Electromagnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Lecture -4
Conservative Field Stoke's Theorem

In the last lecture we had discussed about the curl of a vector field and we had seen that while the divergence of a vector field is a scalar field, the curl of a vector field is a vector itself. Curl of a vector field gives me a vector field. So, in today's lecture we will continue with discussion of curl and if you recall we have talked about Stoke's theorem which related the line integral of a vector field with the surface integral of the curl of the vector after having done, that we will introduce you to what is known as a Laplacian operator and Laplace's equation.

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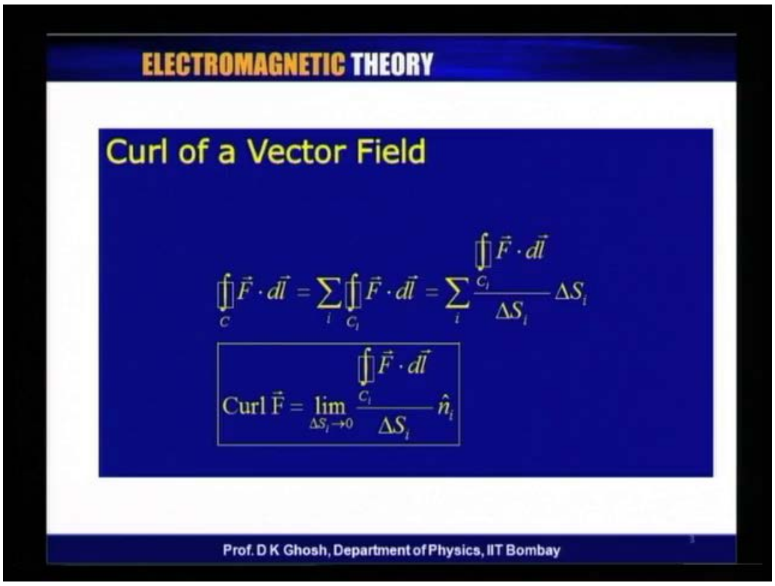

Just to recap what we did last time we had seen that supposing we take an open surface. By open surface I mean the boundary is open. So, I had given a picture like this.

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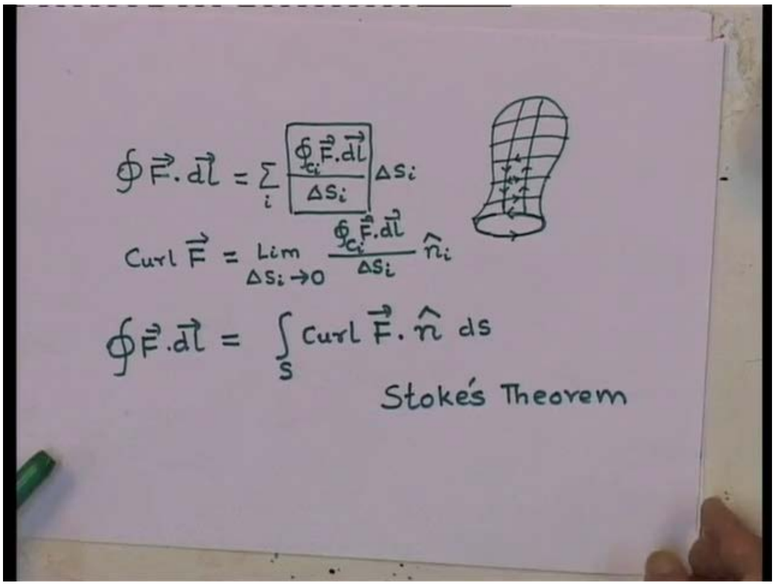

What we said is that this entire surface if I split it up into small elemental surfaces and describe a line integral in the same sense in every such elementary contour. Then what happens is, that all the the contribution to the integral from every line. That is on the boundary between to contours they cancel. Finally, we are left with only the contribution from the curve which defines the boundary of such a surface. Now, with that we had shown, that the integral of the line integral of a vector field F dot d L this quantity can be written as some over I, that is some over each such elemental contours and what I have done is to say the take the contour integral over each such contour. Now, let me divide it by the area of the i th cell and multiply it with the area of the i th cell.

Now, this quantity which we have this quantity, which I have here is a point relationship that is because I am going to let the area enclosed by each one of these contours be as small as possible. We had seen once you know, what is the direction, you know what is the direction of the contour. Then they by usual right hand rule, I define what is the direction of the outward normal to the surface on that contour and that is, what I will call as the direction n i.

The curl of the field at that point, at that point which is on the i th cell will be the limit of delta S i going to 0 of this ratio. Integral over the contour C i F dot d L by delta S I, but curl is a vector. So, I need to define a direction and the direction is the direction n i of the surface element, which is to be defined by the right hand rule. This is what we talked about last time. Now, with this definition you can easily see that, what I have done is to write down the total integral over the, this, they contour which defines the surface the bounding contour of the surface as equal to as equal to the integral over the entire surface, which is obtained by adding them over. Curl of F dotted with n which is the direction of the normal vector at a point times the surface element. So this is, this is what we said is known as the Stokes theorem.

