Electromagnetic Theory
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Module -2
Electrostatics
Lecture -7
Gauss's Law, Potential

In the last lecture, we had introduced the concept of electric field and talked about Coulomb's law as well as the principle of super position that is when there are multiple charges which produce electric field at a point. We had said that the effect is essentially a linear sum of the effects, due to each individual charge. We had also talked about what happens in a continuous charge distribution, where we wrote down an expression for the electric field in terms of integrals, which involve the charge density at a point. In this lecture we will be trying to talk about essentially two ideas.

The first one is known as Gauss's law, which is essentially a law very similar in content to the Coulomb's law, and we will see that in spite of the fact that Coulomb's law and Gauss's law are essentially equivalent statements. But under certain conditions it is much easier to use the Gauss's law, but before we do that let us introduce the concept of a flux.

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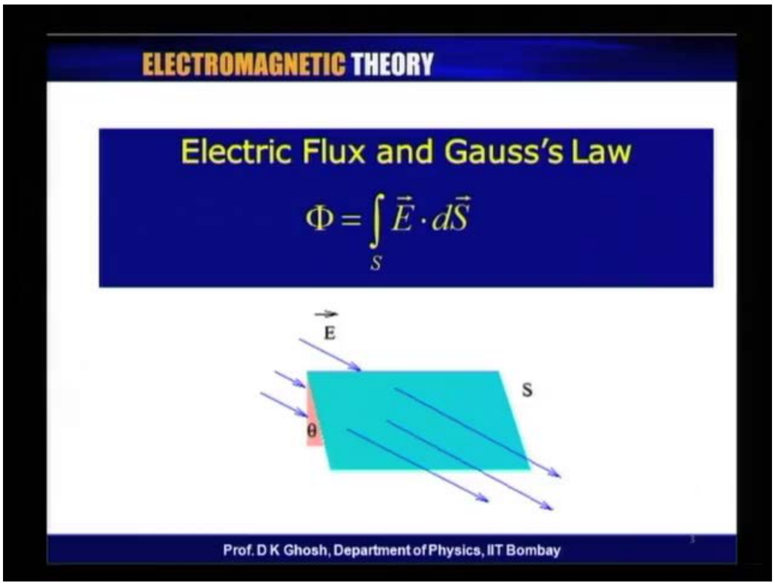

The flux essentially comes in the same way as when for example, you have a flowing water. Now, when a flowing water passes through a surface, basically the the amount of flow that goes out of that surface, it depends upon not only the velocity of the flow, but it also depends upon both the area and the orientation of the area to the direction of the velocity. Now, exactly the same way we define the flux of an electric field. So, you recall that we had talked about the area being considered as a vector. So, if you look at an arbitrary area then you could consider an infinitesimal part of it. The outgoing normal to such an infinitesimal area is the direction associated with such an area, it of course, has a magnitude, so, in that sense an area though it is two dimension, one can consider it as a vector; it is a vector, however there is an convention associated with it namely that of the outgoing normal.

Now, in this particular example what we are showing for instance is a a surface and there is an electric field which is coming. So, with the direction of that surface is perpendicular to this S and as a result the electric field makes an angle with the direction of that surface. One defines the flux of the electric field as surface integral of the dot product of the electric field, with the area limit integrated all over. So, in other words if you look at a particular point where the area element the vector. The normal vector is making an angle theta with the electric field then the definition of the flux would be integral of S E d s cos theta.

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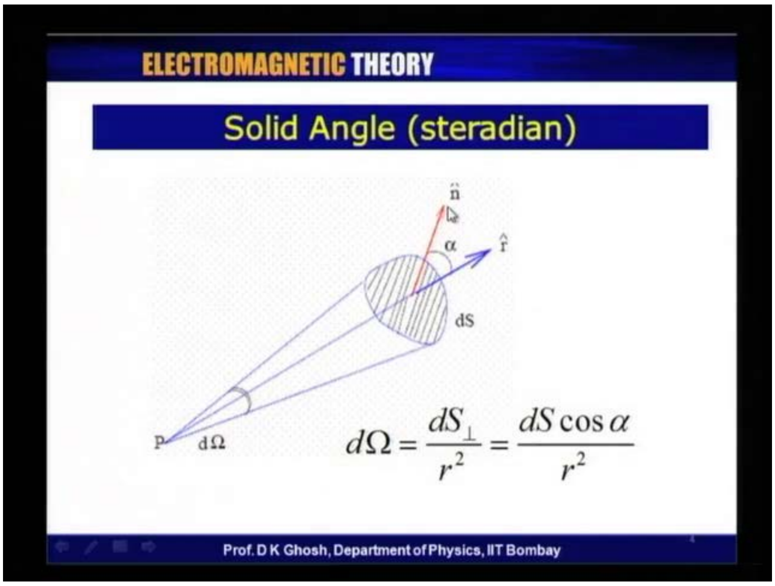

So, this is what we are going to be talking about, but before we continue with it we would like to introduce what is known as a solid angle. You are all familiar with the concept of an ordinary angle. You recall how does one define an angle in two dimension. Now, in angle in two dimension is defined by so it is something like this.

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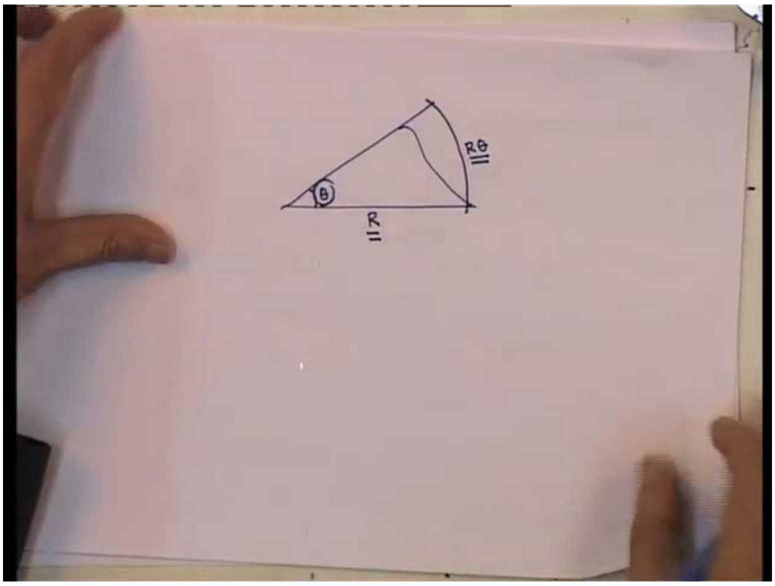

So, supposing you have this angle, so what is this angle? Theta this is you you draw an arc there and this angle theta is related to this length of the arc and this radius. So, if this is radius r then r times theta is this length of the arc. In other words you define this angle theta measured in radiance as the ratio of the length of the arc to the radius. The only thing that you recall is that this arc is a part of a circle. So, for instance if you did not have this as a part of the circle, but you had a curve like this, then I could still define this angle theta by saying that I take projection of this arc along the normal.

