Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 9
Potential and Potential Energy

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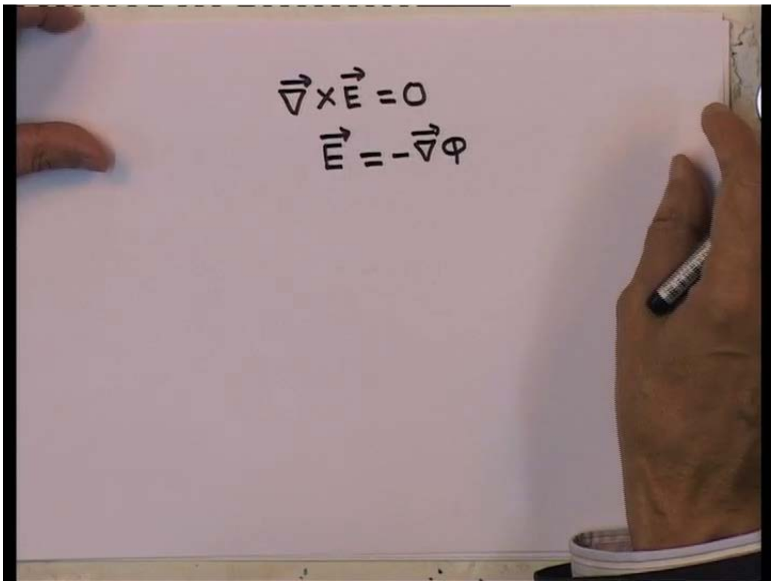

In the last lecture, we had introduced the concept of electric potential. The, what we said is that electric field being a conservative field by definition; what it means is that the curl of the electric field is 0. Now, in that case I can express the electric field itself as gradient of a quantity by convention we take negative gradient of a quantity which you call as the potential. Now, the word potential reminds us of potential energy, but I emphasis again that though the two things are related, potential is not the 'potential energy'. What is the relationship we have talked about last time, and you would amplify on the concept of potential as we go along in this course.

Loosely speaking, as I mentioned towards the end of last lecture, the potential is very similar to pressure in a fluid. For example, if you have water flowing in a tube and at one end of the tube you have a higher pressure than at the other end, then water tends to flow from a region of higher pressure to that of a lower pressure. Similarly, the potential in an electric field essentially gives the measure of a level. So, therefore if a charge, a positive charge is at a higher potential and there are regions of lower potentials around it, then it would tend to move to the region of lower potential.

Today, we will be talking about potential, its connection to the energy, the potential energy, and in general we will see some applications of this. So, that is what is the content of today's lecture.

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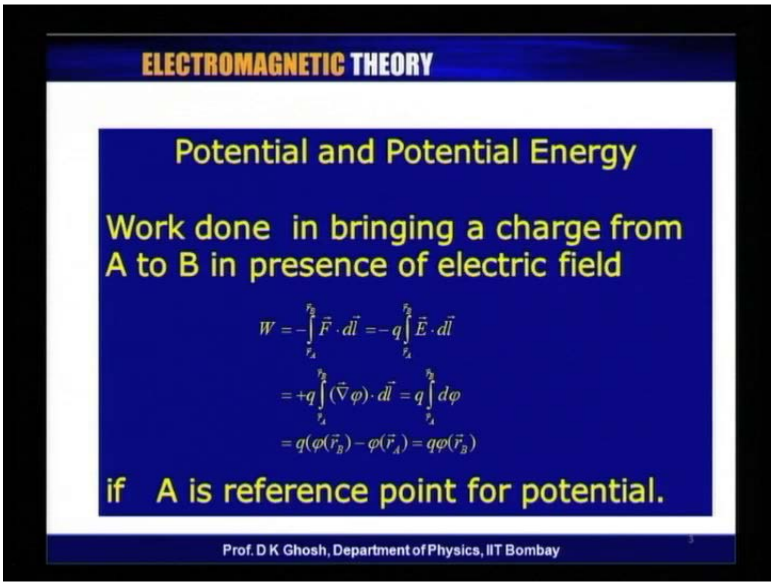

Suppose I want to bring a charge Q, I want to bring a charge Q from a point A to a point B. Now, I know that there is a electric force acting on it. And since the charge is in an electric field, I have to do work to bring this charge, from wherever I am bringing it to this point.

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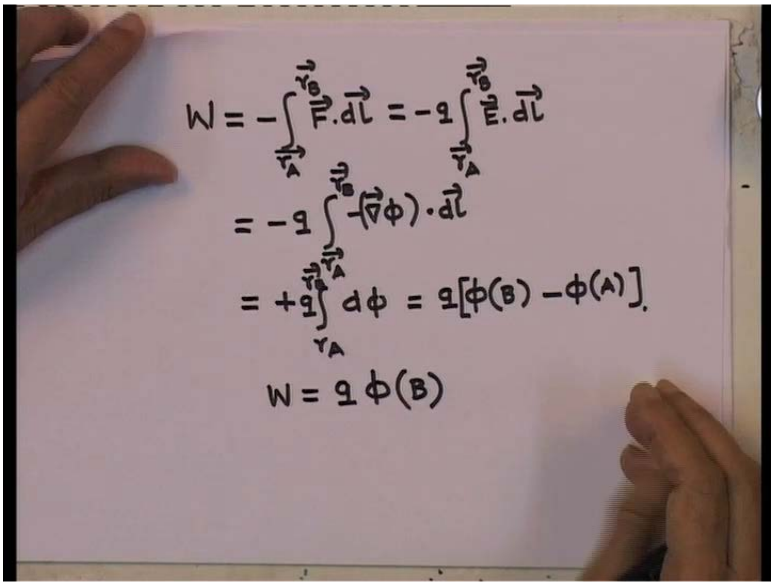

So, I have to do work, which is given by minus; because it is not the work done by the charge, but work done by the external agency. So supposing my reference point is A, having a position vector r A to r B and it is F dot del, this is the work done. Now, since the force on the charge Q is Q times the electric field, this can be written as the integral from r A to r B of E dot d L. We have seen that in terms of the potential, the electric field is negative gradient. So, it is minus grad phi dot d L. And when we discussed the meaning of the gradient, we had seen that this quantity is nothing but the differential d phi. So, therefore what I have, there is a Q there; what I have is Q times phi at the point B minus phi at the point A.

