Special Theory of Relativity
Prof. Shiva Prasad
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 15
Momentum Energy Four Vector

We had started discussing the concept of four vectors. And in our last lecture, we started talking about the velocity four vector. We realized that, if you take the first three components of standard velocity vector, which we call as u x, u y and u z, these do not form the first three components of velocity four vector. The reason for that is that, in order to evaluate velocity, we have to divide the displacement by time or rather time interval. Now, though the displacement does form components of the four vector, the time does not, because delta t is a frame-dependent quantity.

So, we agreed that in order that we want to construct a velocity four vector, we should not divide by delta t, but should divide by delta tau – the proper time interval between the two events of the displacement of the particle. Once we do that, then the standard values of velocity components u x, u y, and u z get multiplied by gamma u.

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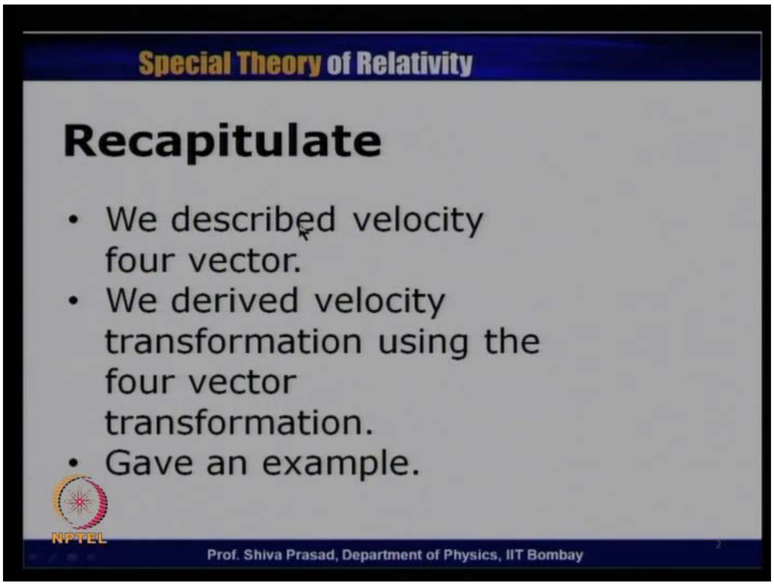

They form the first three components of velocity four vector. And then we also took one example and showed that how to construct a velocity four vector, and also showed that once the same particle is looked by another observer in a different frame, then how this velocity four vector will transform using the standard transformation equation. So, this is the recapitulation of what we had done. We described velocity four vector; then eventually, we also derived velocity transformation using velocity four vector. And, as I have mentioned; then we gave an example.

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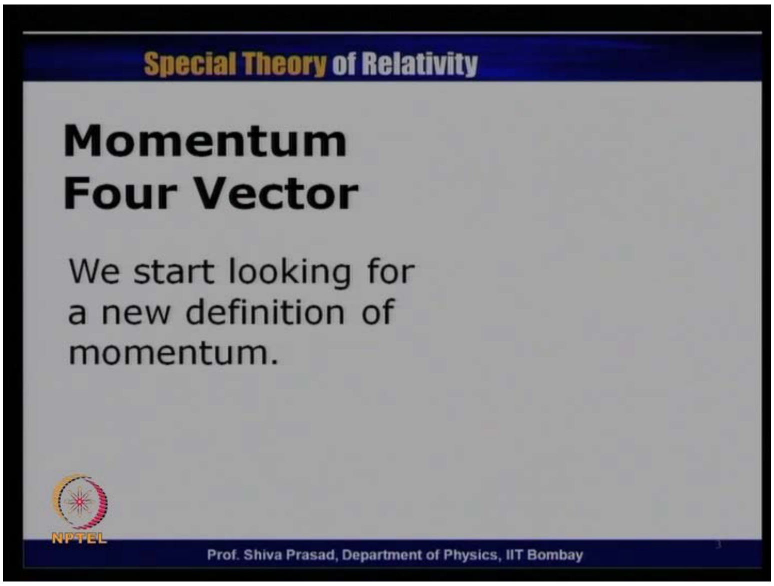

Let us recall that, our idea the way we started looking about the four vectors was eventually to define a momentum four vector. So, now, let us start looking at a new definition of momentum. We have talked about displacement; then we talked about the velocity; then let us talk about the momentum. So, we now start talking about the momentum four vector and try to see the way we have to look for a new definition of momentum.

Now, in order that, we look into a new definition of momentum, I have to multiply this particular thing by something, which has a dimension of mass or we have to multiply a velocity four vector – that is something which has a dimension of mass. And, this particular mass should be frame independent. We do not normally bother about in the classical mechanics about mass, which we always take this to be frame independent. But, let us be specific. And, in order to convert from velocity four vector to momentum four vector, let us define another quantity, which we call as a rest mass of the particle.

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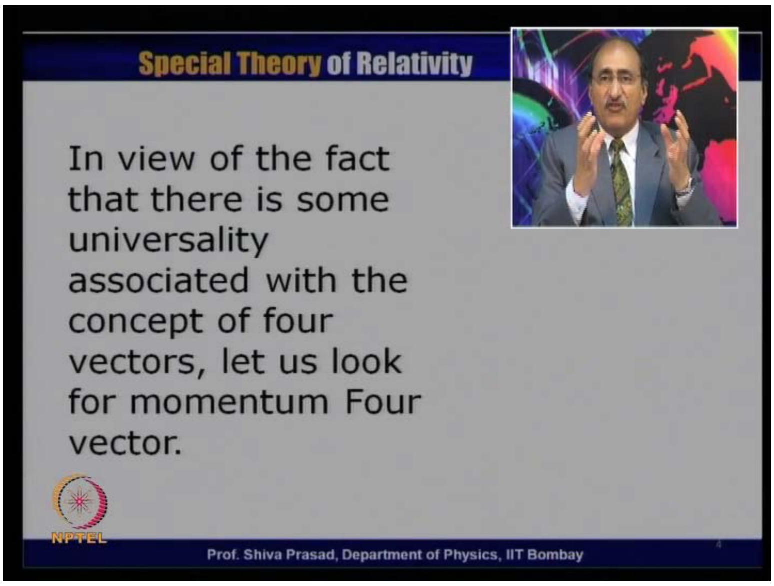

So, in view of the fact that, there is some universality associated with the concept of four vectors, the idea is that, if we define, we could define this particular four vector from the momentum point of view. Then it would be easier to see that, its conservation is obeyed universally means in all inertial frame of reference.

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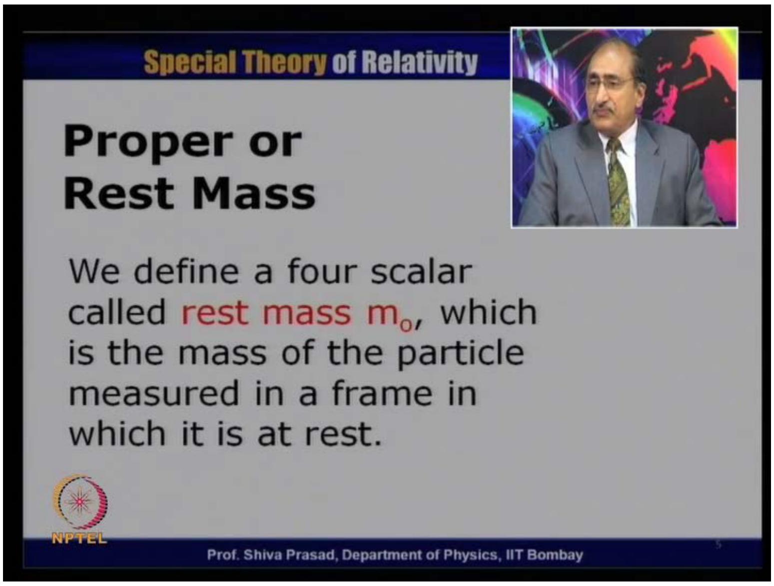

So, as I said, let us define a proper or what we call as a rest mass, which we call as m naught. And, to be specific, let us say that, this mass is the mass of the particle, which is determined in a frame of reference in which this particle is at rest. So, that is the reason.

Popularly, this is always called rest mass of the particle. And, because this is something, which is always measured in a frame in which the particle is at rest, this is a four scalar. So, even if I change, go to different frame of reference, m naught – the rest mass of the particle should not change or does not change. So, we define a four scalar, which is a frame independent quantity called rest mass of the particle calling this as m naught. Once I multiply this m naught by the velocity four vector, I am expected to get something which has ambition of momentum; and let us start calling that as momentum four vector. We will eventually see how to physically interpret and how to see that, conservation of momentum becomes a universal (( )).

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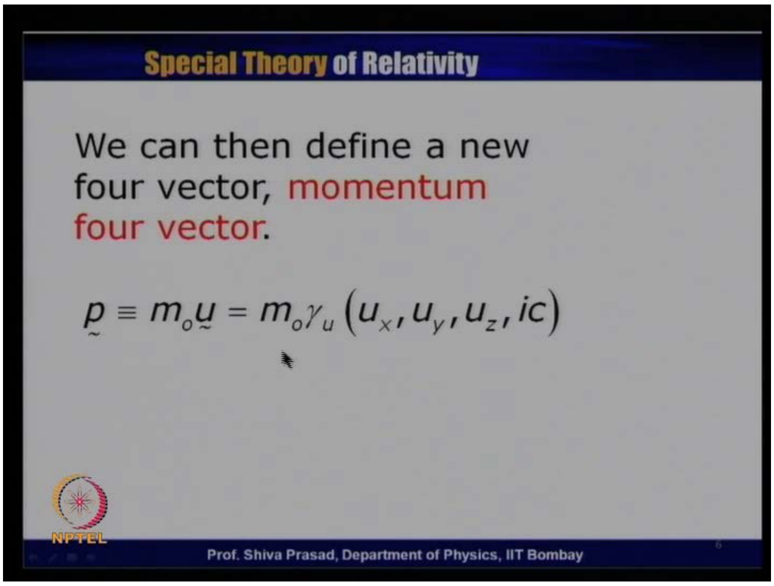

So, what I do; we have already taken the velocity four vector u and the components of that particular four vector as gamma u multiplied by u x, u y, u z, i c. It means the first component is of velocity four vector was gamma u u x; second component was gamma u u y; the third component was gamma u u z; the fourth component was gamma u i c. This entire thing I have multiplied by m naught and called this as a momentum four vector. Again, I repeat this m naught is supposed to be frame independent quantity called the rest mass or the proper mass of the particle. So, let us assume at the moment that, the first three components of this particular momentum four vector the way we have defined are the new definitions of momentum. If we assume that, let us see whether the conservation law really becomes universal, because that is what is our idea; that is what we are aiming at that, the conservation of momentum principle becomes a universal principle. It means if it is obeyed in one frame of reference, it must be obeyed in all other frames of references. So, let us try to look whether this definition can yield me that particular universality.