Now, recall that in spite of the fact that an angle being ratio of two lengths R theta by R is dimensionless. We still conventionally measure it by using if you like an unit call degrees or radiance. So, this angle theta in radiance is given by the length of the arc divided by the radius. Now, the concept of a solid angle is essentially very similar. So, what we do is this, that suppose I am looking at what is the angle, what is the solid angle? That an area makes at a particular point p. Now, what I do is this, I join I draw tangents from that point on to this area. Remember if I take this angle to the small then the direction.

So, I have the following thing that the direction of the length, I which I take to be infinitesimal is perpendicular to the direction or perpendicular to the surface. This is the direction R which is essentially joining p to to that element and extending it and this angle this angle is alpha. Now, one defines the solid angle d omega made by this area element, which is infinitesimally small area element. As the projection the perpendicular component of this surface the perpendicular component this surface divided by R square which means that since, I am taking the projection perpendicular to this R this is d s cosine of alpha divided by R square. So, once again you notice that d omega.

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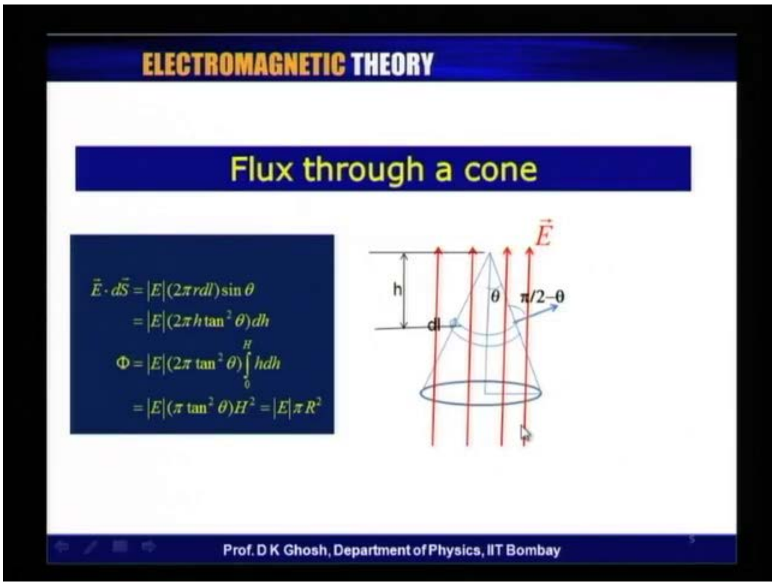

Hence omega is dimensionless, in spite of that we measure it in terms of an unit known as Steradians.

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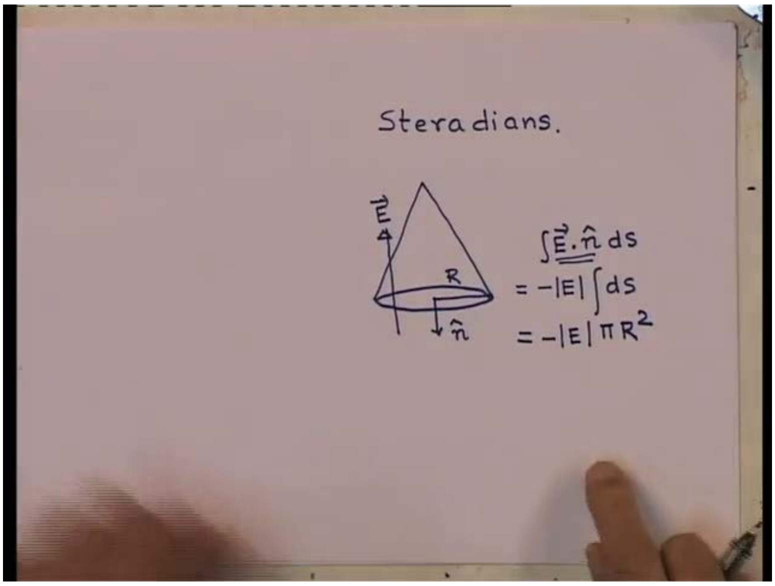

Just to look at this is a little more in detail. Let us look at, let suppose I have a cone. I have right circular cone, having a vertical angle half semi vertical angle equal to theta. Now, let me make a statement. These red things that you see these are the direction of the electric field, which is threading through this cone. From the base of the cone it enters perpendicularly and leaves through the slanted surface of the cone. Now, I am interested in calculating how much is the flux coming out of that cone. Now, before we do that please note out that this cone has essentially two surfaces. So, let me draw a cone here. So, there is a base there and since we define n as the outward normal that would be the outward normal of the base.

Now, suppose I am interested in finding out, how much is the flux through this base? So, this we have seen is integral of E instead of dot d S. Let me write a dot n d S. Since, the electric field is constant and as a magnitude E it will come out of this sign. So, basically and the second thing that you notice is the direction of m and the direction of E are opposite to each other, because electric field is entering the base. So, as a result this E dot n is nothing but minus the modulus of E and I am left with only the integral over d S, hence assuming that R is the radius of the base of the cone. So, this is nothing but minus E times pi times R square. The reason why I have a negative sign is because in my definition, I had talked about outward flux, but as you can see from the picture the electric fields are threading into the cone.

As a result the net outward flux through that base is negative. However, let us look at what happens if I look at the slanted area. Now, actually speaking you could, we could supposing it were a water flowing through you could have got an answer to that without even working it out. The reason is whatever water is entering through a base must leave through the slanted surface of the base and as a result if the flux coming in is E times pi R square, the flux going out through the slanted surface must also be equal to E times pi R square. Let us look at whether that is correct or not?

Now, what we do is this let us consider the following. Let us take a a strip now. This is this is a strip of radius small r this is a strip of radius smaller r at and width d l length, width will be d l at a distance at a height h below. Now, so if I do that look at this. How much so I have a length l and radius small r here.