So with respect to some reference point, the potential of the point B is phi B minus phi A. Now, I could choose I could choose the point, the reference of point A to B, the point where the potential is 0; in which case I find the potential, the work done is simply equal to Q times the potential of the point B, provided phi A is taken to be 0.

So, notice that this amount of work that has been done must now become the potential energy of the system. But this is a potential energy. If supposing, I have a large number of chargers and it is the charge Q which has been brought into the field of these large number of charges, then it is the energy of the entire system, the potential of the energy of the entire system has changed by this amount, namely Q times phi B. But in the total expression for the potential energy, this term has explicit reference to the charge that is being are considered by us, which is Q times phi at the point B.

So, in some sense the potential energy of the system associated with my test charge is Q times phi B and the potential of that charge, at the, when it is at the point B is nothing but the potential energy associated with the unit charge, when it is brought from wherever the zero reference of the potential energy is to the point B. So, as I said that potential is not potential energy, but there is a deep connection between them.

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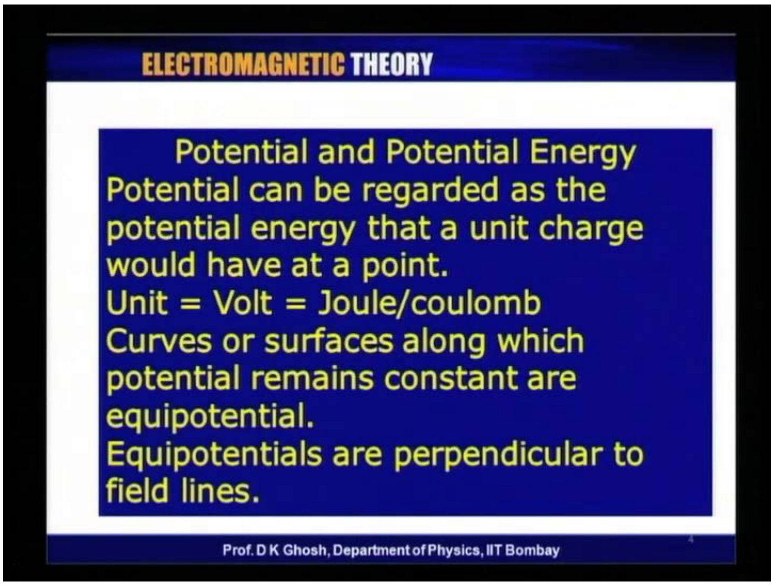

Potential is, potential energy is measured in Joules, but the unit of potential is Joule per Coulomb, which has the name a Volt. Now, often we deal with surfaces or curves on which the potential remains constant. In other words, a charge placed at that point experiences no force. The gradient of the potential is 0. Such surfaces or curves are known as equipotential surfaces or equipotential curves, whatever the appropriate situation is by definition by it is… the way we have defined equipotential surface. They are perpendicular to the electric field lines because there is no work done when you move a charge in an equipotential region.

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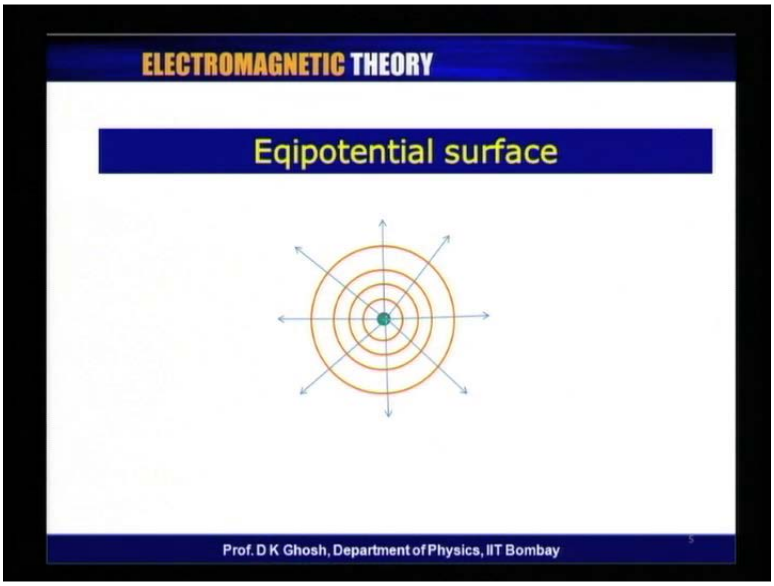

So, here for example, I am showing you the field and the equipotential correspond… to a single positive charge, which is marked by the blue in the picture. As we know that the field lines are symmetrical and are going out, if it is a positive charge. Now, the equi potential surfaces would be surfaces which are perpendicular to it. In other words, they are systems of spheres of various radii around which are perpendicular to these field lines.

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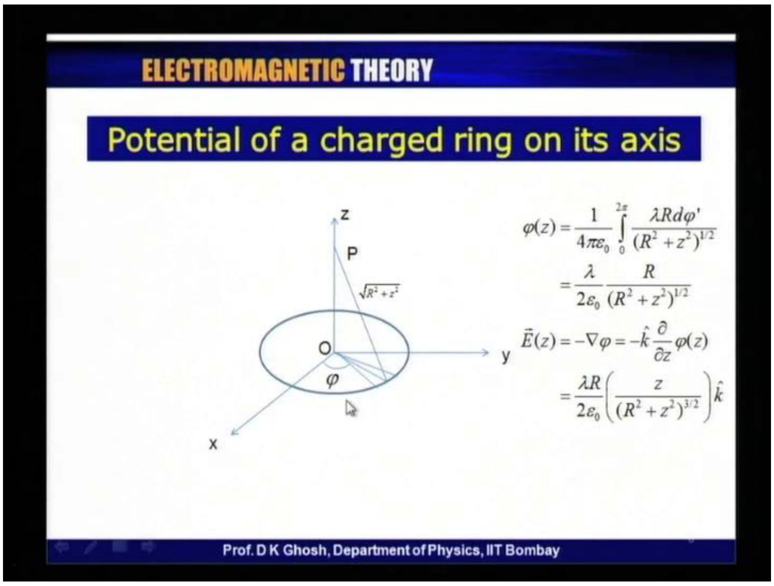

Let me now illustrate the concept of potential by calculating potential for couple of special cases. The first problem that I am going to take up is the potential of a charged ring on its axis the potential of a charged ring on its axis.

