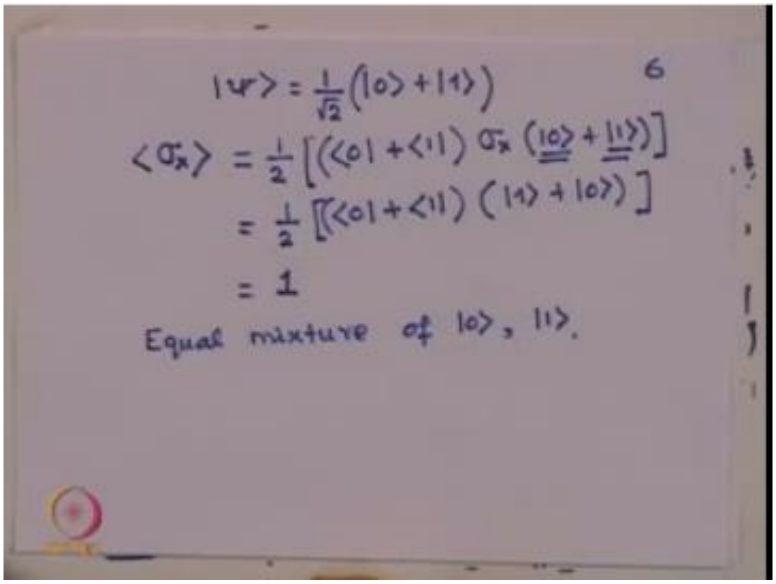

So when I calculate the expectation value I pick up one of them and that will be of course picked up with some probability in this particular case it will be probability of ½. So I will calculate the σ_x expectation value this way either probability ½ of picking up the state 0 so it is ½ of 0 σ_x 0 and an equal probability of picking up a state one and since σ_x converts a 0 to 1 and this is orthogonal so I get 0 + 0 = 0.

So this is the difference there that this is the coherent superposition this is an equal superposition of the states.

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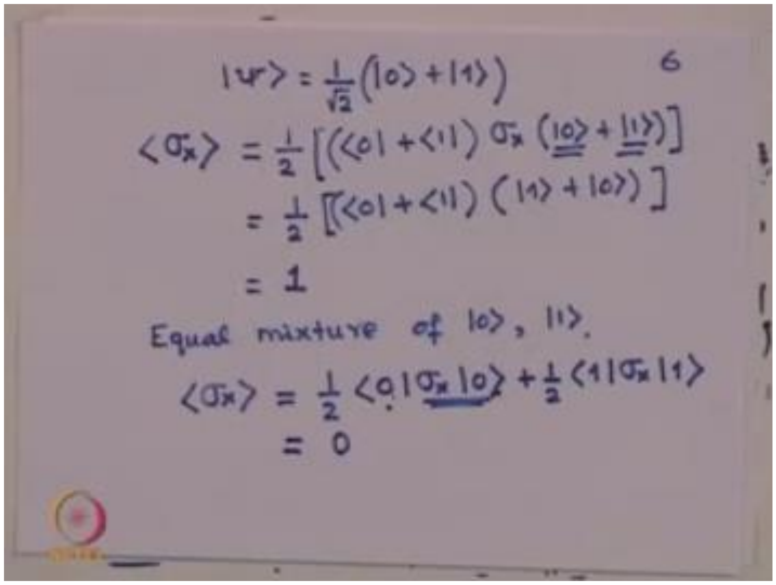

Now so let us look at how does it work? Now in this particular case if you are looking at the density matrix. So look at the density matrix for the equal superposition case the, for the equal superposition case I have got 1/2 that was the probability and I have 00+11. So this is now since my basis state only has two elements the, this by completeness theorem is 1, so I got this is actually it should be I / 2 because completeness theorem gives me this has an identity operator.

So this I should write it is I / 2, I will show in the next lecture that whatever result we proved just now for σ_x that if I have an observable A and a corresponding operator A, then expectation value of A turns out to be trace of rho times A.

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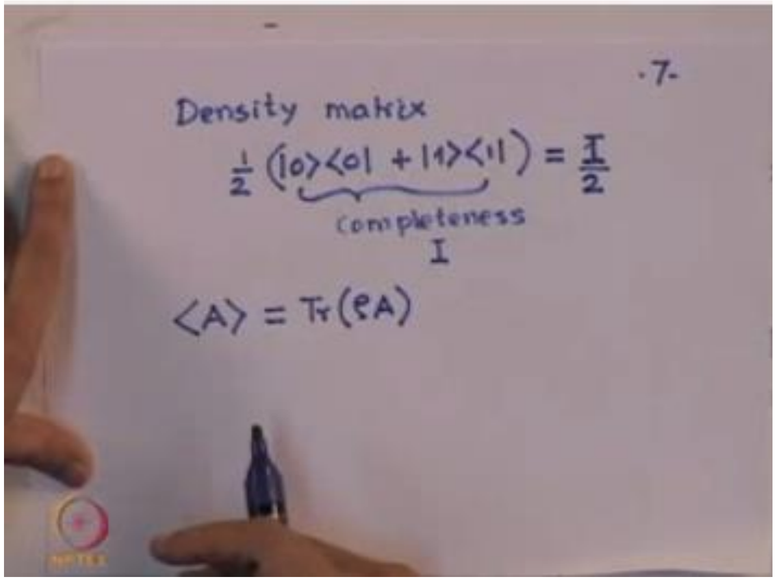

We will go through in our next lecture the properties of the density matrix and its importance in discussing systems which are composite systems or open systems.

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