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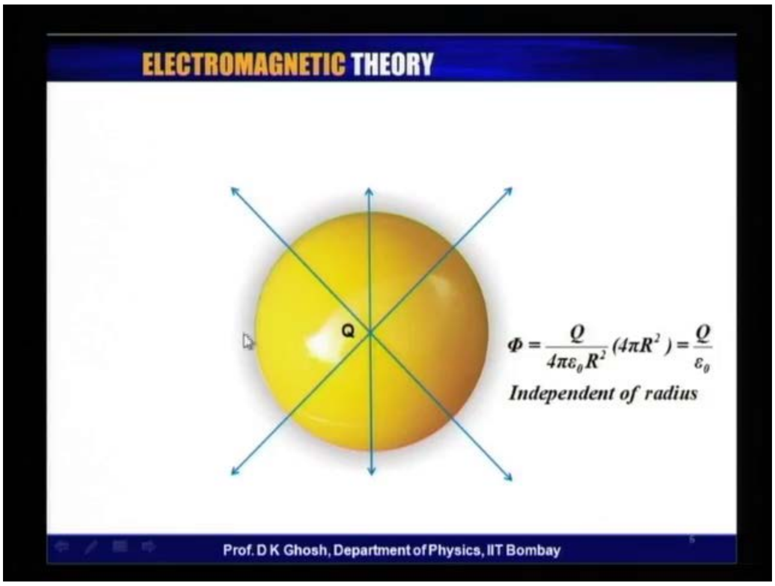

Now, so let us look at this what is E dot d s E dot d sis of course, the magnitude of E the area of that strip area, of that strip, which is equal to 2 pi r because r is the radius there times d l, which is the slanted width. The second thing is that the angle which the normal to this slanted surface is making with the electric field is obviously pi by 2 minus theta. So, I get 2 pi r d l cosine of pi by 2 minus theta. So, let us right it again.

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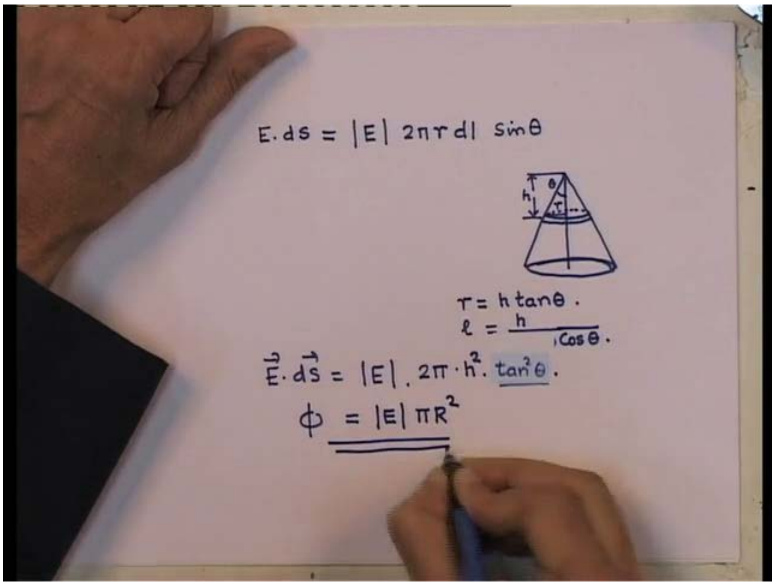

So, E dot d sis equal to magnitude of E 2 pi r d l times sin theta. Now, let us look at this. So, let me redraw this picture on this. So, I have taken this strip here. This height here is h, this angle is theta and this is of course, R I can immediately look up a relationship between them. You can see that the what is R, what is the relationship between l h and R? So, R is obviously equal to h tan theta and l which is this length which is this length is equal to h by cos theta because this is h, so l cos theta is h. So, it is h by cos theta. So, if you plug the name here, what I get is E dot d sis equal to magnitude of E times 2 pi and of course, I have got R d l sin theta. Therefore, you notice there is already a sin theta by cos theta here. I have got this thing so I get h square times tan theta. Now, the tan theta is actually constant tan theta is actually constant. Therefore, what I get here is this.

So, what is my flux? Phi I have to only integrate this quantity there therefore, what I get is after integration, I get E magnitude of E times. If you do it properly and put in the relationship as you have done in the slide times pi R square, this is the flux through a cone by then I already knew the answer, that if the amount of flux that is getting in is E times pi R square. The amount of flux which is going out should also be E times pi R square. Let us look at some results that is coming out of this. Let me take a sphere of radius capital R and let me put a charge q at the center of this sphere.

Now, you already know that the lines of forces which are the directions of the electric field, due to this charge q are all symmetric. If I have since, I have got a sphere, these are all perpendicular to the surface of the sphere and I am only interested in the electric field on the surface of the sphere. Now, since the direction of the electric field and the direction of normal to the surface are parallel. The flux E dot d s is nothing but the E times d s. When I integrate now, remember that electric field strength is the same all over at all points on the surface of the sphere. Namely it is equal to 1 over 4 pi epsilon 0 q by R square.

So, that is my strength of the electric field. Since, the electric field strength is the same everywhere the integral is only over the surface, which gives me 4 pi R square that is that is the area of that surface. If I now, calculate it it becomes q by epsilon 0, notice that the amount of flux I have put in a charge at the center of the sphere, single charge q the flux coming out of the surface is simply given by q by epsilon 0, which is independent of which is independent of the radius of the sphere. Let us proceed with little more, we already introduced the concept of a solid angle.

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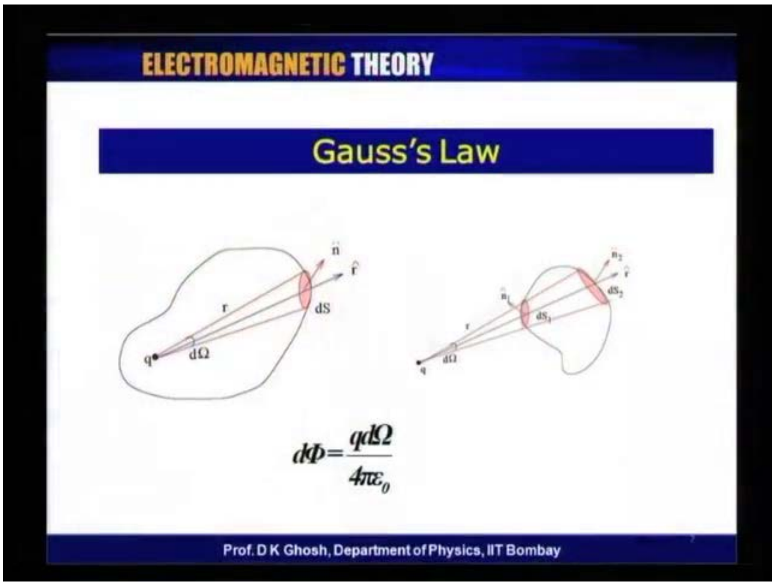

Now, suppose I had a surface of arbitrary shape and there is a charge q there is a charge q, that I have kept somewhere inside that surface. Now, notice supposing I am interested in calculating the the flux through the surface. So, what I do is this, I need to calculate electric field at every point on this surface. So, let me take an infinitesimal surface area d sat a distance r along the vector R from the point q. Now, what I do is this. Using this surface area I try to I connect them to q and find out the solid angle d omega that this surface element d s makes at the point q.