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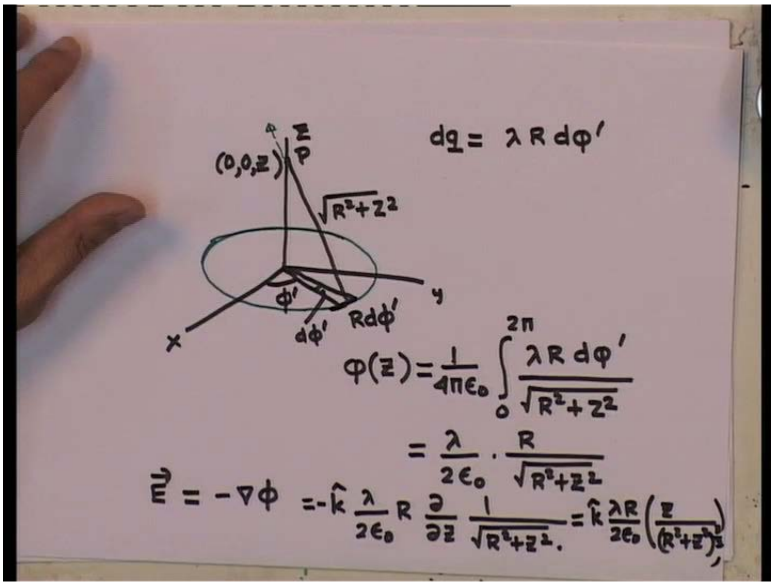

So, let me redraw this picture. So, I take the charged ring in the X Y plane. Say, I take the charged ring in the X Y plane and this is my Z axis and this is a charged ring having a charge density lambda. So, what I do is this. I take an element along this ring. Supposing, this is taken at an azimuthal angle phi and I take d phi as this angle, so that the length of this arc is R d phi.

And what I am going to do is to calculate the potential at a point; here point P, which has the coordinate (0, 0, Z). Since the length of the arc here is R d phi, well, I will put a prime here; phi prime because these are the integration variable. And so let me say that the charge that is there is given by lambda R d phi prime. So, that is the d cube which is the charge on that length element. Now this, if this radius of the ring is capital R and this of course the height is Z, then it follows that this distance is R square plus Z square. So, therefore, the expression for potential at the point Z is given by 1 over 4 pi epsilon 0.

Now, I have to calculate the potential at this point. The potential at this point is I take the charge element here; find out what is the potential due to this, which is 1 over 4 pi epsilon 0 lambda R d phi prime divided by this distance. And I have to integrate this. I have to integrate this from; I have to integrate across the angle which is 0 to 2 pi. So, I got lambda R d phi prime divided by root of R square plus Z square.

Now notice that everything else here is constant, other than the integration over d phi prime which gives me 2 pi. So, therefore I get lambda over 2 pi 2 epsilon 0 times R by square root of R square plus Z square. So, this is the potential. And the corresponding electric field is minus gradient of phi. By symmetry, the electric field obviously is along the Z direction. If you look at the field due to this, the field due to this is directed along this extension of this line. But as I go along by symmetry, the component perpendicular to the Z axis will cancel. And I will be left with only a electric field component along the Z direction.

So as a result, my gradient phi is nothing but unit vector K times d by d Z of phi with a minus sign. So, let me put minus lambda over 2 epsilon 0 R and d by d Z of 1 over square root R square plus Z square. This of course gives me…, minus will go away because of the differentiation K lambda R by 2 epsilon 0 into Z divided by R square plus Z square to the power 3 by 2. So, this is this is the final expression as we have obtained for the potential by due to a charge ring along its axis.

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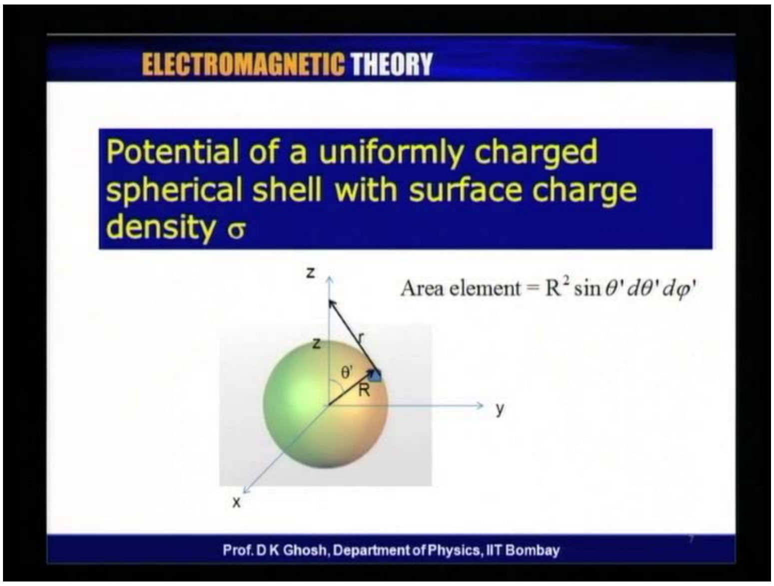

Now, let me continue with some more examples. Here, I am trying to calculate the potential of a uniformly charged spherical shell. So, I had a line charge; now, I have a surface charge; uniformly charged spherical shell with a charge density with a charge density sigma. So, what I do is this. That this is my sphere and it is a shell, so the charges are all on the surface. So, I take an element area element on the surface.

Now you recall that, in the spherical polar coordinate the area element is given by R square sin theta d theta d phi. Here, we have used for running variables, primes. So, it is R square. This R is the radius of the sphere because everything is on the surface. So, therefore that does not change; sin theta prime d theta prime d phi prime.