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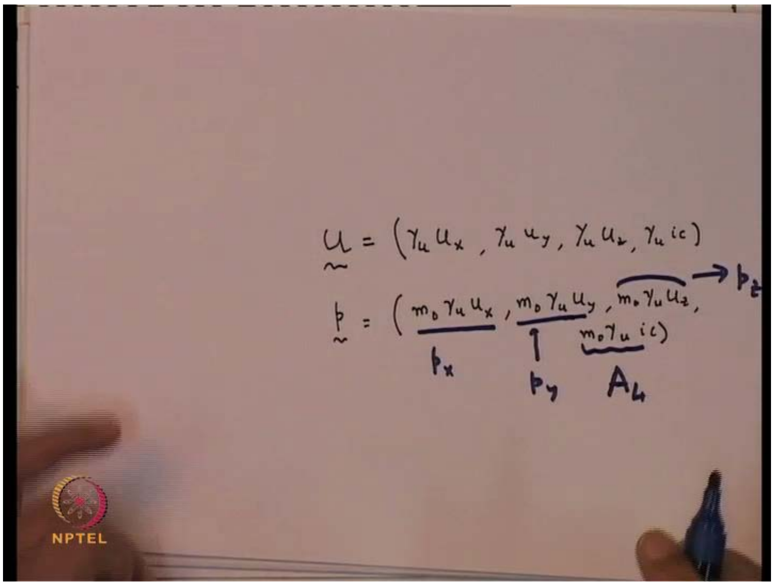

Before we do that, let us write these components. If you remember, we had written the velocity four vector as gamma u u x – that was the first component; the second component was gamma u u y; the third component was gamma u u z; and, the fourth component was gamma u i c. So, I have multiplied this by m naught. So, my first component of momentum four vector becomes m naught gamma u u x, m naught gamma u u y, m naught gamma u u z, and the fourth component is gamma u m naught gamma u i c. As we had agreed that, the first three components of this are the first three components of momentum; the new definition of momentum as per special theory of relativity. It would mean that… I would call this quantity here as p x, which is the x component of the momentum; this quantity I will call as p y; this quantity I will call as p z; and, this fourth quantity – at the moment, we have to find out what it is. Let us suppose this fourth component; I am just writing this as A 4. We will look at this particular A 4 little more clearly in the later part of this particular lecture.

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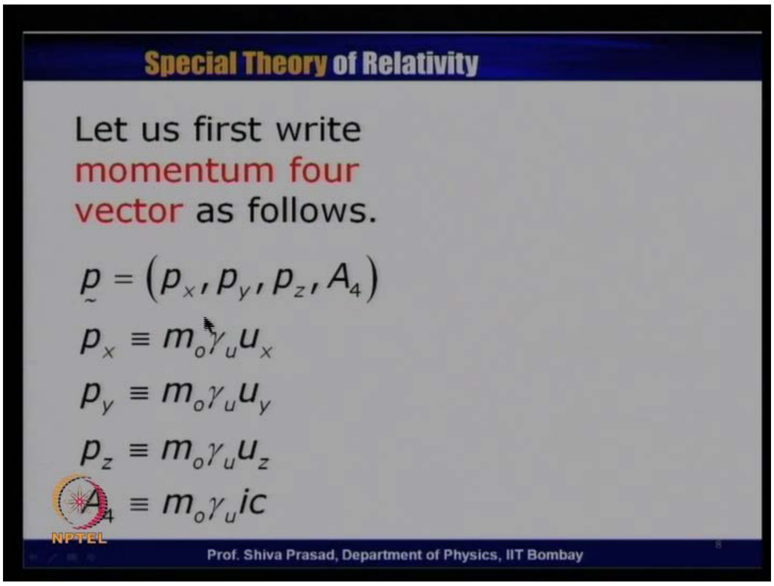

So, this is what has been written in this particular transparency. Let p – I am writing in the form of p x, p y, p z and A 4; while p x is equal to m naught gamma u u x; p y is equal to m naught gamma u u y; p z is equal to m naught gamma u u z; and, A 4 – something which we do not know is m u gamma u i c. These three I will call as the x, y and z components of the momentum of the particle. So, if we assume this particular thing, let us see whether conservation of momentum would be a universally accepted phenomenon. It means all the observers in all different frames would find the conservation of momentum law to be holding good with these new definitions of momentum.

What is conservation of momentum? The conservation of momentum tells that, if you have a system of particles and if they interact and if there is no external force on the system of the particle; in fact, we differentiate between internal forces and the external forces. So, we have a system of n particles and each are interactive with each other; and therefore, each is influencing, applying a force on other particle. So long the force is being applied on a particle by another particle of the system; that particular force I will call as an internal force. But, any force which is being applied on any of the particle by something, which is not a part of the system, we call that as an external force.

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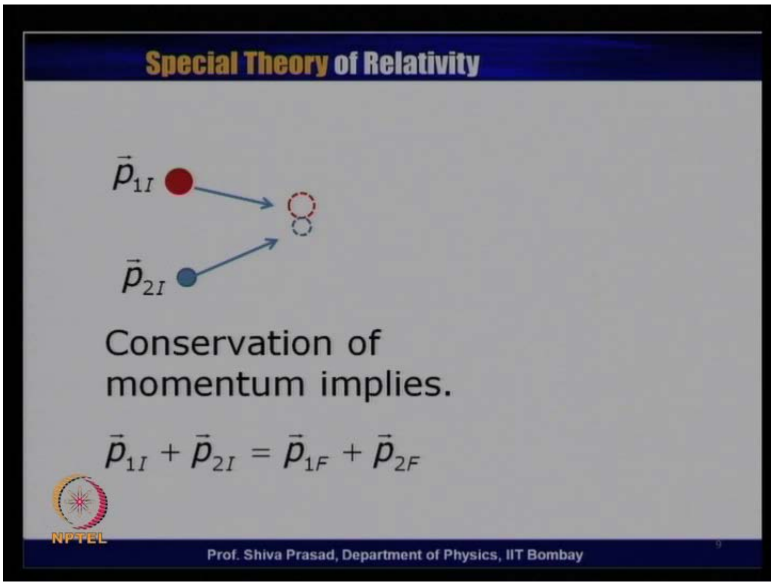

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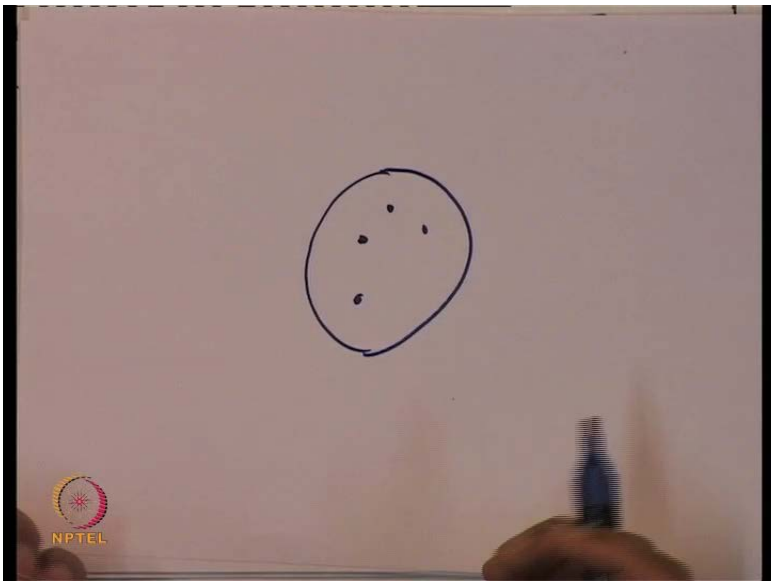

So, if you have a large number of particles interacting – something like a gas molecule; you have one particle here, you have one particle here, one particle here, one particle here; they keep on coming; somewhat collides here, something collides here. This could be for example, that matter even charged particles. So, there could be electrostatic force between these two; electrostatic force could be between these, two between these, these. So, they are all interacting. But, if I call this as the system, any force which is caused by particle within the system; that is what we will call as internal force. But, assume that, this whole thing is kept under gravitational field and there is an earth here, which is attracting all these particles. And, this earth is not a part of the system, then that particular force will be external force to the system.

So, conservation law – conservation of momentum law says that, if external force is 0, then the net vector momentum of all the particles must always remain constant. That is what is called conservation of momentum; momentum will be conserved. So, if no external force exists on a system, then the total momentum – the vector sum of all the momentum of all the particles must remain identical; it should not change. So, this is what I have shown in this particular picture in which I have shown only two particles.

So, let us suppose this is one particle; there is another particle. These two particles interact. And, in this particular case, I have assumed that, they just collide; collision is actually one of the form of interactions. So, once they collide, this particular particles velocity may change; which I have not shown; it is not important for our argument here. Similarly, the velocity of this particular particle is also likely to change after the collision. We have normally seen the collisions of particles, collisions of billiard balls; so many collisions. These are very natural things. Collision is a very standard classical mechanic phenomenon.

Now, what conservation of momentum would mean that, if I take the initial momentum of the first particle, which I am writing as p 1 I and the initial momentum of the second particle, which is I am writing as p 2 I; even though their momentum would change after collision, I must have p 1 F. If p 1 F is the final momentum of the first particle plus p 2 F; where, p 2 F is the final momentum of the second particle, because there is no external force to the system. Therefore, momentum must be conserved. Therefore, according to conservation of momentum law, this initial momentum must be equal to the final momentum. Now, what is our requirement? Our requirement is that, if the same collision is being seen by some other observer in some different frame, he may find different values of initial momentum of the two particles. But, whatever he or she finds, immediately after the collision, the final momentum in his frame or her frame should also be the same. So, though they may disagree on the value of the momentum like they disagree on the value of the velocity, but they should not disagree on conservation of momentum; initial momentum must be final momentum whatever is the value of initial momentum may be. That is what is conservation of momentum. And, that is what I wanted to happen. With this new definition of momentum, we have realized earlier that, if we just use Laurens transformation and do not change any other definition of momentum, then this will not be the case. We had given an example earlier.

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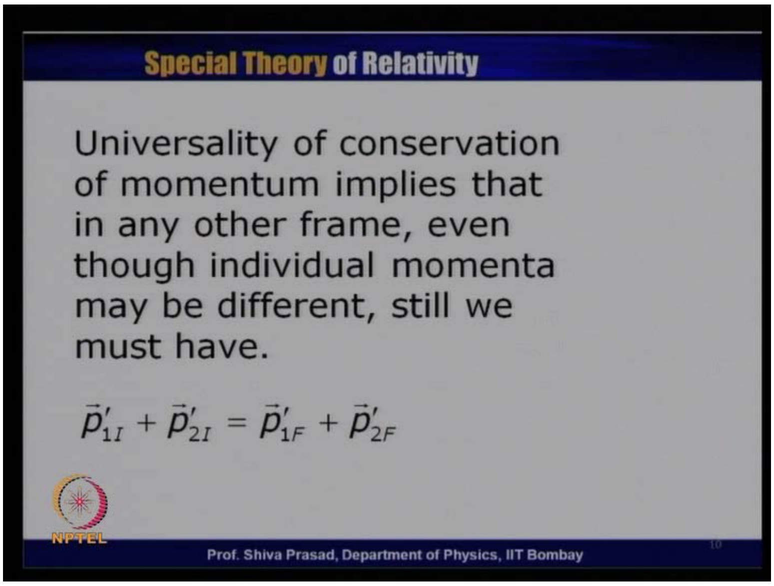

So, this is what I have written here. Universality of conservation of momentum implies that in any other frame, even though individual momenta may be different, but still we must have p 1 I – I have put a prime here to show that this is a different frame – plus p 2 I prime. This could be different from p 1 I, could be different from p 2 I, because they are primed.

So, the momentum – individual momentum value may be different in different frame. In fact, even this sum may be different in different frame. But, whatever might be this value, immediately after the collision that particular observer sitting in this particular frame of reference must also conclude that, the final momentum of the first particle plus final momentum of the second particle must be equal to the initial momentum of the first particle plus initial momentum of the second particle. This is what you should also conclude. He or she should also conclude that, conservation of momentum is valid in his or her frame also.