Now, let us look at what do I get out of it? Now, so far as the electric field is concerned, the electric field at the distance R is q by 4 pi epsilon 0 1 over R square along the radial direction along this direction. I have to now, take the dot product with d S. So, I multiply with d s times cos theta, whatever is the angle that makes, it makes here. But you remember that d s cos theta is nothing but the projection of this area element d sin a direction perpendicular to the R. So, that is nothing but in our previous notation d s perpendicular. So, as a result what I get is that d s perpendicular divided by R square that gives me d omega. So, the d phi is q times d omega divided by 4 pi epsilon 0. So, notice that the flux due to the area element d s depends only on the solid angle, that it subtends at the point q.

Now, when I do the integration all that I need to do is to take the area limit all over the surface. You notice that every place the direction of the radial direction is outward and the normal to the surface will make an acute angle with it and as a result I get the total solid angle that is subtended at the point q, which obviously is 4 pi. So, that I get the total flux equal to q times 4 pi divided by 4 pi epsilon 0 namely q by epsilon 0. Now, let us suppose let us suppose this charge q is not inside that surface, but is outside that surface, at this point. Now, I do exactly the same thing, but notice that the radial if you draw rays from this point q, it will cut the surface in two places.

One is here let us call it d s1 another is here that is called d s 2. The normal the normal direction to this is that way, whereas the normal direction to this one is in the opposite direction. Now, so as a result as you go around this surface for every positive solid angle, there would be a negative solid angle and the net effect would be 0.

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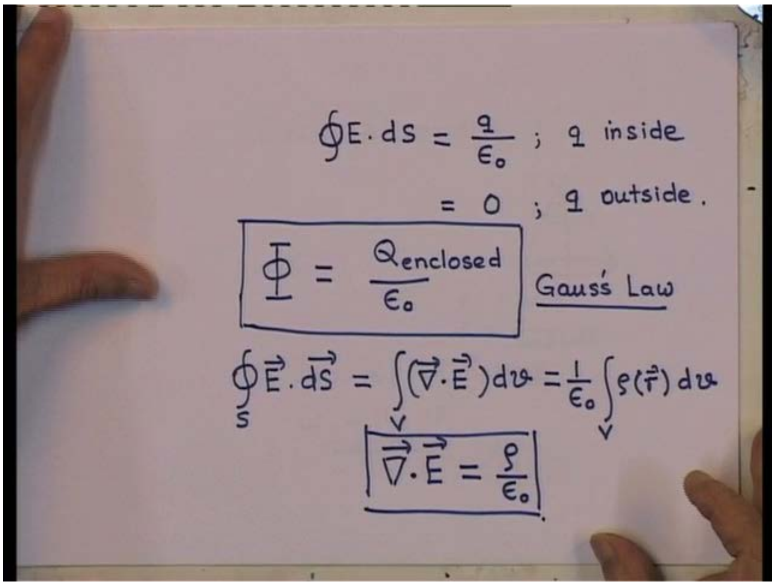

So, what are we actually got, what we got is that integral of E dot d s over a closed surface is equal to q by epsilon 0, when the charge q is inside the surface is equal to 0 if q is outside. Now, this is in general true. If I have multiple charges as well because the principle is the same it is a super position principle. So, what you actually find is this that the flux phi electric flux phi which is the same as the integral E dot d sis given by the amount of charge that is enclosed by the surface divided by epsilon 0. This is the statement of Gauss's law. If there are no charges enclosed inside it is equal to 0. If there are some then of course, you have to simply talk about the amount of charge that is enclosed.

So, that is my Gauss's law stated in the form of flux phi which is integral E dot d s. Now, let me let me now recall for you that suppose, this surface S encloses a volume V. So, this integral E dot d s by divergence theorem, is same as the integral over volume V and the surface integral will become a volume integral. I will be left with del dot E that is divergence of E d V and that is equal to the amount of charge that is enclosed divided by epsilon 0. The amount of charge that is enclosed, can be written as integral over the volume, the integration over the density. Now, since this is true for an arbitrary volume, I get a corresponding differential statement of the Gauss's law, which says del dot E is equal to rho by epsilon 0. So, this is the differential form of the Gauss's law.

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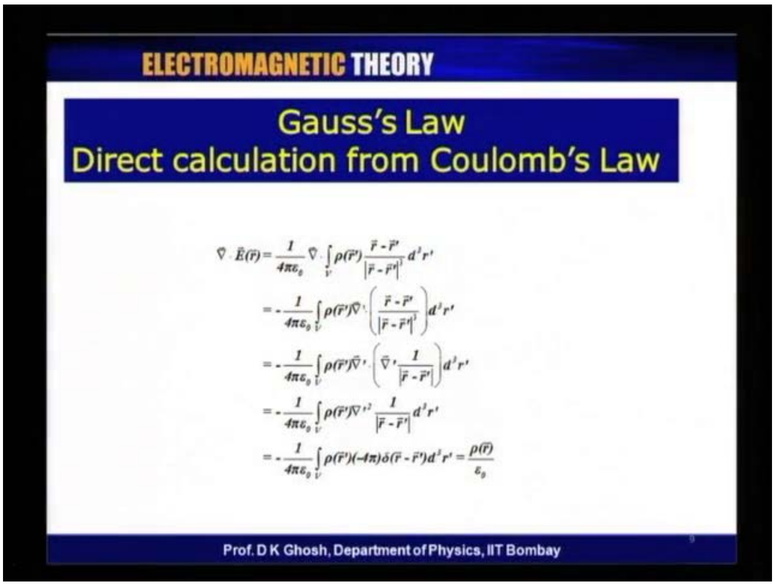

So, let us do a little bit of detailed arithmetic and try to see whether one can directly derive it from Coulomb's law. So, for that I need the expression of Coulomb's law for a continuous charge distribution. So, notice that that we had earlier written down.