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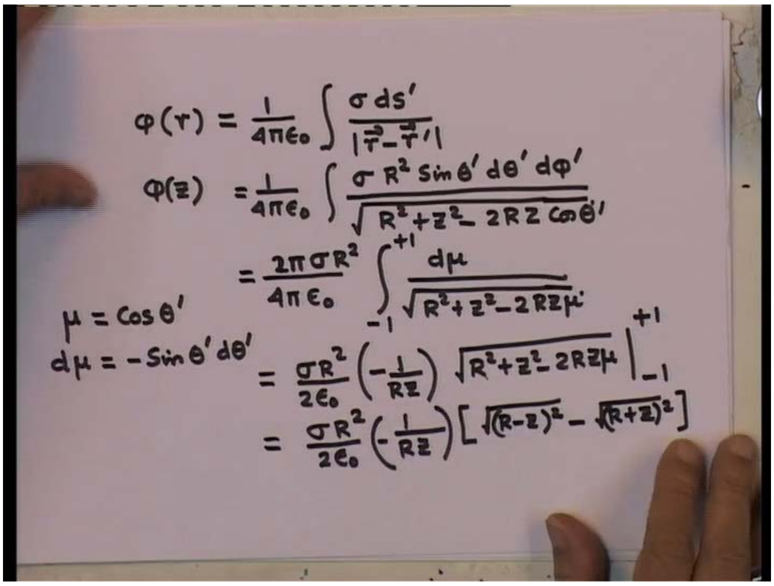

So as a result, the potential is at the point R. Let me go back a little bit. So, notice this that whichever point, now sphere is perfectly symmetrical, whichever point you want to calculate the potential, let me join that to the origin and call that as my Z axis call that as my z axis. So, the distance along the Z axis is Z, but I could… well, later on replace it with the R because it is, since sphere is perfectly symmetric with respect to the distances around it.

So, therefore my phi of r or phi of Z is 1 over 4 pi epsilon 0. I have my charge that is there, which is sigma d s prime divided by r minus r prime. And this we have seen. This we have seen is 1 over 4 pi epsilon 0, sigma of course is constant, d s prime is R square sin theta prime d theta prime d phi prime. And I need r minus r prime. So, r minus r prime; if you look at this diagram again, you find you find r minus r prime. See, this is R which I have been calling as Z and this is capital R which is my r prime.

So, what is wrongly labeled here as r is, actually the vector r minus r prime. So, r minus r prime vector by triangle laws happens to be equal to its magnitude; is square root of R square plus Z square minus 2 R Z cosine theta prime. So, this is equal to, let us write it down; R square plus Z square minus 2 R Z cos theta prime. So, this is this is same as phi of Z because I have defined my Z axis that way.

So, notice this that the integral has no dependence on phi phi prime. So, as a result the azimuthal angle integration gives me a factor of 2 pi. So, let us pull this out. So, I have got one over four pi epsilon 0; 2 pi sigma R square; these are all constant. I am left with sin theta prime d theta prime divided by this. And let me introduce a change of variable by taking mu is equal to cos theta prime; which means d mu is equal to minus sin theta prime d theta prime.

The limits, the recall that we are talking about theta; so theta limits are from 0 to pi. And since I am, now I am doing cos theta prime, so 0 to pi means 1 to minus 1, but I can make it from minus 1 to plus 1 by accommodating this minus sign there. So, therefore I have my integral from minus 1 to plus 1, sin theta prime d theta prime is d mu over square root of R square plus Z square minus 2 R Z mu. Well, this is an integral which is easily done.

So, I have got sigma R square by 2 epsilon 0. And if I integrate this, I get minus one over… actually, I get 2 R Z because there is a 2 R Z there; so but there is a square root there. So when I integrate it, I get to the power 3 by 2… 1 over, so this is raise to power minus half. So, I get minus raise to power half divided by half. So, that 2 and this 2 will take care of. So, we will be left with minus R Z. And the integration then is simply R square plus Z square minus 2 R Z mu from minus 1 to plus 1. And that is easily evaluated, which is sigma R square by 2 epsilon 0 1 over R Z into, this is square root of this is square root of R square plus Z square minus 2 R z; which is R minus Z whole square minus square root of R plus Z whole square. There is a reason; I have written it like this because we have to take care; while taking the square roots that the positive sign square roots are taken.

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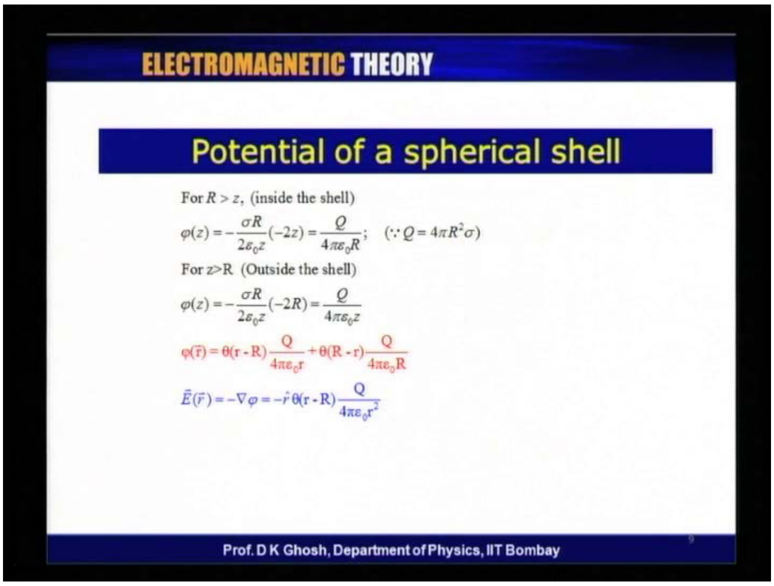

So…So, let us look at what it implies. So, this means that if I am talking about points inside the shell.