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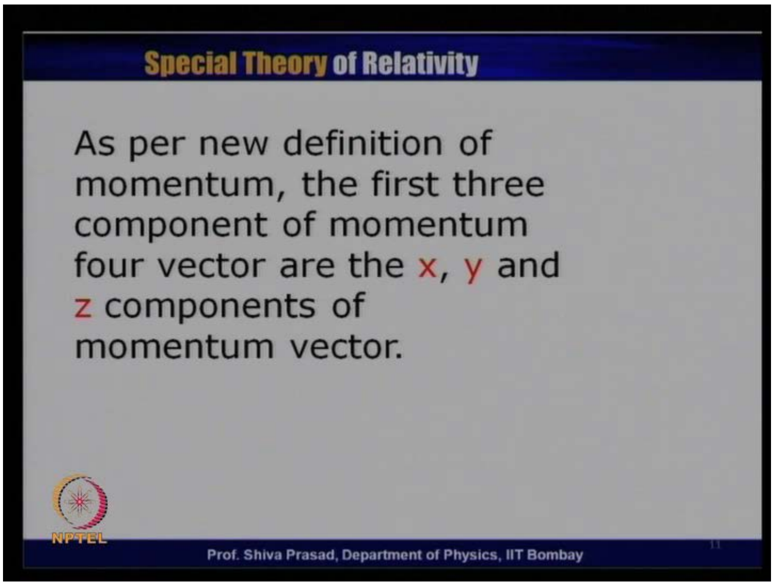

Now, as per new definition of momentum; in fact, they have given new definition of momentum; and, we want to see whether this obeys this universality condition. Then as per the new definition of the momentum, the first three components of the vector – four vector are x, y and z component of momentum vector. That is what we have discussed.

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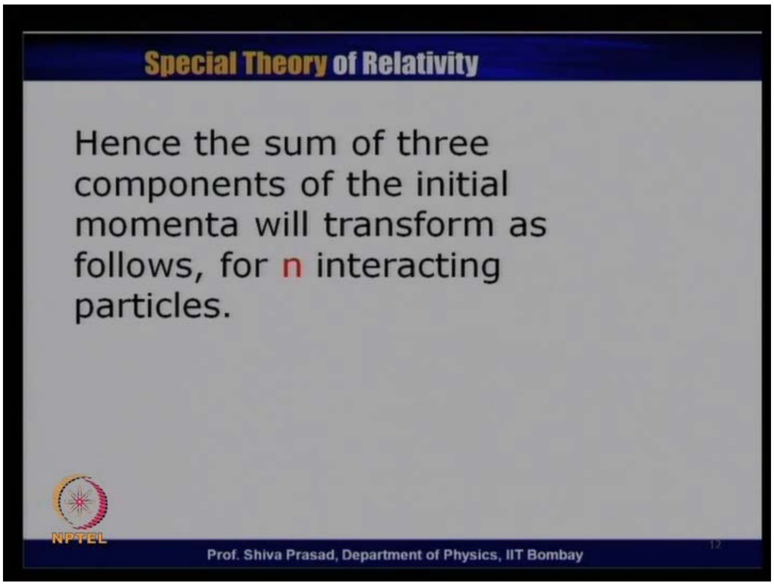

Hence, the sum of the three components of individual momentum would transform as follows. And now, I make it general; instead of talking two particles, I talk of n particles. And, remember when I am talking of conservation of momentum as a vector quantity, it means all the individual x, y and z components must also be conserved. A vector does not necessarily means that is only magnitude should be same; it also means its direction should be same; it means its x, y and z components individually must be conserved. So, even the summation of x value of momentum, summation of y component of momentum, summation of z component of momentum should also conserve.

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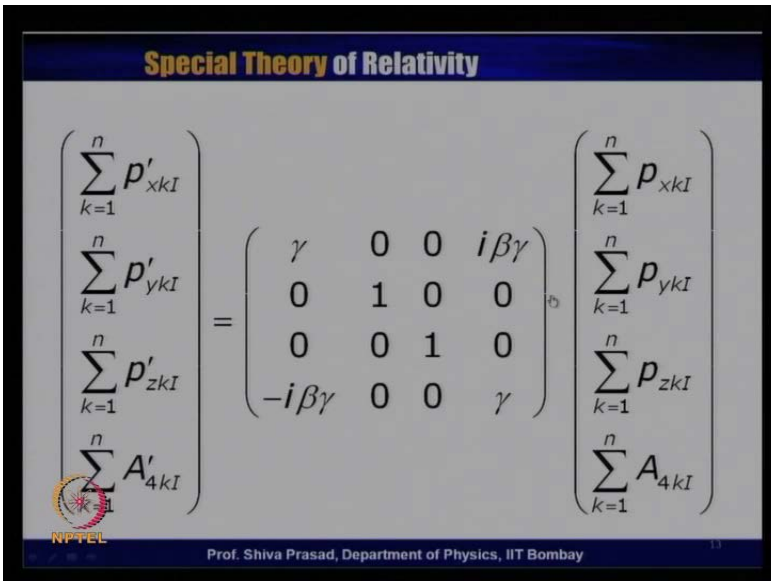

Now, let us look at the transformation equation, because you have started with the four vector momentum concept. Let us see whether this universality of conservation law will hold good with this new definition of momentum or not. So, this looks a somewhat a complicated equation on this transparency, but let us try to spend (( )) of time to understand it. So, I have assumed that, the first three components of the momentum are p x, p y and p z utilizing the new definition of momentum. Now, I assume that, there are n particles. And initially, at a time when I started observing, I find that, the x component of the first particle is p x 1, of the second particle is p x 2, for the third particle is p x 3 and this I signifies that, these are the initial values at the time when you started looking at the observation. So, if I take for all the n particles, I have to sum at all those values. So, I will get summation of k over 1 to n p x k I; where, k would vary from 1 to n, because there are n particles. So, this will give me the sum of the x component of momentum for all the particles. Similarly, this will give me the sum of the y component of the momentum of all the particles; this will give me the sum of the z component of all the particles – z component of momentum of all the particles. A 4 – as of now, I do not know what it is. So, I will just say A 4 k I. Of course, I know the value of A 4 in terms of gamma u; I am not i c. That is a different question. But, the physical… As of now, we do not know what does physically it represent and what is its role as far as conservation of momentum is concerned.

Now, we had earlier agreed that, if p x, p y, p z and A 4 are the components of four vector. If I take two four vector and add them, they will also be the components of the four vector; it means these summations must also transform by the same transformation equation that we have written earlier; it means if I go to a different frame of reference, which I call as S prime frame of reference, look again at all those particles and find that in S prime frame of reference, the summation of x component of momentum is given by this, y by this, z by this, and A 4 by this. Then if I open up this equation, that should give me this value – sum of the x component of the momentum in S prime frame of reference in terms of this particular initial values of sum of momentum in S frame of reference. This is the transformation equation, which is has to be used to transform the initial value of momentum to initial value of momentum in S prime frame of reference using the transformation equation.

Let me just expand this matrix. It is rather easy to expand. We have done some examples earlier. So, let us not spend too much of time here. What I will do here, I will just expand this particular thing here; this should be equal to gamma times this particular expression plus 0 times this plus 0 times this plus i beta gamma times this. This is equal to this as we have seen it earlier number of times. This is equal to this again as we have seen number of times. Now, as far as the fourth component is concerned, this will be minus i beta gamma times this plus 0 times this plus 0 times this plus gamma times this. So, I am just expanding it in the next transparency to write these things into four different equations. This is what it is?

So, sum of the x component of the momentum of all the particles in S prime frame of reference would be given by this particular expression, which is here. Summation of k from 1 to n p prime – prime, because this is S prime frame of reference x k I – k varies from 1 to n should be equal to gamma times whatever was the value of this corresponding quantity in S frame of reference plus i beta gamma times summation of k is equal to 1 to n A 4 k I whatever this A 4 be.

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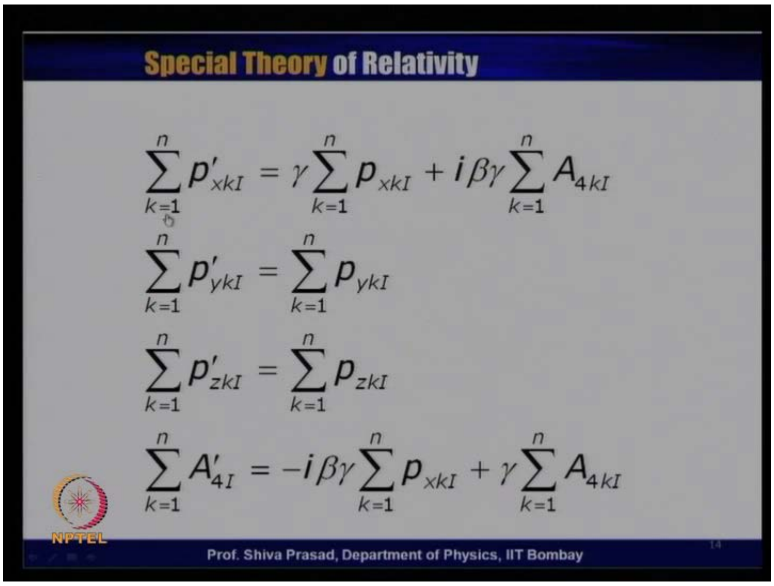

This versus straight forward equation; or, we also see the transformation of A 4 prime. This is equal to minus i beta gamma times the first component plus gamma times the fourth component. So, these are the equations that we get after expanding that particular matrix or expanding that particular matrix equation. Now, as far as these two things are concerned, these are straight forward. Let us look little more closely at this.

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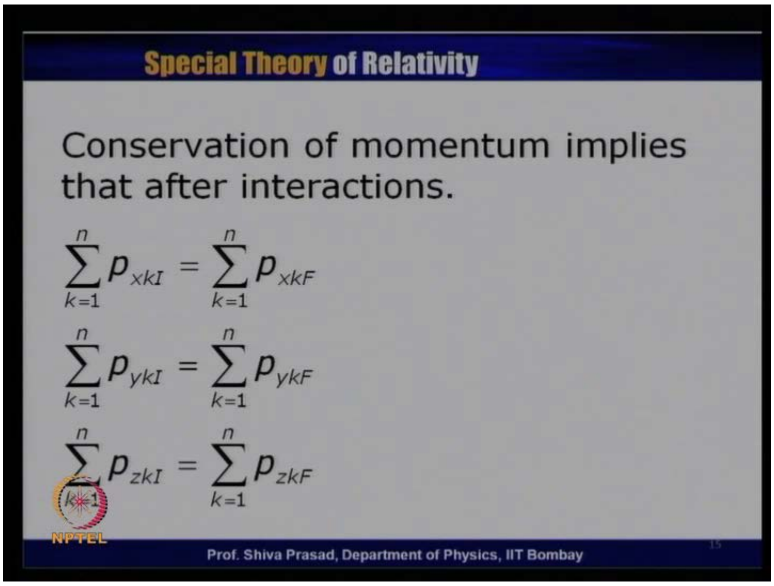

What is conservation of momentum imply? That in S frame of reference… Let us assume that, in S frame, the momentum is conserved. If in an S frame, the momentum is conserved, it means, after the interaction, they are all colliding with each other, hitting each other. And, after sometime you will find that, the individual value of momenta of one of them has changed. But, because there is no external force to the system, which we are assuming, then the sum of the final value of the momenta of the x component must be same as the initial value, similarly for y; similarly, for z.

Now, I want this conservation. I assume that, this conservation law is holding good in S frame. Now, I want that, when I transform these things to S prime frame of reference, there also the conservation of momentum must be holding good. So, this is given to be that, this is equal to this; the initial value is equal to final value, which will be the case if there is no external force to the system. Similarly, this is equal to this; this is equal to this. And, I want to have a case; where, an S prime frame of reference, also equivalent equations hold good with p being replaced by p prime.