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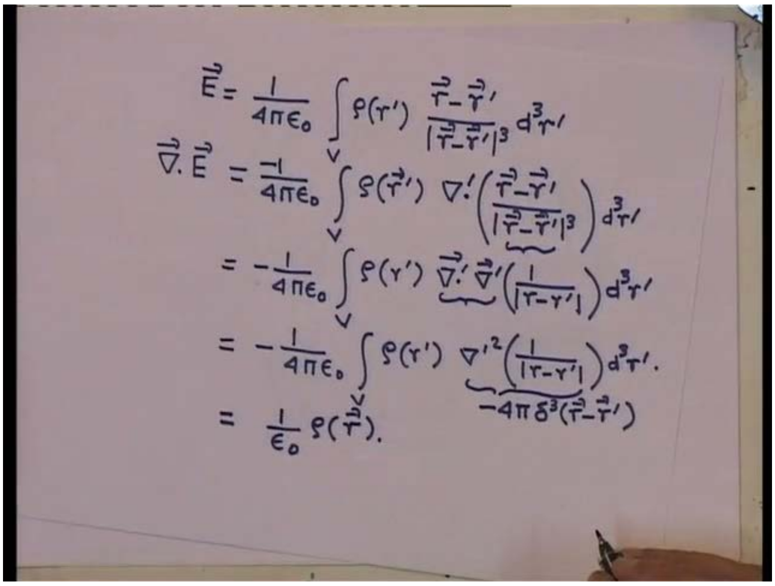

The electric field E is given by 1 over 4 pi epsilon 0. There is electric field at a point r integral over the volume rho at an arbitrary point inside that volume r minus r prime vector by r minus r prime cube and the integration is over the prime variable. So, if I want now, del dot of this, if I need the del dot of E this remember this del operator is with respect to the variable r, which is the point at which I am trying to find my electric field. So, this can be written as 1 over 4 pi epsilon 0 integral over the volume rho r prime does not have any dependence on r. It is a quantity dependent on r time prime. Then I write down del dot of r minus r prime divided by r minus r prime cube.

Now, so this is this is del dot of this thing. Now, I can do a bit of a slight of hand and I notice that this quantity here depends on r minus r. So, instead of doing the derivative over the variable r supposing, I have to take the derivative with respect to r prime, then all that I need is just a minus sign. Therefore, I put a minus sign here and put a prime there. So, that difference now is that this quantity now is del prime dot r minus r prime divided by this. Now, this is equal to minus 1 over 4 pi epsilon 0 integral over the volume rho r prime del prime dot. Now, notice this is essentially a vector like vector r by r cube. So, I can write it as a gradient, I can write it as a gradient but because I am writing it as a gradient over the primed variable, I do not pick up that minus sign.

So, this is del prime dot prime of 1 over r minus r prime d cube r prime, but this is nothing but del prime square, this is nothing but del prime square. So, this is equal to minus 1 over 4 pi epsilon 0 integral over the volume rho r prime del prime square of 1 over r minus r prime d cube r prime. If you recall when we talked about the Laplacean operator, we said del square of 1 over r is a magnitude is equal to minus 4 pi times, the delta function. This is actually three dimensional delta function. Therefore, this quantity here is minus 4 pi times delta function.

Now, since the delta function is a three dimensional delta function, I indicate it by delta cube r minus r prime, but Once I have a delta function this integral is easy because at all points other than r is equal to r prime the delta function vanishes. So, I only have contribution from r is equal to r prime there is already a minus sign 4 pi cancels. So, I will be left with 1 over epsilon 0 times rho of r, which is exactly what we derived using the definition of the flux, but you notice this is a hard work. Let me illustrate the calculation of flux taking a few examples.

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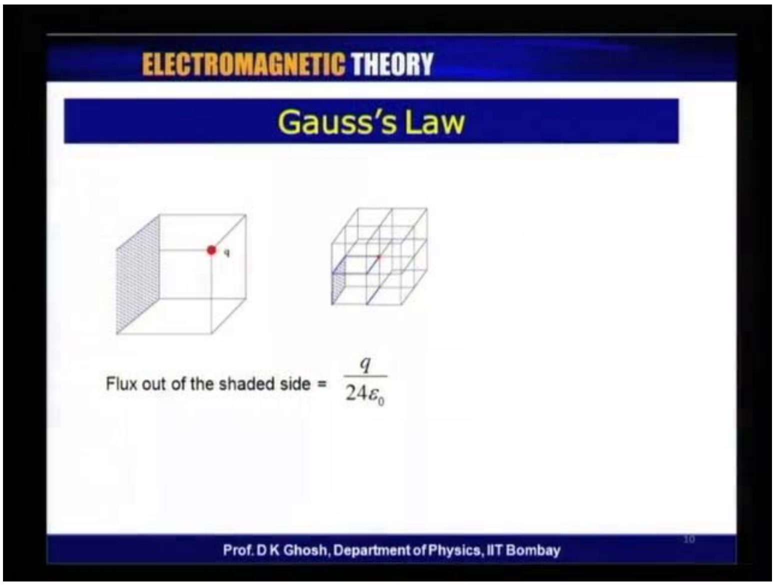

I have here a cube. Now, I have a cube and at the corner of the cube at the corner of the cube I put a charge q, but before I illustrate this notice a couple of things. The physical content of the Gauss's law is the same as that of the Coulomb's law that is more nothing nothing less. However, we can sometimes use Gauss's law to our advantage, particularly where the symmetry makes it easy for us to write the left hand side namely integral E dot d s as a function, explicit function of the electric field E, now then if I can calculate that I can get an expression for the electric field. This obviously cannot be done, accepting in cases, where there is significant amount of symmetry and that is exactly what we are going to talk about now.

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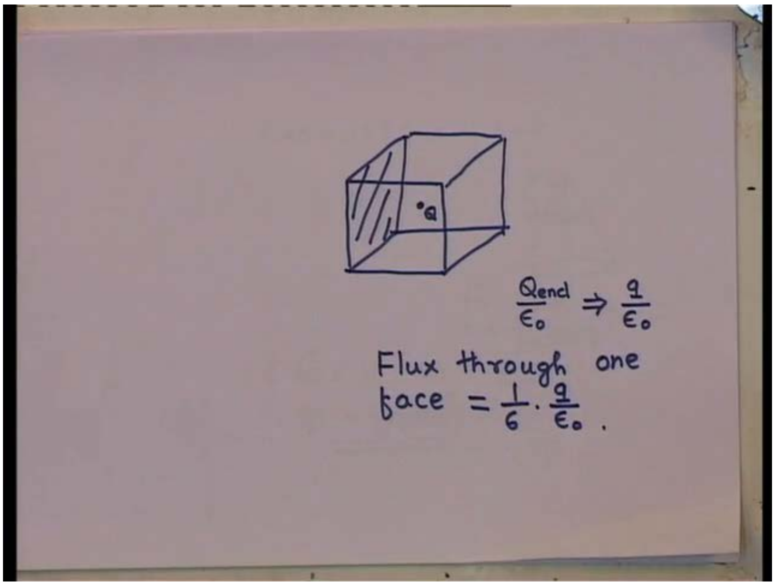

For instance, let us take first a cube. Now, supposing I put a charge q at the center of the cube. Now, I ask the question how much is the flux that is going out of any of the sides? For instance this side now, first you in vogue symmetry there is no reason. Why since the charge q is at the center one side of the cube should be different from any other side? Therefore, whatever is the flux through all the six surfaces of the cube, should be divided equally by its surface. What is the amount of flux? So, the amount of flux that I have according to Gauss's law is nothing but charge enclosed q divided by epsilon 0 q enclosed which is equal to small q by epsilon 0.