So, what does the stokes theorem do? The stokes theorem connects these surface integral over an an open surface, with the line integral over the contour which defines such a surface. Now, obviously you realize that with a given contour, with a given curve I can have different surfaces. Just just to give you an idea of what I mean by that. Let us check just a circular contour, this circular contour is in the plane if the paper. Let us take that as an x y plane.

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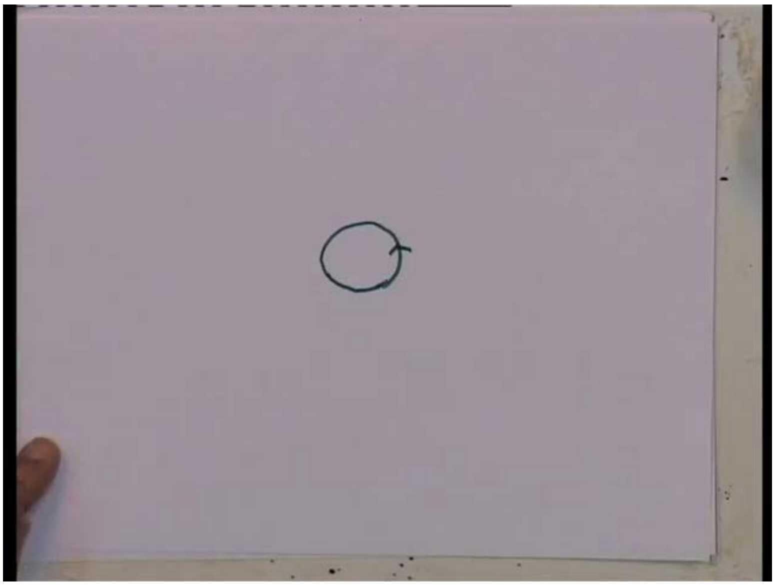

Now, this contour can define this contour can define a surface which is a disk, which is enclosed by this. Now, the other possibility for example, is to make a a cylinder open cylinder on which whose boundary will be given by such a contour. So, while the contour is defined by us, the surface is left to us to chose. This is this is exactly what we do, while using the Stoke’s theorem. The idea is this that when you are required to evaluate a contour integral over a given contour, instead of evaluating the line integral directly which may turn turn out to be difficult sometimes, you can convert this to a surface integral over the curl of the function. What is the advantage, because this surface you should choose in such a way, that the surface integral becomes easy to calculate. We will be giving some examples of what we mean by that.

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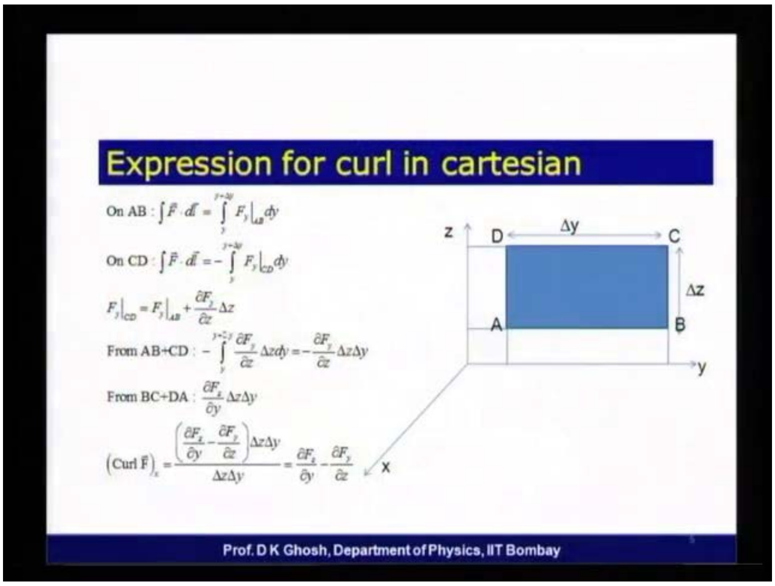

Before we do that let us get an expression for the curl, which we had earlier defined has the ratio of a line integral to a, the surface enclosed by the contour. Let us get an expression which we can easily manipulate. Namely we will try to get an expression for the curl of a vector field in a Cartesian coordinate to begin with. So, what I have done is this that, I have taken a in the y z plane a small rectangle. The this rectangle is of width delta y, this is delta y by delta z. Delta y is the along the y direction and delta z along the z direction.

Now, obviously if I am going on this contour along A B C D that is the anti-clock wise fashion the direction of the normal, the outward normal will be along the x direction which is perpendicular to the plane of the screen. Now, let us get an expression. This is small delta y and delta z are infinitesimally small. Now, first let us compute the contribution of the line integral from the from A to B. Well notice that A to B is along the y direction. Therefore, the direction of the vector d l is along the J direction and is given by J d y. So, when I take the line integral F dot d l, I get F y. So, let me repeat that here so, this is my y direction, z direction and of course, the x direction is out of the plane of the paper.