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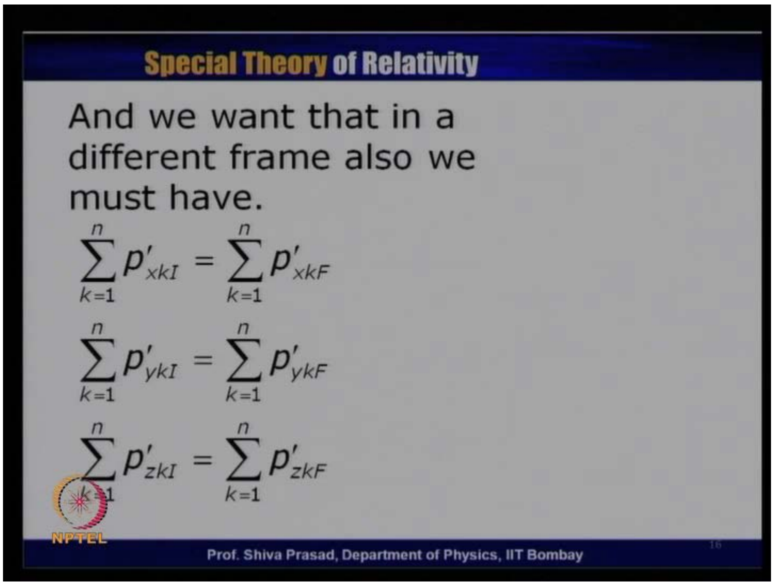

Now, out of this, as I have said, x equation is much more… This is what we have written as we want that, this also to be true in a different frame. We must have p prime x I should be equal to p prime; should be p prime; should be equal to p prime y component, p prime z component; should be equal to prime z component. This is what I have been saying.

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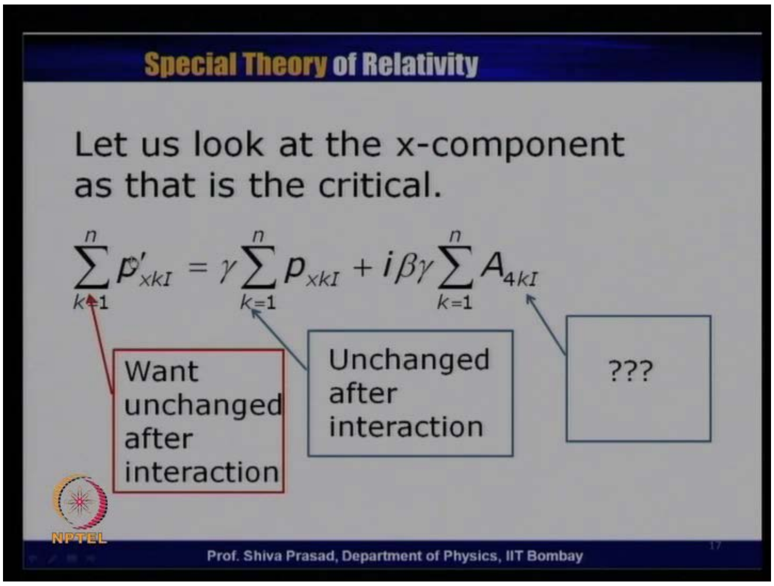

Let us look at the first equation, which is much more critical, because for the y components, they are equal anyway. So, this is the first equation, which I have picked up from the transformation equation, which tells me what is the sum of the x component of the momentum in S prime frame of reference. Then I transform this. When I know these values, how they will transform to S prime of reference? And, as we can see very easily that, this summation would be equal to gamma times this summation plus i beta gamma times summation of this A 4 whatever this A 4 be. Now, we have said that, after collision or after interaction, the particles have all collided with respect to each other. And, after sometime, we are observing. Then conservation of momentum is obeyed good in S frame. It means the final value must be the same as this particular value. So, once interaction has taken place, I have been given that, this remains unchanged. After collision, this does not change. After multiple collisions, there could be multiple collisions; these are the particles. But, this does not change.

Now, using this particular value, I have obtained that, this will be the value of the sum of the momentum in S prime of reference. And, I want that, after the interaction, because this has not changed, this should also not be changed. But, I realize that, this quantity does not solely depend on this, but it also depends on the A 4. If it so happens that, after interaction, this does not change – this summation does not change, but this summation changes; then I will not be assured of conservation of x component, because p prime would then change; summation of p prime would change. It means after the interaction, the value of p prime – summation of x component of p prime would be different from the initial value. And, that is what we do not want. It means if I want that, this particular conservation should be holding good in S prime frame of reference, I must look much more carefully at this particular value here. If this value also does not change after the interaction, then only it is possible that, this particular value will also not change after the interaction. Hence, it is extremely critical to know what is this particular A 4 component. And, only if this particular A 4 component is conserved, then only I am guaranteed that, the momentum will be conserved in all other frames.

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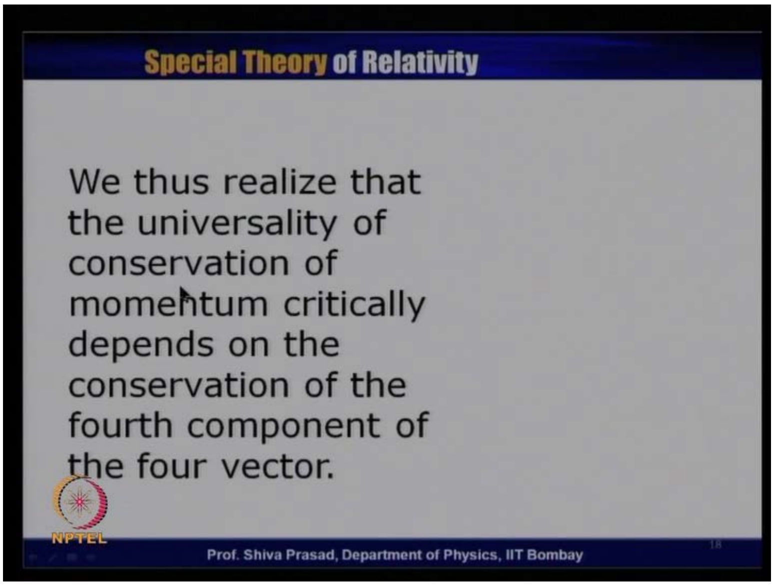

So, we thus realize that, the universality of conservation of momentum critically depends on the conservation of the fourth component of four vector. Now, if we go back to our thinking over classical mechanics, we have two specific conservation laws: one is what is called conservation of momentum, another is conservation of energy. However, in classical mechanics, we have come across situations, where what we call as mechanical energy need not be conserved. We have given in earlier example of completely inelastic collision, where two particles come and hit each other and then they get stuck. We find in such a situation that, mechanical energy is not conserved. But, momentum is conserved, because there is no external force in system. So, internal forces do cancel out; and therefore, there is definitely a conservation of momentum, though lot of energy. So, first, we realize that, in this particular case in the new definition in the relativity, conservation of momentum critically depends on the conservation of the fourth component. Let us think whether this particular fourth component can be thought of somehow related to another important quantity, which is called energy.

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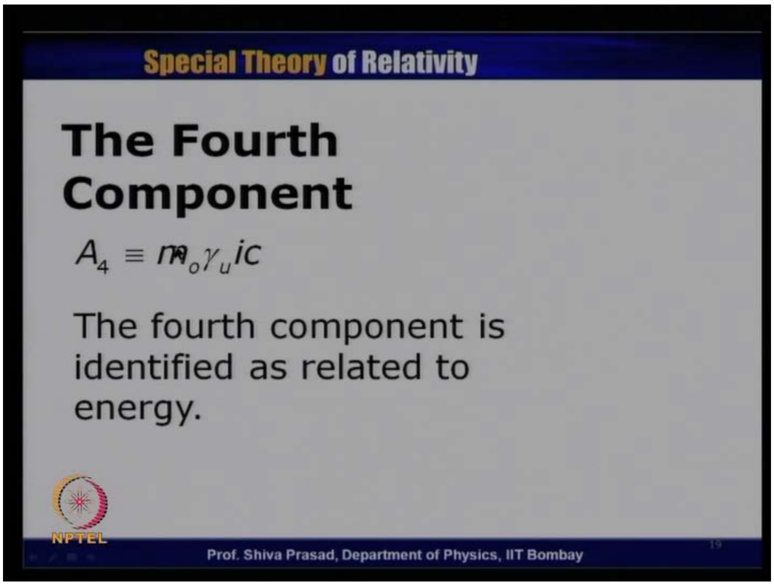

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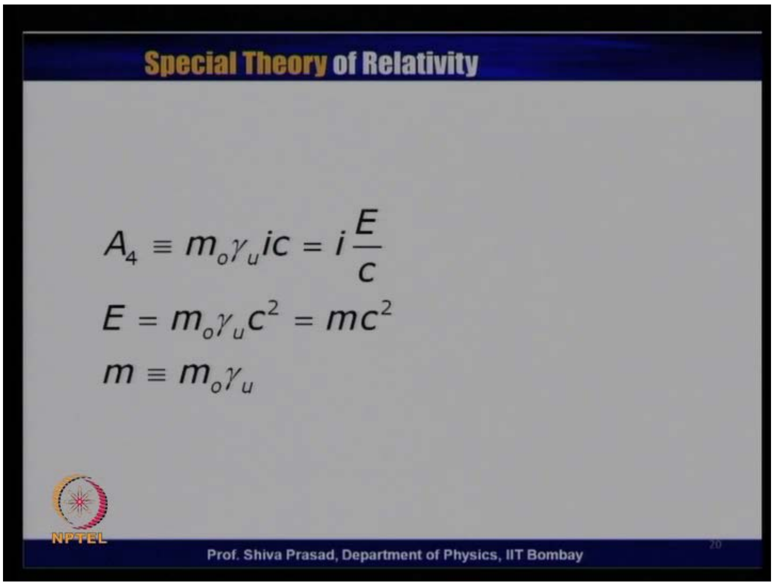

This was the fourth component – A 4 was equal to m naught gamma u i c. So, let us try to look whether this fourth component can be related to energy. That is what was the main role of Einstein to realize this particular component as the force component of course. Historically, that may not be the case of the… because the concept of four vector probably came much later. Einstein had a different probably way of looking into this new definition of energy. But, now, we can talk in this particular term.

See energy must have dimension of energy obviously. And, we realize that, mass into velocity square is the definition, is the dimension of energy. So, I write this A 4 as m naught gamma u i c; then write E is equal to m c square, because that will give me the dimension of energy. If I write this E is equal to m c square divided by c; where, m of course, is defined as m naught gamma u; then I can write the fourth component A 4 as i E by c. Let me just take little bit of time to explain once more.

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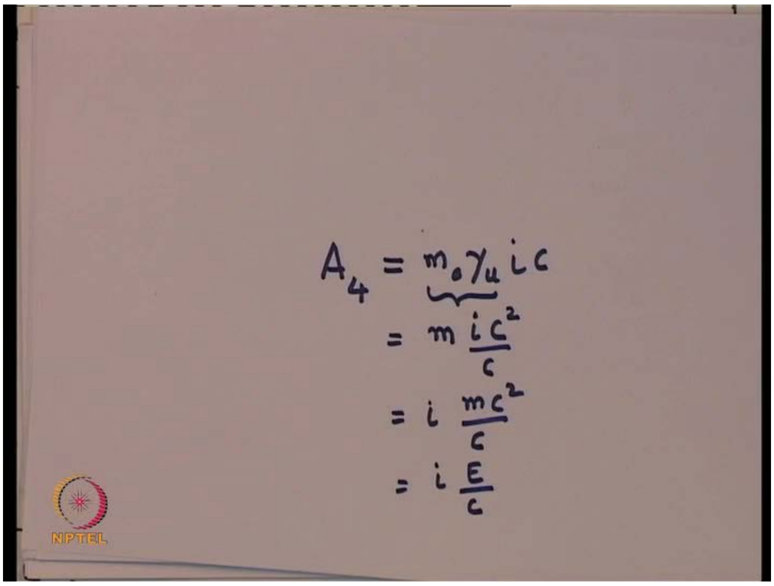

So, I have A 4, which I wrote as m naught gamma u i c. Now, this m naught gamma u – I call as m; I write this as m i c. Now, I want to call E is equal to m c square, because that has the dimension of energy. So, I multiply this by c and divide this by c. So, this becomes i m c square by c. And, this new definition – this becomes i E by c. So, the fourth component now becomes related to the energy of the particle; and, is written as i E by c.