This is the amount of flux that would go out through all six surfaces. So, if you want to talk about one surface when the charge q is at the center, I get flux through one surface or one face of the cube has to be equal to 1 sixth of q by epsilon 0 that is good. Now, now let us ask what if this charge was not at the center of the cube, but at one of the corners the symmetry is obviously lost the symmetry is lost, because that charge is at the corner intersection of three surfaces. So, these three surfaces are a different from the other surfaces, but we can do something better. So, notice that I am interested in calculating the outward flux through the shaded surface. Now, what I do is this a charge is at the corner.

I already know how to solve the problem, if the charge happen to be at the center of a cube. So, why not artificially make this charge at the center of a cube, but please note that these surfaces that we are talking about on which we apply Gauss's law there are known as Gaussian surfaces. A Gaussian surface does not have to be a physical surface, it could be an imaginary surface as well because the the physical property of a surface never came into our discussion. So, I could be talking about artificial volumes artificial surfaces. Of course, the charge is real.

So, let us look at this. So, what I now do is this that I put in eight such cubes. I I put in I put in cubes like this I I so over this cube, I put in another cube and put two on the side and things like. Therefore, this cube the charge which is shown in red is now, at the center of a cube artificial cube, but this cube has twice the side of the original cube. Now, as we have just now seen that if a charge is at the center of a cube, the flux going through any one of phases is q by 6 epsilon 0. This expression did not depend upon the size of the side the the length of the side. So, whether it is a or it to a if the charge is at the center of a cube, the flux through any phase is q by 6 epsilon 0, but I am not interested in calculating flux out of the entire 2 a by 2 a phase. I am only interested in my original phase.

So, this original phase now, you notice is 1 fourth of one of the phases here and there is perfect symmetry there as well. So, you had q by 6 epsilon by 0 through the 2 by 2 a by 2 a phase through my original phase of the size a by a. It is 1 fourth of that namely q by 24 epsilon 0. Let me come to yet another example.

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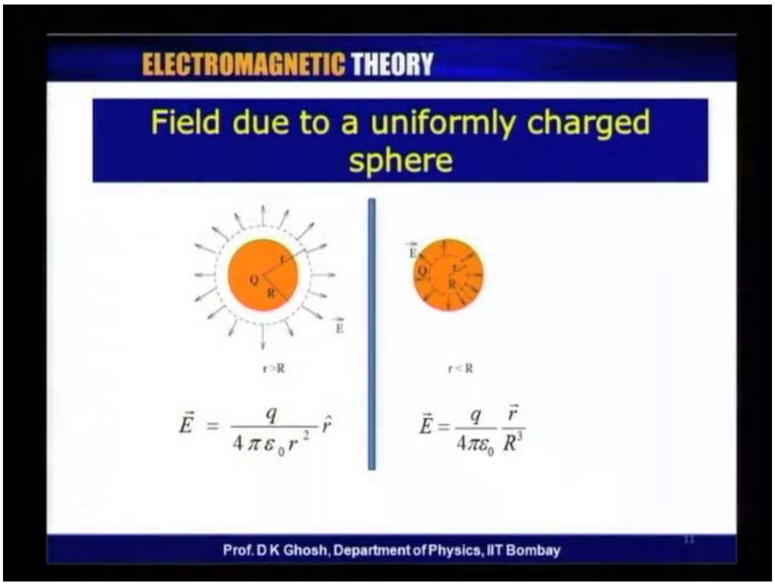

So, let me take a uniformly charged sphere. So, there is a uniformly charged sphere containing charge q. Now, if you are looking at so let me illustrate this.

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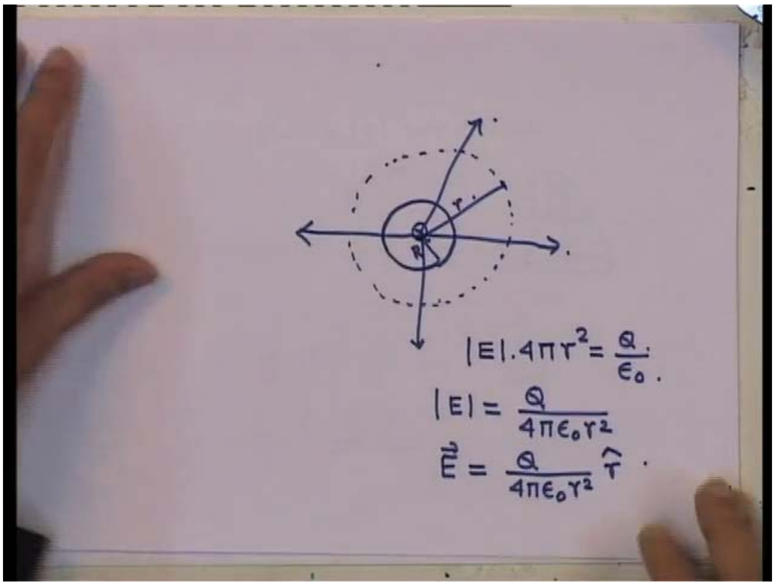

Here supposing, this is my sphere here containing a charge q and I am interested in finding out a field at a distance r from the center. Now, if this r is greater than the radius capital R of the sphere. What I do is, I enclose this in what I call as a Gaussian surface, which is a concentric sphere of radius R. The electric field of course, is outward from here. Now, notice that at a distance r, I am interested in calculating the electric field. The magnitude of the electric field is everywhere the same because it is sphere this is the Gaussians sphere imagine it is sphere.

So, that my flux is E times 4 pi r square. The direction of electric field is the radial direction which I will put in by hand at a later stage. That is equal to q divided by epsilon 0. Therefore, the magnitude of the electric field is q by 4 pi epsilon 0 r square and if you want to write down the electric field itself. Then of course, you put 2 by 4 pi epsilon 0 r square times. The unit vector r notice this expression is identical to what you would get if all the charges of the sphere, instead of being uniformly distributed were to concentrated at the origin of the sphere.