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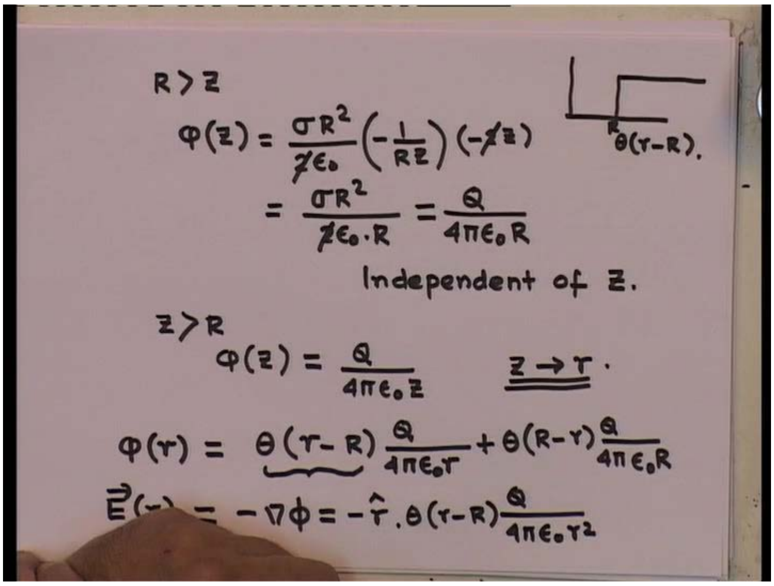

The for points inside the cell, radius R is greater than Z. If the radius R is greater than Z, then phi of Z is given by… now, you recall what did I have? I had sigma R square divided by 2 epsilon 0 into minus 1 over R Z. I have to be careful, a little careful that I have a square root of R minus Z square minus R plus Z square. So, I have to take the square root properly.

So, in this case I should take R minus Z because Z is less than R. So, therefore I get here minus 2 Z and that cancels giving me sigma R square by 2 epsilon 0 times R. Recall that, area of the sphere is 4 pi R square. So as a result, so this 2 and this 2 actually cancels out. So, I am not I do not have a 2 there. So I add, I multiply and divide it by 4 pi. I get 4 pi epsilon 0 R and 4 pi R square times sigma is the total charge in the shell. This is the field inside.

So, the field inside is given by Q; Q by 4 pi epsilon 0 R. You notice that, this is independent of the distance this is independent of the distance. On the other hand if we are talking about points outside the shell, Z is greater than R, and then I have to write down phi of Z by taking appropriately the square root. And I would get, instead of this factor being minus 2 Z, I will get minus 2 R. And as a result, I get this as Q divided by 4 pi epsilon 0 Z. So, it goes as 1 over Z.

Now both these both these expressions, I can combine into a single expression by writing this in terms of what are known as theta function; theta of r minus R times Q by 4 pi epsilon 0 r. Notice, what I have done is I have I have made now Z going to small r because I told you that, I could do that because of symmetry. So in principal, since distances are usually written as r, let me write it as r. And then plus theta of R minus r Q divided by 4 pi epsilon 0 capital R.

Now, this theta function is known as a step function, whenever the argument r is greater than R. If r is, small r is greater than R, then this function is equal to 1 and if capital R is greater than R, then this function will be equal to 1. So, this is the way to write it. And the electric field is obtained as a negative gradient of this potential. This is a constant field. So, it does not really give me any contribution. So, so a theta function is something like this. So, this is a theta. So, this point is R, this is theta r minus R. So, this is given by… This is obviously along the radial direction times theta of r minus R Q divided by 4 pi epsilon 0 r square; which implies that for points outside this sphere, it would seem like all the chargers are concentrated at the center of the sphere.

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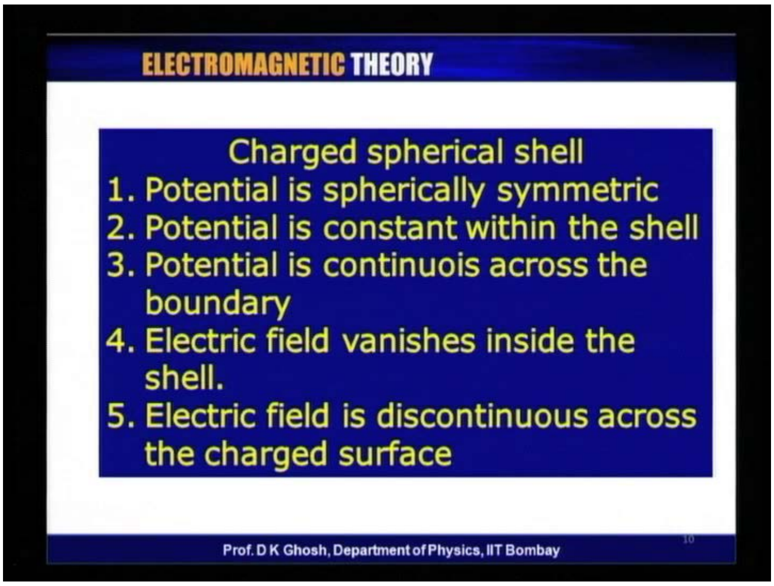

So, let us let us summarize what we have learned from this example. This is a uniformly charged spherical shell. First thing to notice as I expect is that, the potential is spherically symmetric. It depends only on the distance from the center. Second thing is the potential is constant within the shell potential is constant within the shell. This also follows from the fact that, since all the charges are on the surface of the sphere, using Gauss's law you can easily show that the field inside is equal to 0, because Q… 0; E dot d s is 0. Now since the field is 0, the potential has to be constant and which we take to be the 0. Potential is constant inside.

The other thing is that, if you look at the expression for the potential, so what I meant is that the field inside field inside is 0, the potential is constant. Now, if you look at the expression for the potential, then you notice you notice that as small r becomes capital R, this expression is the same as that expression. In other words, the potential is constant across the surface. So, potential is constant inside; spherically symmetric outside, going as one over R as if all the charge was concentrated at the center.

The potential is constant across the spherical surface, but the electric field is 0 inside the shell, but the electric field outside goes as one over R square; Q by 4 pi epsilon 0 into 1 over R square. So on the surface; very close to the surface where r is equal to capital R, but just outside, the electric field magnitude is 1 over 4 pi epsilon 0 Q divided by capital R square; where r is the radius. But the moment you come infinitesimally inside, the electric field is 0; the potential is constant. What it tells me is that across the charged spherical surface, the electric field has a discontinuity the electric field is a discontinued.