Remember, we have used E is equal to m c square and m as m naught times gamma u. Remember it is m naught, which is a four scalar, not this m, because this m is m naught multiplied by gamma u. And, gamma u will be different in different frames, because this depends on the speed of the particle; and, speed is indeed a frame-dependent quantity. And, as we have all known that, this concept of new definition of energy E is equal to m c square is so popular that, people may not know physics, but they know E is equal to m c square. This has created huge impact on society just by name of Einstein; E is equal to… We always associate with Einstein E is equal to m c square; it had a huge impact.

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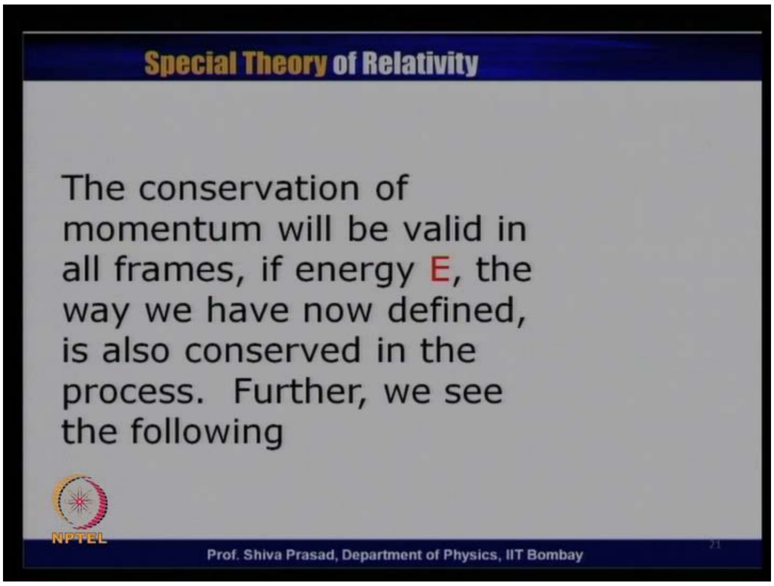

So, what we said, let us repeat now. The conservation of momentum will be valid in all frames if energy E, the way we have now defined is also conserved in the process. So, you must have both momentum and energy conservation unlike the case in classical mechanics, where their certain phenomenon in which we were not conserving the energy; and, in certain phenomena, we were conserving energy. In fact, strictly speaking in those phenomena, also, the energy is conserved; energy is not lost, but it is converted from the standard mechanical energy to some different form of energy. But, strictly speaking, the energy is not lost; it is always conserved. But, now, in relativity, whether the type of collision is the one which we described earlier with the two particles come and get stuck to each other or whether is what we traditionally called it an elastic collision; in all these cases, we will conserve both energy and momentum if we want them to be universal phenomena.

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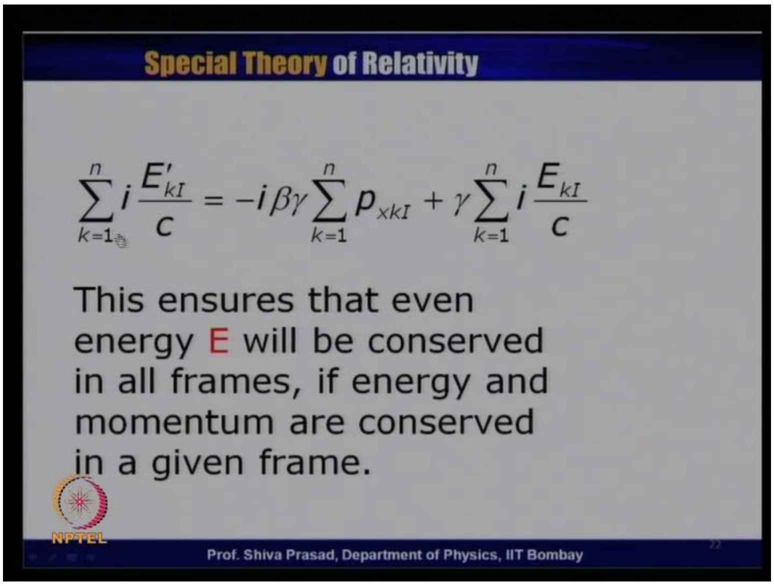

Now, we also see… Let us look at the fourth component of the (( )) transformation. The fourth component – now, I have replaced that gamma u m naught with E prime. So, we have new definition of energy here. So, this is the transformation of the fourth component minus i beta gamma times the first component plus gamma times the fourth component. Now, if this is conserved, this is conserved; this automatically means this is also conserved. So, this ensures that, even energy E would be conserved in all frames, if energy and momentum are conserved in a given frame.

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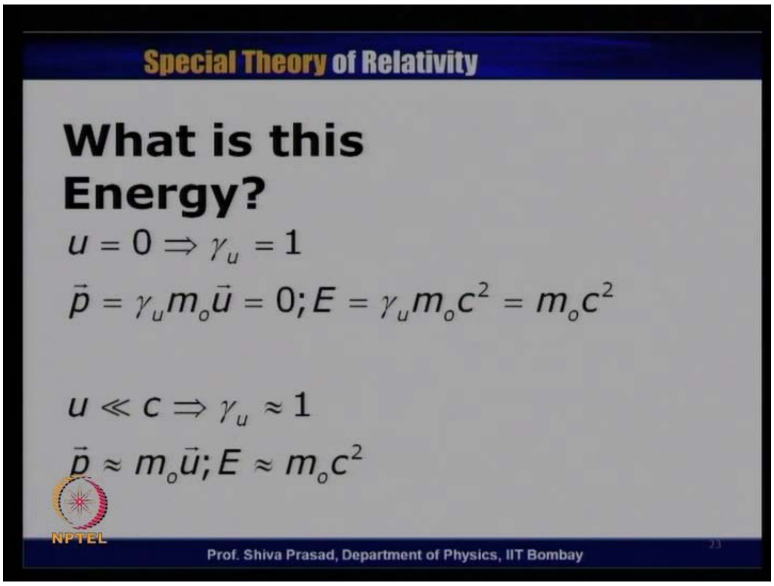

So, if momentum is conserved, I need energy to be conserved, if this momentum conservation has to be universal law. But, it also ensures that, momentum and energy are conserved in one frame of reference; both momentum and energy will be conserved in any other frame.

Now, let us look at this particular energy. What is this energy? See we always think that… We have always been telling that, if you go to the non-relativistic limit; it means if the particle speed is very small in comparison to speed of light; that you must get back to the classical mechanics. That is what we have always been telling. So, if u as 0 for example, gamma u equal to 1; the momentum, because u is 0 here, will be 0. But, this energy will not be 0, because this becomes 1; this will be just m naught c square. Similarly, in the limit, let us assume that, u is not 0, but very small in comparison to speed of light. So, in the limit, u being much smaller than c, gamma u will again tend to 1; which means p naught will be approximately equal to m naught u, because gamma u is more or less close to 1; which gives me back the classical definition of momentum, which is mass multiplied by the velocity.

But, E will still remain m naught c square. So, we do not have a really classical analog of this particular energy. We had never thought that, this – if a particle is at rest, when you have speed equal to 0, is still the particle will have some energy. In the classical mechanics, we have not taught; this is a totally new revolutionary concept of energy, where we say that, even if there is a particle, which is at rest, this particular particle has a rest mass energy. That is what we call a rest mass energy of m naught c square.

Now, the concept of energy is little more general in the eye of Einstein or in the eye of special theory of relativity; it contains all the form of energy. If the particle gains speed, its energy increases. So, this will be on the top of the m naught c square. Similarly, the particle becomes hot, means it is gaining some energy; its mass should increase; its m naught should be increased. So, it contains information about the entire energy of the system. Whatever might be the energy, for example, if you have two particles and these two particles have their own rest mass energies and get bound; then when they get bound, certain amount of energy is released and that release of energy would eventually lead to a decrease in its rest mass, because as far as relativity is concerned, there is no difference between mass and energy, because the two are related by a fundamental constant c square.

So, a rest mass of the particular particle will tell the entire energy contained in it; if we are talking for example, the rest mass of hydrogen atom, the rest mass of hydrogen atom will be slightly smaller than the sum of the rest masses of proton and electron, because in order to form a hydrogen atom, certain amount of energy has been released. And therefore, the mass must have come down.

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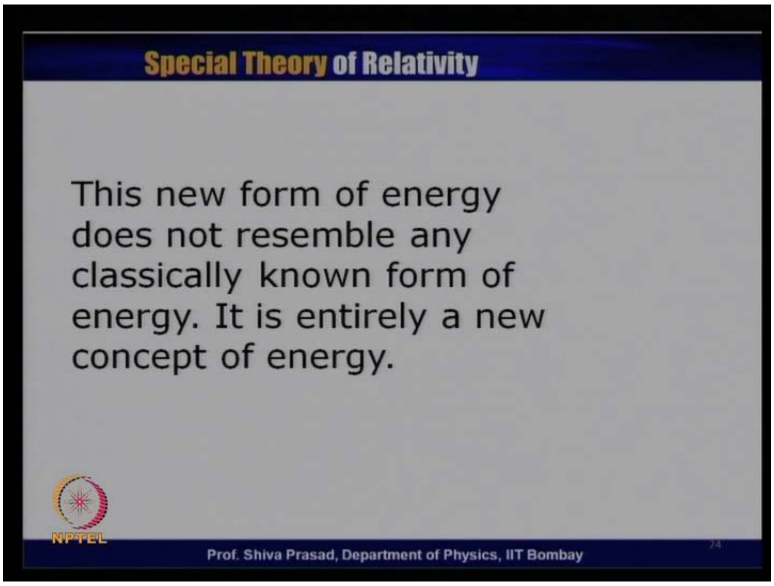

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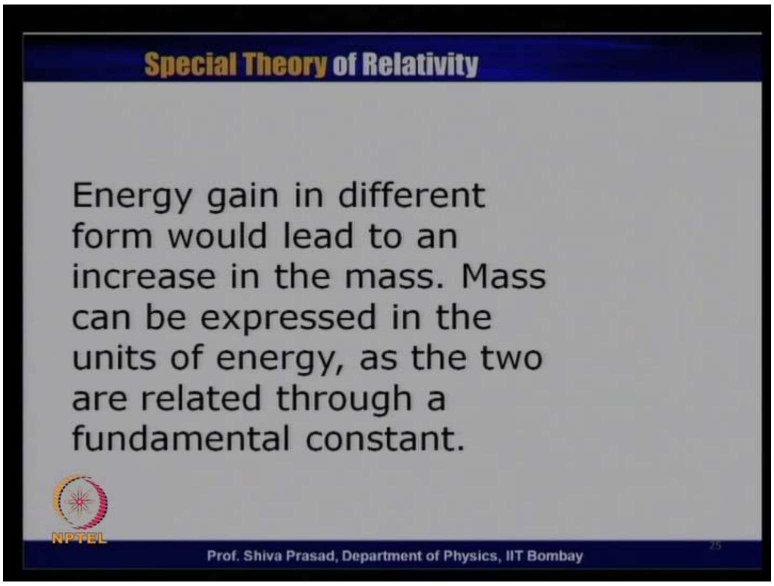

Similarly, if I take hydrogen atom and release it, so that we give it certain amount of energy; so hydrogen… so that the electron and the proton becomes separate; then we will find that, the masses of electron and proton, their sum would be slightly larger than the mass of the hydrogen atom; and, that increase will be just related to the binding energy of the hydrogen atom. So, this is what I have written. This new form of energy does not resemble any classically known form of energy; it is entirely a new concept of energy.