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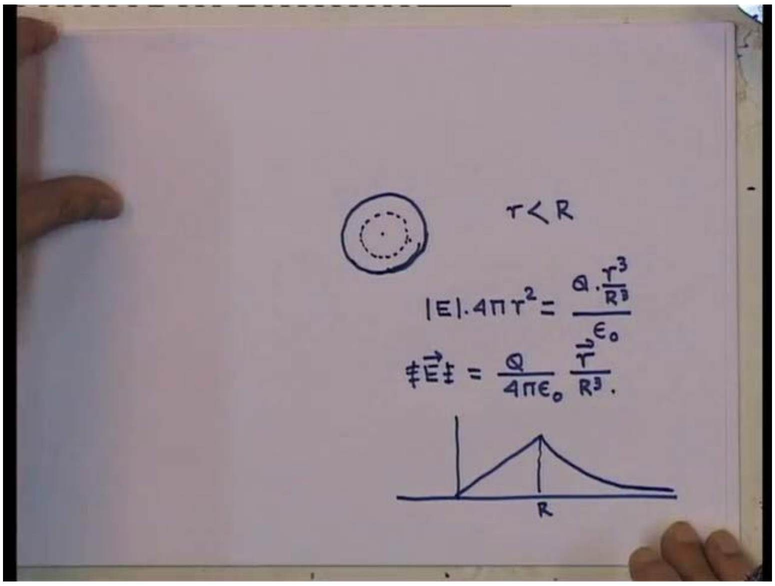

Now, suppose instead suppose instead we have this sphere, where the charge is uniformly distributed radius capital R, but I am interested in finding out the field at a distance r less than r. Now, once again I take a concentric Gaussian sphere of radius small r, the field directions are perpendicular to the surface of this imaginary sphere. So, the left hand side still calculates to be the same, namely E times 4 pi r square, but remember in our right hand side I had q enclosed the amount of charge, that is enclosed in this Gaussian sphere is not the full charge q, because the volume is less by an amount small r cube by capital R cube.

So, the amount of charge enclosed is small q capital Q times r cube by r cube. This is the q enclosed this divided by epsilon 0. So, electric field magnitude is q by 4 of 4 pi epsilon 0. Notice it is now linear r by capital R cube. The electric field direction is still the same, sorry this is so let me put this as a vector field. Therefore, what happens in this case is this, that up to the radius r it goes linear and then it falls off quadratically, following Coulombs law.

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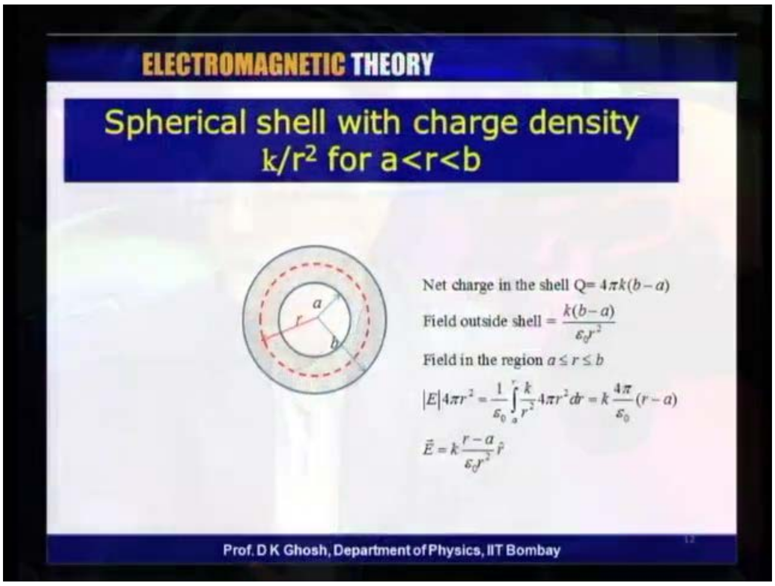

One more exercise let us look at a spherical shell spherical shell of inner radius a, and an outer radius b. Now, what is the field at three different distances. First if you look at totally outside the shell somewhere in this region my calculation is identical because the amount of charge that is contained is all that we have to calculate. I have given that this is a spherical shell with charge density k by r square for a less than r less than b.

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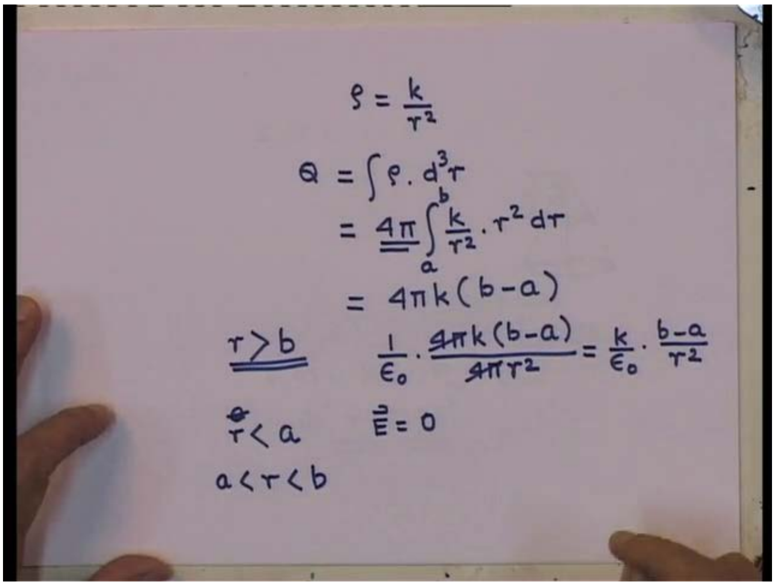

So, rho is some constant k by r square. So, how much is the total charge? The total charge then is integral rho d cube r. Now, notice this only depends upon r. Therefore, the theta and phi integration do not come into picture. That gives me 4 pi angle integration gives 4 pi and I have a rho, which is k by r square and d cube r, which is r square sin theta d theta d phi the sin theta d theta d phi have been taken care of already in my 4 pi. So, this times r square d r from a to b. So, this nothing but 4 pi k into b minus a that is my charge. So, if I am looking at a radius r greater than b, the directions are all radial. So, the magnitude of the electric field will be 1 over epsilon 0.

The amount of charge enclosed namely the whole charge 4 pi k b minus a divided by 4 pi r square that came from that E dot d S. So, this also goes as the 4 pi will cancel out and I, you will be left with this expression k by epsilon 0 into b minus a by r square. So, once again it is as if the entire amount of charge is concentrated at the origin. Now, let us look at the region. First let me look at the region r less than a. Now, if r is less than a. I take a Gaussian sphere of radius r and inside that there are no charge, because it is within the hollow region of the sphere. So, my electric field will be equal to 0. No no field in the hollow space, now what about a less than r less than b. So, if a is less than r less than b, the calculation is identical. Accepting now, you have to worry about what fraction of charge is included there in.