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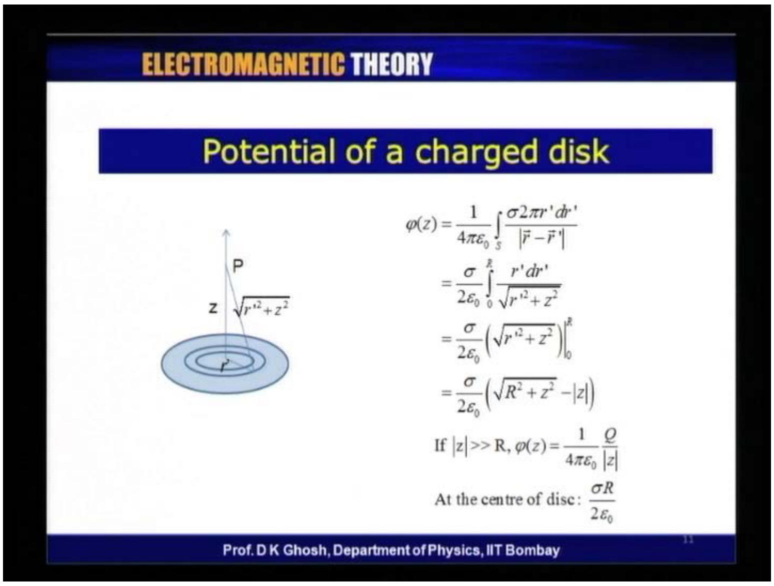

Now, we will see that this is true in many other cases as well. Let us got to another problem, which is calculating potential for a charged disk. So, I have shown here a charged disk. So, this is a… in principal, a much simpler problem. So, I have a disk and I am interested in calculating the electric field along its axis. Now, there is the problem, which obviously has cylindrical symmetry. So if this distance, here now what I do is I take a concentric or I take a shell of radius r prime and width d r prime, so that the area of that shell.

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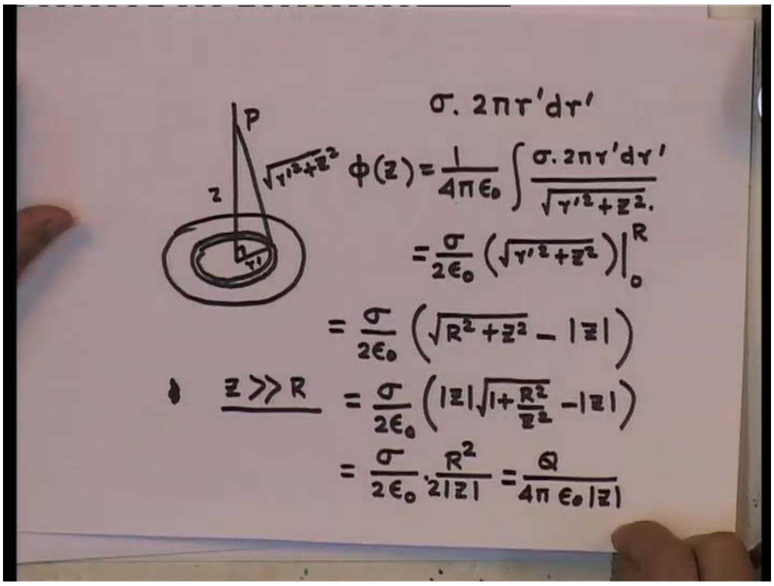

So, let me redraw it on the paper. So, this is a disk and I am looking at the axis on the center of this disk. So, what I have done is to take a concentric annulus here of radius r and width d r; radius r prime, width d r prime so that if the charge density is sigma, the total charge contained in that annulus is sigma 2 pi r prime d r prime.

Now all these points, all these annulus is symmetric with respect to this. And so therefore the potential at the point Z; what I need is just to take any of these points and you notice if this distance is r prime, this is Z. This is simply square root of, this is actually the this is a perpendicular. So, this is r prime square plus Z square.

So, phi of Z becomes one over 4 pi epsilon 0 integral this charge which is there, sigma 2 pi r prime d r prime divided by square root of r prime square plus Z square. Now, this integral is trivially done; 2 pi, 4 pi goes, I am left with sigma by 2 epsilon 0. This is already root of r prime square Z square, plus Z square; r prime d r prime is differentiation of r prime square. So, this is nothing but square root of r prime square plus Z square only. So, once again for the same reason, the factor of 2 cancels out; because there is a 1 over, there is one over square root there and r prime square gives a factor of 2. So, this is from r prime equal to 0 to capital R.

So, let us look at what does it give me. So, this is equal to sigma by 2 epsilon 0 r prime. So, this is R square plus Z square minus, I have to put r prime equal to 0, so you notice what I get is square root of Z square, which is nothing but modulus of Z. So, this is my potential at the point potential at the point Z.

Now, let us take some specific limits. Supposing this point Z, this point, let us say P is far away, so that Z is much greater than R. If Z is much greater than R, I can write this sigma by 2 epsilon 0. What we do is this. That, in this case since Z is much greater than R, I have to do a binomial expansion of this one so that, what I get is I take modulus of Z out; square root of 1 plus R square by Z square minus modulus of Z. So, this is equal to sigma by 2 epsilon 0 R by modulus of Z.

Now look at what it actually means, it is the potential, the expression for the potential that we are finding at a very large distances at very large distances. It is the… it is given by… now, notice that the total area if I put in because this is just pi r square, this is pi r square, so I can take out sigma times pi r square is equal to Q and write it in terms of charge Q. And I will get; then the expression that is the at large distances it looks like a single point charge.

But, more interesting point is what happens if this is actually there is a factor of two there; because there is a binominal expansion. So, it is 1 plus 2 times R square 1 plus half R square by Z square. And so therefore it is there and accordingly we could do that. This is R square by Z; because there is a Z with Z square canceling out and I am saying that pi times sigma R square is the charge Q divided by 4 pi epsilon 0 Z; which means that at large distances, at the point p, the potential is same as that due to a single point charge. Now, that is physically meaningful, because when the distances are very large. Then obviously the disk which is of small size, it appears like a point charge.