Energy gain in different form would lead to an increase in the mass. So, if we take one particular, in principle, heat it; then its mass should go up. Mass can be expressed in the units of energy as the two are related through a fundamental constant. In fact, it is often told, many of the faces especially, those who are working in the particle physics area, they will always express mass in the units of energy. What is a mass of electron? It is 0.51 Mev. See Mev – mega electron volt is the unit of energy.

But, one says, the mass is equal to energy. Similarly, what is the mass of proton? It is approximately 940 Mev. Again, we are expressing mass in terms of energy; effectively means it is the value of m naught c square, which is equal to that energy. So, there is no difference in relativity in the mass and energy, because the two are related by the (( )) They are dimensionally different; no doubt, they are different dimensionally. But, as far as the relationship is concerned, they are related by a fundamental constant speed of light square.

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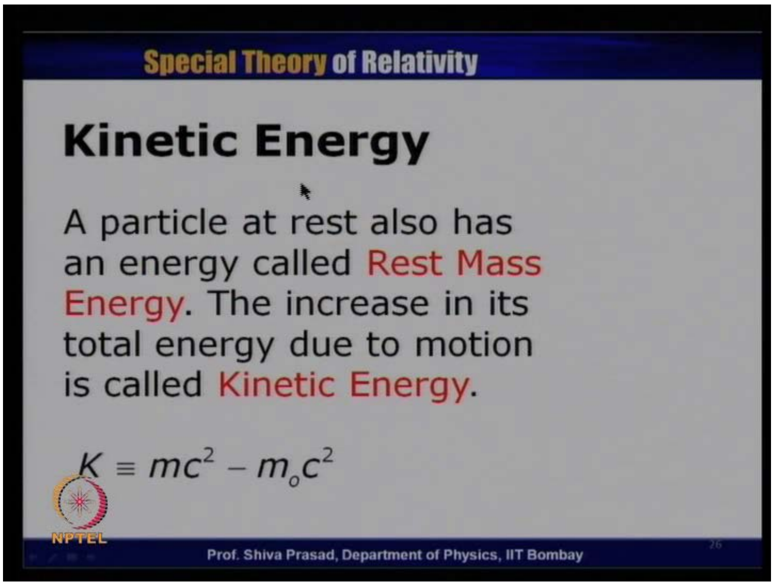

Now, let us talk of kinetic energy. This is something which we know in classical mechanics. So, let us define kinetic energy with this new definition of energy. As we have seen that, a particle at rest also has energy, which we call as rest mass energy; now, if this particular particle has started moving, then its energy would have gone up. Then whatever is the increase in the energy – that is what we call as kinetic energy.

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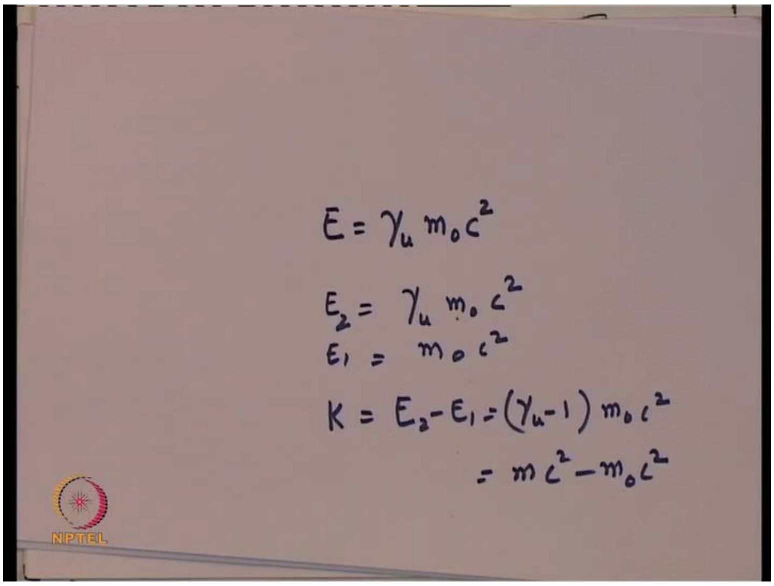

See if the particular particle has started moving, then the energy will be given by gamma u m naught c square. When the particle was at rest, this gamma u was equal to 1 and the energy was m naught c square. Was the particle has picked up certain speed, the value of gamma u has gone up whatever might be the slight value, but it has gone up. And therefore, this energy E has increased. Whatever this increase in the energy E… This was the final energy; this was initial energy; this is what we will call kinetic energy as E 2 minus E 1 is equal to gamma u minus 1 m naught c square. We can also write, because this is what we have defined as m.

So, I can also call this as m c square minus m naught c square. So, kinetic energy in relativity is defined as either gamma u minus 1 m naught c square or as m c square minus m naught c square. And, we will just now show that, this kinetic energy, which will reduce to the traditional definition of kinetic energy in the limit of low speeds or low values of u.

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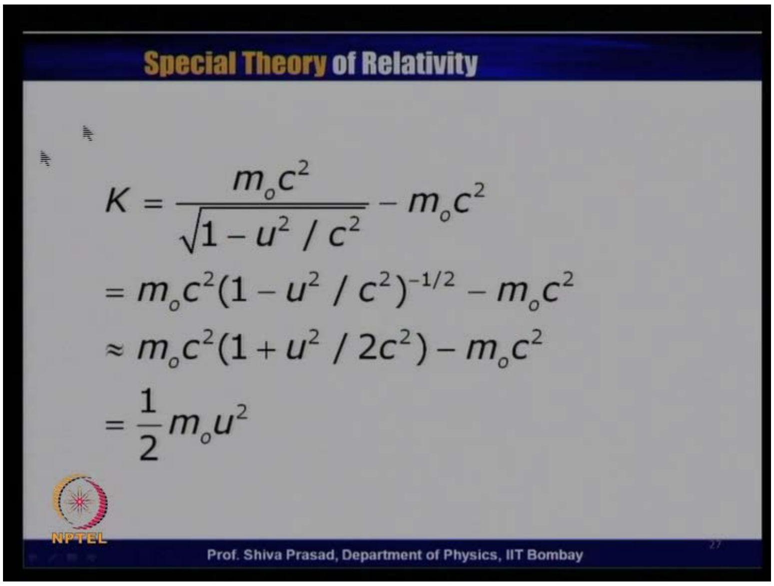

Let us see how. I have written K as m naught c square; and, this was the value of gamma u; gamma u is 1 upon under root 1 minus u square by c square. So, this is the value of m c square minus m naught c square. So, I take this particular quantity in numerator. If I take this particular quantity in numerator, I can write this as m naught c square multiplied by 1 minus u square by c square, which is here to the power of minus half. This term is identical, which is minus m naught c square. Now, this particular thing… because in the limit, u is very small in comparison to c.

This can be expanded into a series and you can retain just the first term and neglect higher order terms, because u is very small in comparison to c. So, when I expand here, I will get this quantity multiplied by here or there is a minus half. So, this will become plus half. So, this will become approximately m naught c square multiplied by 1 plus – this sign becomes plus, because of this negative sign here; and, there is a half; so this becomes 1 plus u square by 2 c square under the limit that u is very small in comparison to c. This minus m naught c square is a kinetic energy.

If you expand this, you are getting this m naught c square. This will cancel with this m naught c square. This c square in the second term would cancel with this c square. And then you will just get half m naught u square, which is the standard classical definition of kinetic energy. So, this new value, this new definition of kinetic energy does yield me to the classical expression in the classical limit. That is what we had expected. But, let us realize that, the E, which we now call as total energy or to be more precise, total relativistic energy; that does not have classical analog. That will not reduce to half m v square when the speed of a particle is slow. It is the kinetic energy, which will reduce to half m naught u square. So, we should be differentiating the kinetic energy and the total relativistic energy and remember it is the total relativistic energy, which must be conserved in a process along with the momentum.

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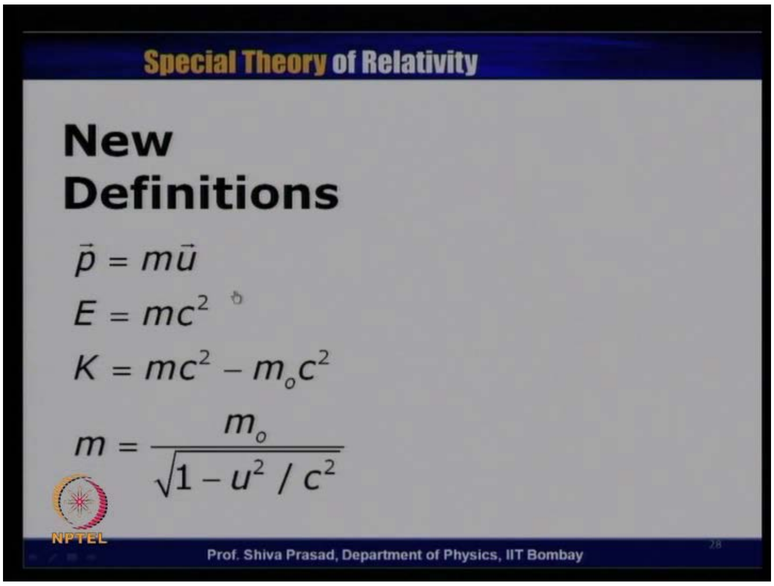

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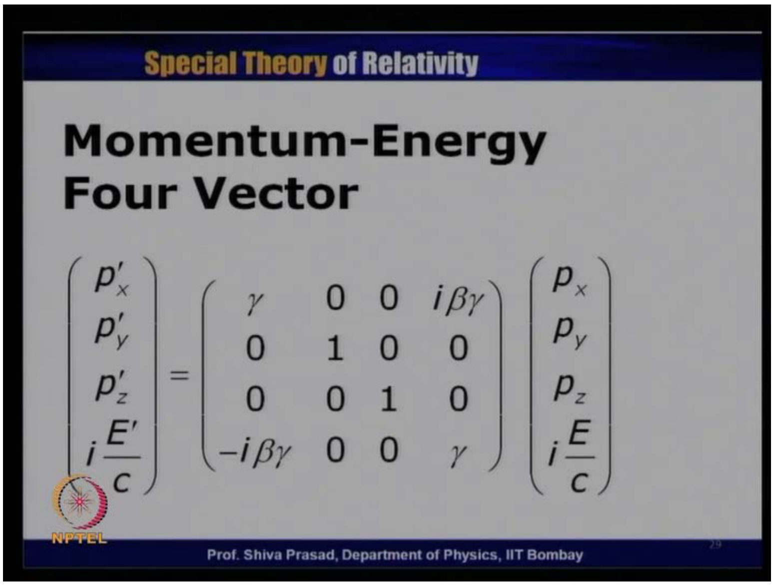

So, these are the new definitions just to summarize. p is equal to mu; E is equal to m c square; K is equal to m c square minus m naught c square; and, m is defined as m naught divided by under root 1 minus u square by c square.