So, this is the amount of charge that is contained between this sphere of radius small a and this Gaussian surface marked with red. I can calculate how much is the volume of this, simply the difference between the volume of the radius r sphere of radius r and the volume of sphere of radius a. Repeat the same calculation we will find the electric field is given by k times r minus a divided by epsilon 0 r square times r. So, this is this is something which we this is an application of the spherical shape. Now, let me take and now, we get another example. Let me take the example of a charged sheet, an infinite sheet containing charge, which I will take to be the positive charge.

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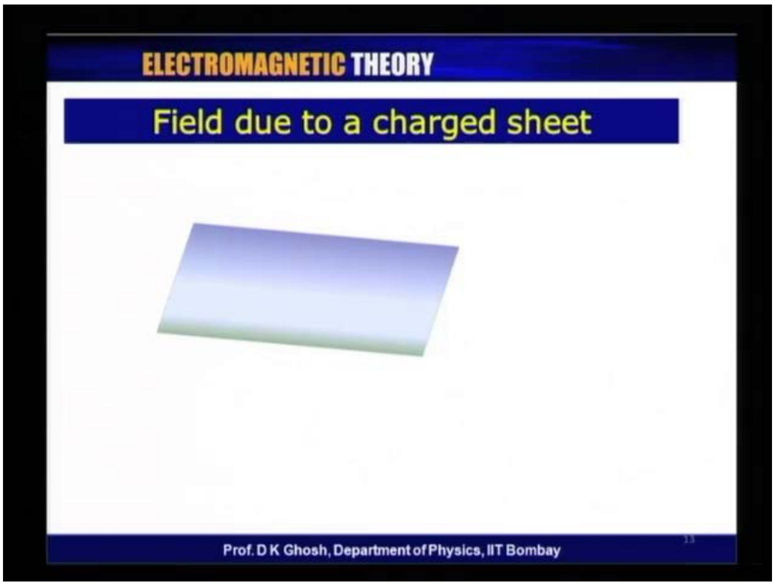

So, this is a sheet which is positively charged. I am interested in calculating how much, what is the field? First let us look at, what we, what statements we can make without actually working out anything? It is an infinite sheet, therefore the field can only depend upon the distance from the sheet. So, field depends upon the distance from the sheet only. The other thing is if this charge is taken to be positive, then in the upper half of the plane on the above that plane, the direction of the electric field will be outward perpendicular to the field below the thing. It will be in the reverse direction because the direction of what is outward changes.

If it is above it is upward, if it is below it is downwards. So, the electric field is oppositely directed. We will be discussing about it later, but you notice that there is obviously a discontinuity of the electric field at the charged surface.

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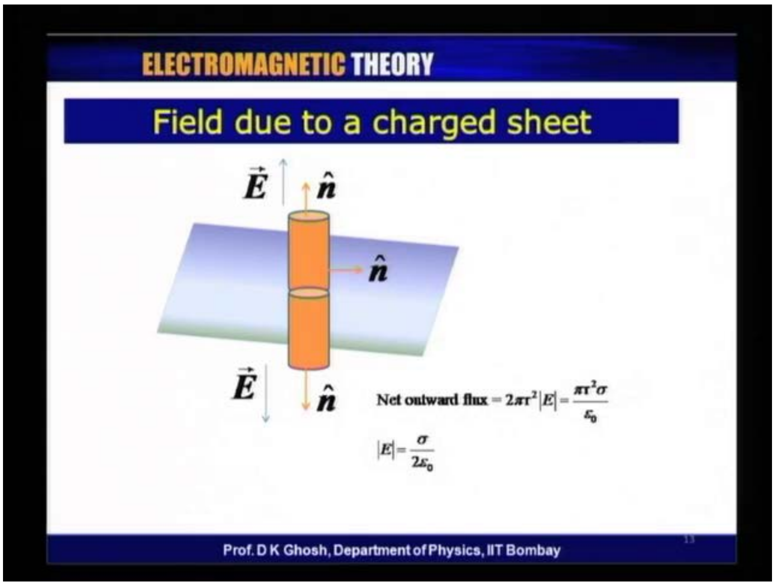

Now, what I do is this supposing I am interested in calculating the electric field at a distance r, what I do is I take a cylinder, I take a cylinder. This is a Gaussian cylinder and let it intersect let it intersect the sheet. Now, look at the various directions. The electric field on this phase here on the top phase, is upward on the bottom phase, is downward on the side phase, is it is like this perpendicular to the this. Now, so when I calculate E dot d s there is no contribution from the slanted phase.

Since, the direction of n and the direction of E are parallel on the top and the bottom phase, the net outward flux that I get there are two two phases pi r square each. So, I get 2 pi r square times E and that is equal to the charge enclosed, which is pi r square times sigma. Sigma is the surface charge density by epsilon 0. So, that the magnitude of the electric field is given by sigma by 2 epsilon 0.

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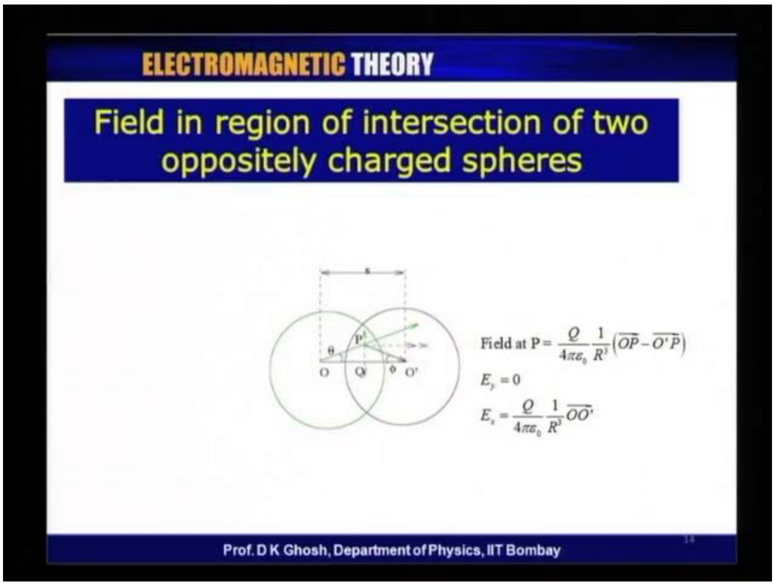

We will continue with more applications of Gauss’s law, and introduce the concept of potential in our next lecture.