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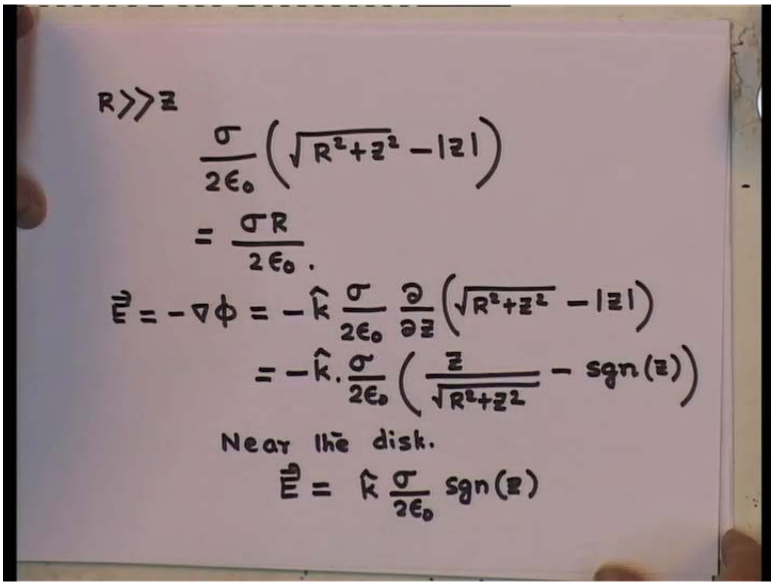

On the other hand, the more interesting limit is when the radius R is much greater than Z. What does it mean? It means that I am very close to the surface. Now when you are close to the surface, the expression is sigma by 2 epsilon 0 square root of R square plus Z square minus modulus of Z. If Z is very small, then this is simply equal to sigma R divided by 2 epsilon 0; this is sigma R divided by 2 epsilon 0.

Now, notice that when you are very close when you are very close to a surface, so far as that point, the observation point is concerned, the disk looks essentially of infinite extent. So, the expression for the potential that you would get would be identical to that due to the infinite charged plane. Now, the thing becomes obvious.

If using this you have to calculate the electric field, which is minus gradient of phi, and we use the cylindrical symmetry so that, this is given by sigma by 2 epsilon 0 d by d Z of this quantity here; square root of R square plus Z square minus modulus of Z. And that is equal to minus K sigma by 2 epsilon 0 Z by root of R square plus Z square, which is the differentiation of R square plus Z square.

Now, this is the modulus of Z. So, its differentiation with respect to Z should give me 1. If Z is just it is Z, minus one. So, if Z is greater than 0 it is minus Z. So, either I just get 1, and if Z is less than 0, then this Z is negative. So, I must write plus Z and differentiate it. So, I get sign of Z; sign is s g n, namely the signature; the positive or the negative sign of Z.

Now now notice that, near the disk where Z is equal to 0 the electric field, this term vanishes; because there is a Z in the numerator and I am left with K sigma by 2 epsilon 0 sign of Z. So, if Z is positive it is K by sigma by 2 by epsilon 0 and if z is negative it is minus k sigma by 2 epsilon 0.

Once again you notice that, while the potential was continuous across the surface, the electric field is not. The electric field has a discontinuity the electric field has a discontinuity; because it is plus K sigma by 2 epsilon 0 plus sigma by 2 epsilon 0 above the plane. And below the plane, it is minus sigma by 2 epsilon 0.

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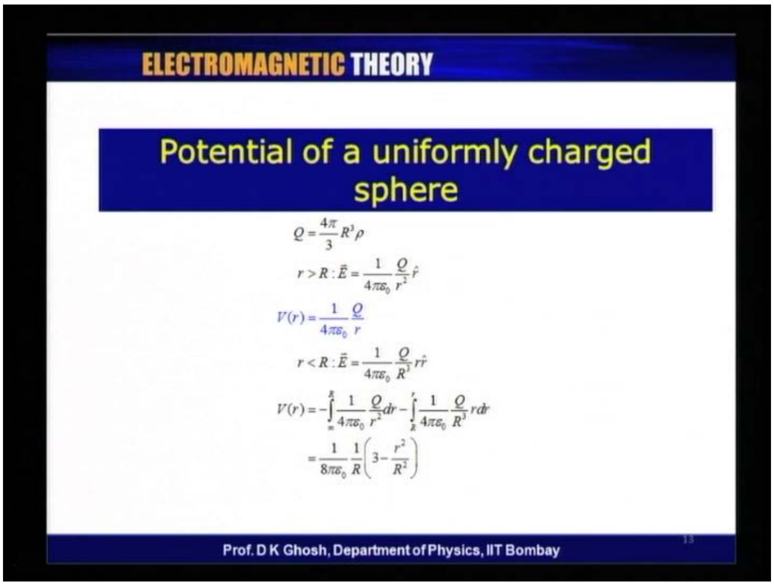

Before I go to this other example, let me make one observation. There are two cases that we have talked about. One was the charged spherical shell; another was a charged disk. In both cases, we have charged surfaces. And what we noticed is that whenever there is a charged surface, I am generalizing it because I will prove it in general. Whenever we have a charged surface, the potential across it is continuous. But the electric field suffers a discontinuity as I go from above the charged surface to below the charged surface. We will see that this can be proved as a general property.

Let me let me continue example of a… Or calculation of potential for a three dimensional case. So let, in this case I am talking about a uniformly charged sphere containing a charge Q.