Now, looking at these new definitions let me rewrite the momentum energy four vector. We had earlier agreed that, the first three components are p x, p y and p z. We have also discussed that, the fourth component is i E by c. So, what it means that, if I find a particular particle in a frame of reference, with the new definition of momentum and energy (( )) value of momentum or x component of momentum is p x; y component of momentum is p y; z component of momentum is p z; and, its energy is E. Then these four components will transform to a different frame of reference as prime of reference. It means the new values in a different frame; the values of the x component of the momentum, y component of the momentum, z component of the momentum and the energy would be given by this same transformation equation, which we have used earlier.

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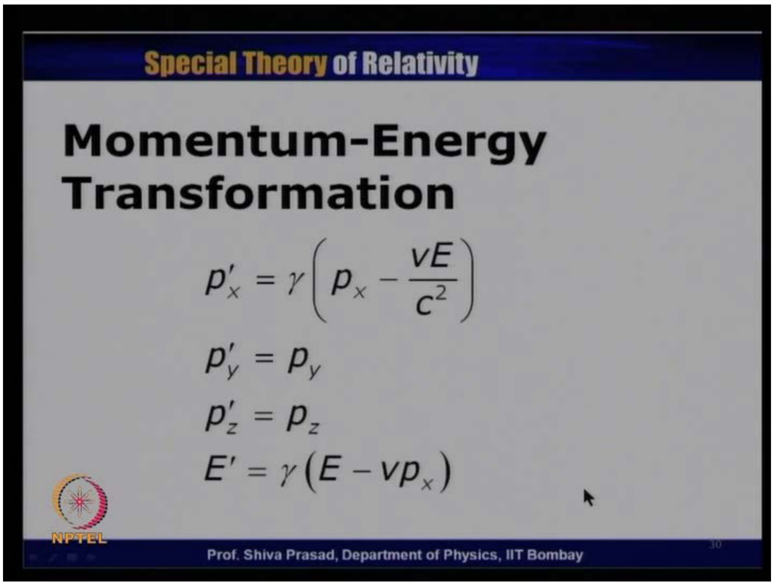

In fact, (( )) involve both momentum and energy. We generally call this as momentum energy four vector, not just a momentum four vector. So, if I have to transform the momentum, then all I have to do is to expand this particular matrix. So, I can find out that, p x prime for example, here – p x prime will be equal to gamma times p x plus i beta gamma times i E by c; p y prime of course will be p y; p z prime will of course be equal to p z; and, i E prime by c will be equal to minus i beta gamma times p x plus gamma times i E by c.

If I just make them just simplify this equation, this becomes the momentum-energy transformation. So, p x prime would be given by gamma times p x minus v E by c square; p y prime is equal to p y; p z prime becomes equal to p z; and, E prime becomes gamma E minus v p x. Probably, one would have noticed this symmetry in the arguments – symmetry in these transformation equations. See in this type of equation, x minus v t was appearing for x component; here now appears for the fourth component, which is energy in the momentum, which was the first component. This equation is gamma p x minus v E by c square. Similar type of equation was for t – t prime was turning out to be equal to gamma t minus v x by c square; you just remember. But, we realize, there thing, which was the fourth component was totally different here; the fourth component is i E by c. So, that makes a difference. There it was i c t; here its i E by c. So, that makes this particular difference.

Now, let us look at the length of its energy-momentum four vector for a single particle. Same principle we can write this particular momentum four vector for a set of n particles. We can add all these momenta. We will work out one or two examples for this particular case later; not in this lecture, but some of the later lectures. But, at the moment, let us just assume that, there is one single particle. And, if it is so then the length of this particular particle should be a four scalar; it means even if I change the frame of reference, this should not change. It is too in principle for any four vector. Even if we would have written for the sum, the length should be frame-independent quantity. But, I am interested in looking at a specific expression, which eventually is obtained by obtaining the length of energy-momentum four vector for a single particle. So, let us just evaluate the length of energy-momentum four vector for just a single particle, which gives me one expression, which turns out to be very useful expression for solving the problems.

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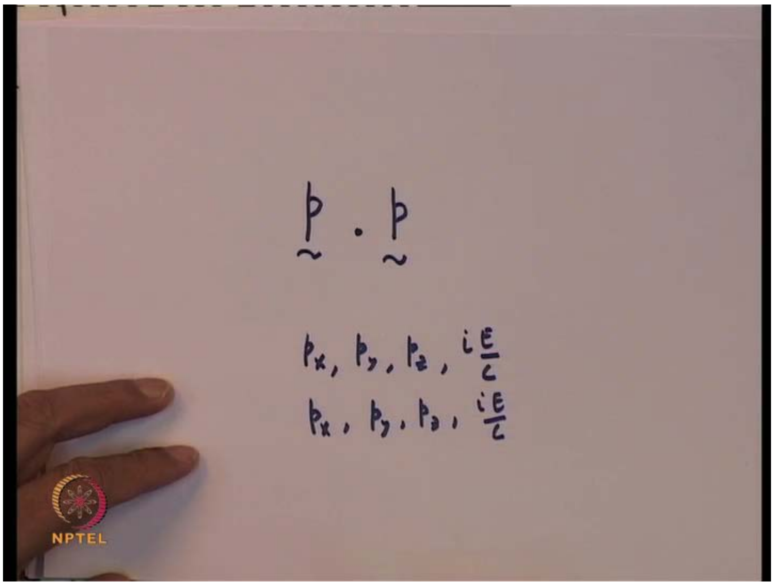

See if I have to look at the length, what I have to do? I have to take the momentum four vector and take a dot product with its own… This will give me the square of the length of the energy-momentum four vector. So, let us evaluate this p dot p; which I am doing in the next transparency.

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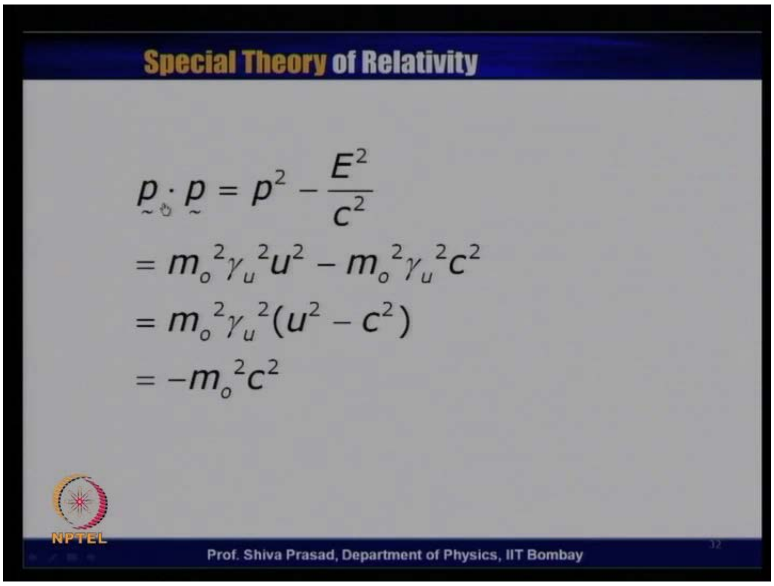

So, this was the p dot p. Remember the first three components of p are p x, p y, p z; and, the fourth component was i E by c. Now, if I have to take a dot product, it means we take with its own self. So, I write the same vector the component wise as i E by c. I take the dot product; it means this multiplied by this plus this multiplied by this plus this multiplied by this plus this multiplied by this. That is the way we have defined the dot product between two four vectors. See if I multiply this by this, I get p x square. So, this will turn out to be p x square plus p y square plus p z square, because I have to add all these three. So, this multiplied by this plus this multiplied by this plus this multiplied by this plus this multiplied by this. This gives me p x square plus p y square plus p z square. When I multiply the two, i square will give me minus 1. So, this will be p x square plus p y square plus p z square minus E square by c square. So, this is what I have written here – p dot p is equal to p square minus E square by c square.

Now, we have already seen that, the value of p is related to m naught gamma u u. So, u x square plus u y square plus u z square; I can write as u square. So, this becomes m naught square gamma u square u square. And, as far as E is concerned, this was also m c square; and, m was m naught gamma u. So, this becomes m naught square gamma u square c square. I can take this m naught square gamma u square out common. So, this becomes m naught gamma u square.

This becomes equal to u square minus c square. Now, you can very easily see that, if I write this gamma u square and expand it, this will be 1 upon under root 1 minus u square by c square. If you just simplify this equation, which we have in one of the cases earlier, this will just become minus c square and this is minus m naught square c square; which indeed is a four scalar, because m naught we had defined as a four scalar. See I know its constant in all frame of reference. So, the length of the four vector is indeed unchanged when I change the frame of reference. But, this particular equation gives me something, which is interesting; it tells me that, for a single particle, p square minus E square by c square must be equal to minus m naught square c to the power 4.

This is what I have written. This is obviously same for all the frames. This leads to the following useful relationship – E square is equal to p square c square plus m naught square c to the power 4. Let us just look at here. This was p square. So, what I take, I multiply it by c square.

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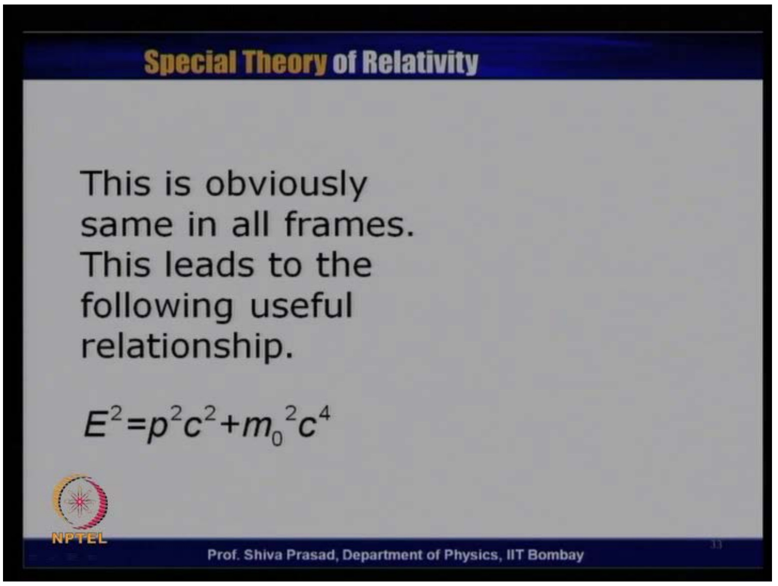

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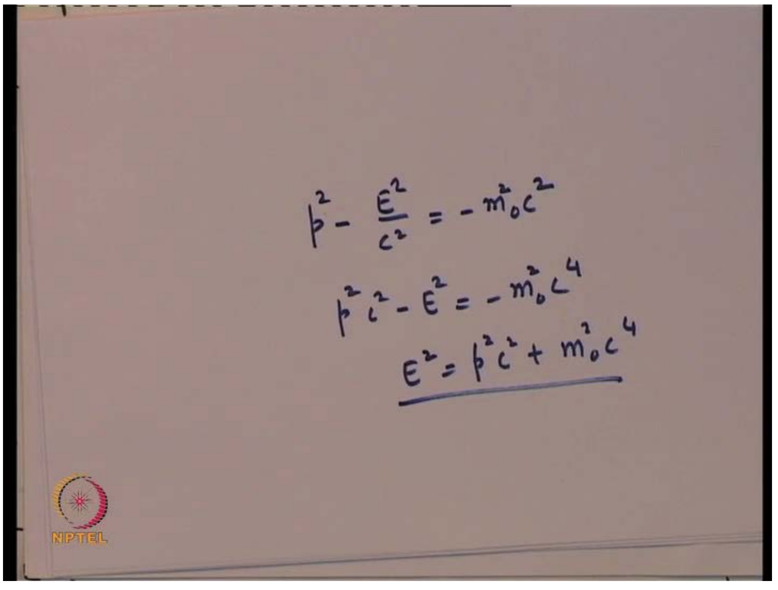

Let me write it here. p square minus E square by c square is equal to minus m naught square c square. So, if I multiply the whole thing by c square, p square c square minus E square is equal to minus – there was m naught square here – m naught square c to the power 4. I take this on this side; this on this side. So, this becomes E square is equal to p square c square plus m naught square c to the power 4. This is a very useful equation, because this gives you relationship between energy and momentum. See if I know the energy of the particle, I can find out what is the momentum of the particle. If I know the momentum of the particle, I can find out what is the energy of this particle. Hence, this particular equation is useful, because it gives me a relationship between the energy and momentum.