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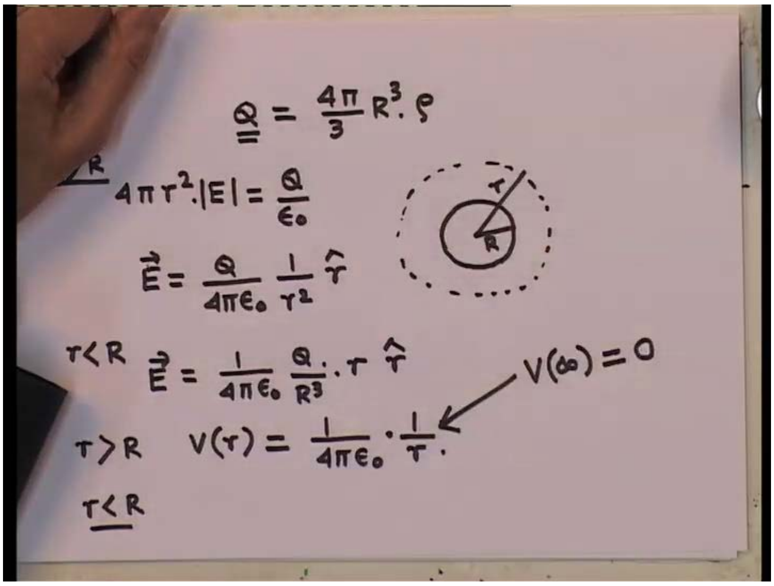

So, my charge Q is given by the volume 4 pi by 3 R cube times the charge density rho, which is being taken as uniform. Now, this is a problem which we have seen in the past as an application of Gauss's law. Remember that, Gauss's law tells me that the flux through any surface, real or imaginary, is given by the charge enclosed divided by… the flux through any surface is given by charge enclosed divided by epsilon 0.

So as a result, if the point at which you are you are calculating the field; so let us suppose this is my sphere, this is of radius R. Now, if I am calculating the field at a distance r, small r, then I enclose it by a an imaginary Gaussian surface of radius small r. And I find that the flux then is 4 pi r square, which is the area, times magnitude of E which is equal to the amount of charge contained, which is this Q, full Q divided by epsilon 0 which gives me the electric field. This time, I will put its radial direction Q by 4 pi epsilon 0 1 over r square unit vector r; which is like Coulomb's law; as if the entire charge is concentrated at the origin.

However, if radius R, so this is R greater than r. If the radius r is less than R, the electric field is given by 4 pi R square E equal to Q enclosed by epsilon 0. And the Q enclosed is not the entire Q, but Q times small r cube by capital R cube; because that is the fraction of charge that is included within an imaginary surface of radius small r, which is less than capital R.

So, this will give me… if r is less than R, the electric field will be given by 1 over 4 pi epsilon 0 Q by R cube times small r. And of course, the direction is still along the radial direction. Now, what I will do is this. I will integrate to find out I will integrate to find out the potential in the two cases. So, in the first case for r greater than R, the potential V of r, remember negative gradient of the potential is electric field, so this is simply given by 1 over 4 pi epsilon 0 1 over r.

For r less than R; now this already assumes this already assumes that the zero of the potential is at infinity. Now having chosen the reference once, I cannot change the reference for the second part of the problem. So, one has to be careful when you calculate or you determine the potential at the point small r, which is less than the radius.

So, I need the potential of that point with respect to the surface of the sphere. So, in order to find out what is the potential on the surface of this sphere, I go from infinity to the radius of the sphere, which is of course the result that I already know. So, that is one part. And the second part is to go from the surface of this sphere to the point r and take this expression and integrate it. These are both of them are trivial integration. If you calculate it properly you will find this is given by 8 pi epsilon 0 1 over R 3 minus R square by R square.

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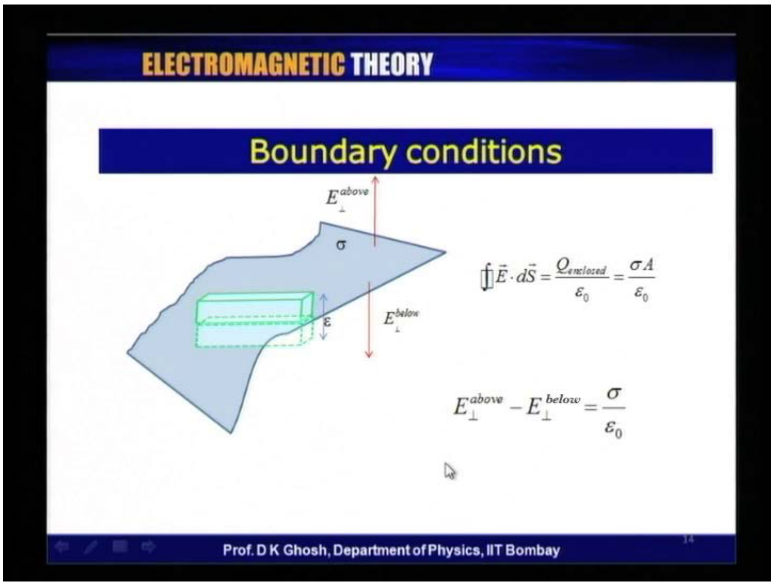

We have taken a few examples of how to calculate the potential in different cases. Let me return back now to this point that we have been making. That, whenever we found that there was a surface charge, the electric field had a discontinuity and the potential did not.

Now, in other words I am looking at the behavior of the field and the potential at the boundary. What happens at the boundary? So, let us look at a situation like this. I have an arbitrary surface here. Now I already know, supposing these are infinite charged surfaces, I already know how to calculate the electric field above and electric field below. Now, notice we can obtain the thing like this. Let me enclose this by a Gaussian surface.

So, I take a rectangular parallel pipe of height epsilon. Half of it is above the surface and half of it is below the surface. Now, I know that the flux through this is equal to Q enclosed by epsilon 0. And how much is Q enclosed? Remember the charge is only on the surface. So, if the surface area of the top happens to be equal to A, then I get sigma A is the charge divided by epsilon 0. This is flux.

So therefore, this is E dot d s. And as I make this, epsilon becomes smaller and smaller. And I find that the contributions are only from the top and the bottom phase; which means that normal component of E above the charged surface minus normal component of E below, this should have been below, is sigma by epsilon 0.

So, this reemphasizes the fact that across a charged surface, we have discontinuity of the electric field. I will elaborate on it in our next lecture. Summarizing; we have looked at potential, its meaning and calculated it in a few cases. We have talked about a charged ring, a spherical shell and a uniformly charged sphere; three different types of problems. Important point to find out was that whenever there is a charged surface, we have found electric field has a discontinuity. Now, this is a general problem. These are known as electrostatic boundary condition, which I will take care of in the next lecture.