So, this is what is the equation, which we normally use in many of the equations requiring interactions of various particles giving me E square is equal to p square c square plus m naught square c to the power 4. Similarly, I can find out a relationship between kinetic energy and momentum. Though kinetic energy has no very special meaning in relativity, but many times now we talk in terms of kinetic energy. So, if we are interested in finding out a relationship between kinetic energy and momentum, that also I can do.

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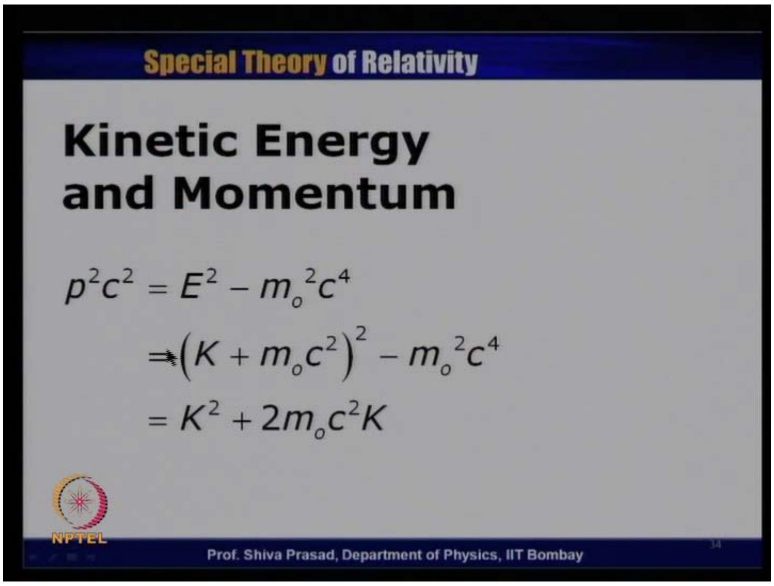

So, this becomes p square c square is equal to E square minus m naught square c to the power 4. The same equation I have written in a slightly in different fashion. And, this energy E – I have written as K plus m naught c square, because we had defined K as m c square minus m naught c square; and, E was m c square. So, this I can write as K plus m naught c square whole square minus m naught square c to the power 4. You just open this up. This becomes K square plus 2 K m naught c square plus m naught square c to the power 4. That m naught square c to the power 4 cancels with this. This equation comes out to be equal to K square plus 2 m naught c square K. So, this is the relationship between kinetic energy and the momentum.

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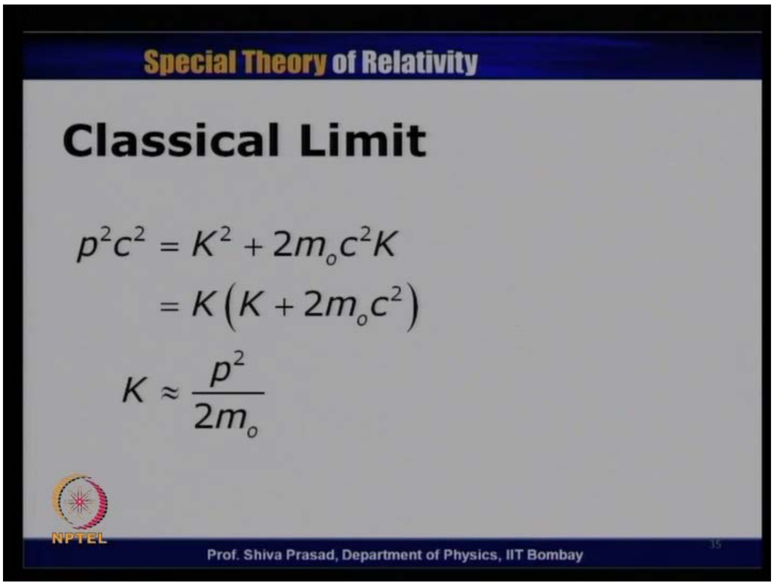

Let us just look at the classical limit of this particular equation. The classical limit of this particular equation, I just take K out. So, p square c square becomes K multiplied by K plus 2 m naught c square. So, if K is very small in comparison to the rest mass energy of the particle, I can neglect K. And, if I neglect K, then I get K is equal to p; this c square cancels here; I get K is equal to p square by 2 m naught; which is the standard energy-momentum relationship in the classical mechanics or rather kinetic energy-momentum relationship in the classical mechanics. p square by 2 m gives me the kinetic energy. This also tells you one more thing.

Often people ask when we should be sure that, we should apply special theory of relativity without… unless we will make a big mistake; and, when we need not apply, because we are really in a classical limit. In case of… If we are talking about the velocities, we always say that, gamma should be very close to 1. It means you must be talking… you must less than 0.1 c of that order. In terms of energy, this always tells, so long, the kinetic energy of the particle is much smaller in comparison to the rest mass of energy of the particle; then I can apply classical mechanics. For example, if I am talking of hydrogen atom, where the particle has the energy order of the order 13 electron volt, we rely that rest mass energy is half Mev; then of course, I can apply classical mechanics.

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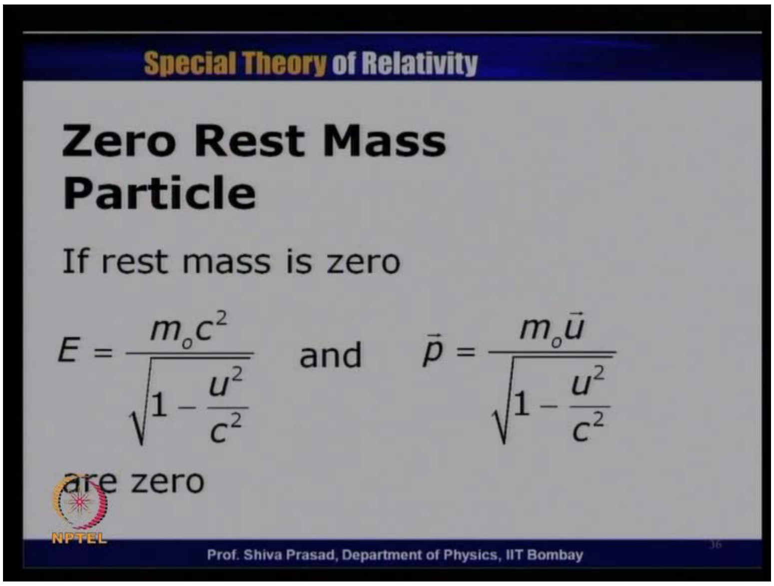

The last thing which I would like to describe in today's lecture is a totally new concept, which comes out of relativity; which is a presence of a zero rest mass particle. In the classical mechanics, we never can imagine a particle, which has no mass. But, in relativistic mechanics, we can imagine that, a particular particle may have zero rest mass; m naught can be 0. Of course, if m naught is equal to 0, using these two expressions for energy and momentum, energy should be 0 and p should be 0 provided this particular quantity is not 0.

And, this quantity will be 0 only when u becomes equal to c. So, this gives you a possibility that, even if m naught is equal to 0, E can be a finite nonzero; p can be a finite nonzero. But, in that case, you must have u equal to c; means that particular particle must travel with speed of light. So, relativity gives you a possibility of presence of a zero rest mass particle. But, that particle must travel with the speed of light. And, we know a typical example is light itself. Photon is considered as a particle, which moves obviously with the speed of light and has zero rest mass.

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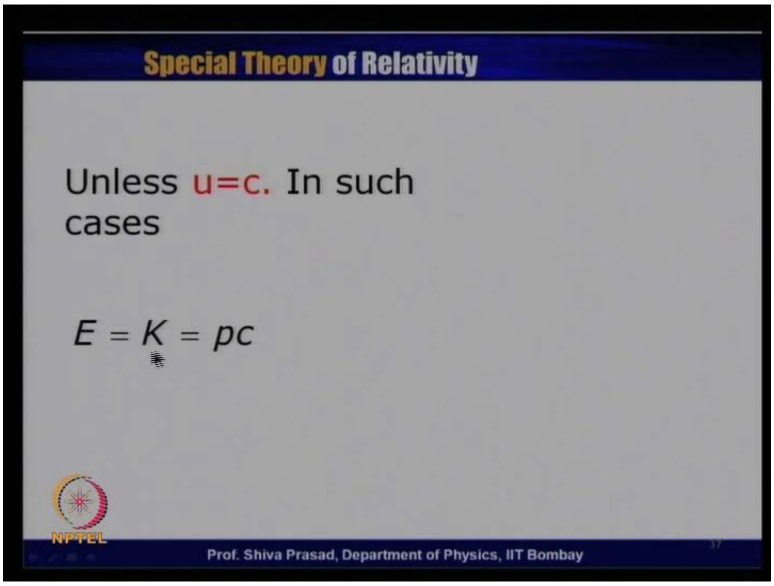

So, this is what I have to say. Unless u is equal to c, in such cases of course, because rest mass energy is 0; so total energy is equal to the kinetic energy is equal to pc, because E square is equal to p square c square plus m naught square c to the power 4. That m naught square c to the power 4 term does not exist. So, it just becomes E is equal to pc.

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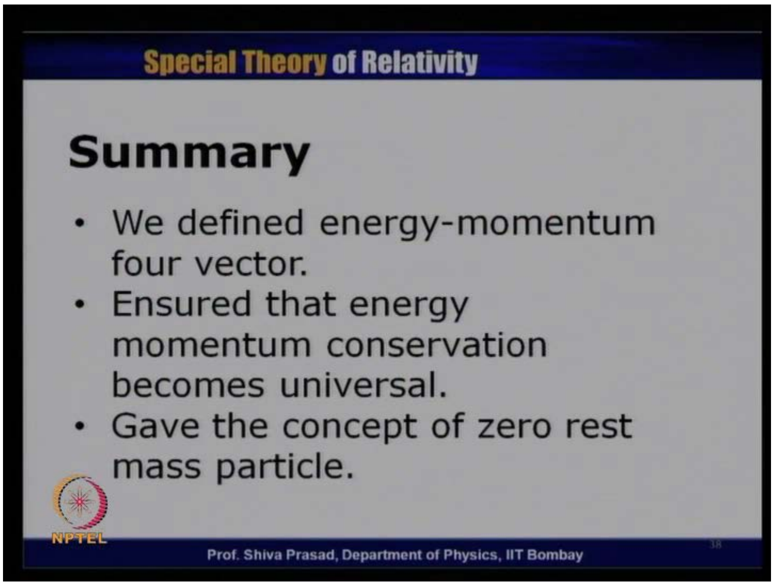

So, finally, I will summarize whatever we have discussed. We defined energy-momentum four vector in this particular lecture; then ensured that, energy-momentum conservation becomes indeed universal; and finally, gave a totally new concept of zero rest mass particle.

Thank you.