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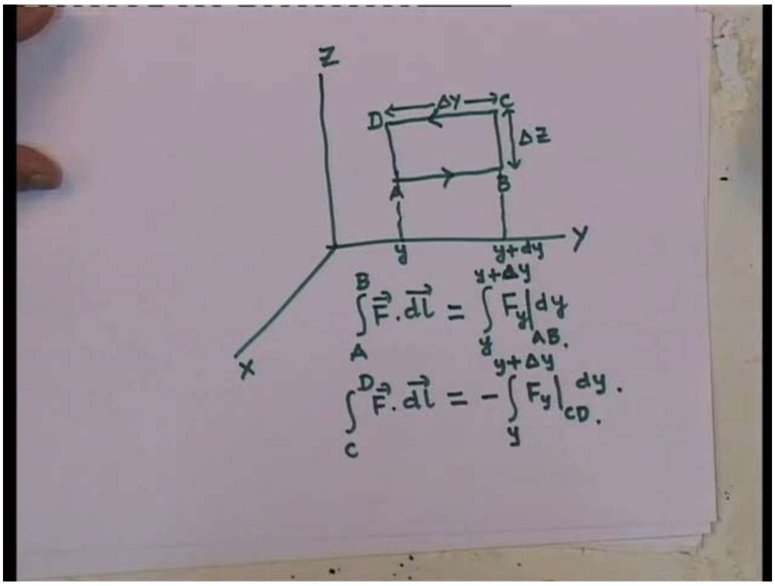

I have taken a small rectangle, this rectangle is of is delta y by delta z and this is A B C D. So, the contribution to the line integral F dot d l along the path A to B along the path A to B from A to B is given by F y d y and this will be form the value. Supposing the A as a coordinate y and this as a coordinate y plus d y. So, this will be from y to y plus to let delta y lets say. So, F y d y. Now, since my this element of the rectangle is rather small, I can safely assume that F y F y does not vary much does not vary much from A to B. So, the value of. So, what I am now doing is this, that let us then add with this. The contribution from C D. So, when I add the contribution from C D again F dot d l, you have to realize that this contour is traversed in the opposite direction. Hence a minus sign.

So, this will be from C to D. Now, C to D is simply means that I have already taken care of the minus sign and so, it is y to y plus d y and again F y, but this time evaluated on C D d y. How do I get the contribution contribution from A B and C D taken together. Now, notice that A B and C D they differ in in their value of z. So, on A D, I have a one particular value of z. On C D I have the z different by an amount delta z. Therefore, what I can do is to write down that, I can write down F y along C D is same as F y along A B plus the variation of F y plus the variation of F y with z, when the z coordinates increases by delta z.

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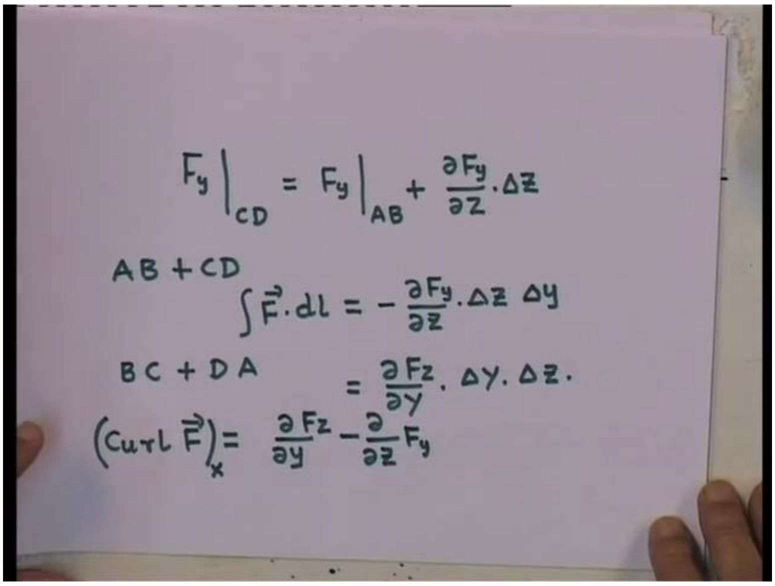

So, the contribution from A B and C D put together, the contribution to integral F dot d l will be given by because the C D is traversed in wrong direction, in the reverse direction. It will be given minus delta F y by delta z. delta z This multiplied by of course, delta y because I have to do an integration. This is the difference in the value F y, but this is to be integrated over y from y to y plus delta y. Therefore, it is delta z delta y. Now, I can I can repeat a similar calculation for B C and D A and notice the difference here is that B C and D A differ by their y coordinate.

Similarly, I will get the contribution from B C plus D A, this will be given by delta F z by delta y into delta y delta z. So therefore, the curl of F the curl of F which if you recall is the integral F dot d l over this contour divided by the area enclosed which is delta y delta z is given by. Now, I have to worry about I am calculating curl F's x component, that is because the direction of this area is along the x direction.

So, this is given by, delta by delta y of F z minus delta by delta z of F y. So, this is the Cartesian expression for the x component of the curl of the vector. Now, I can essentially extend the same argument to the other components and you notice, the way it is done that I am saying curl which as I told you.

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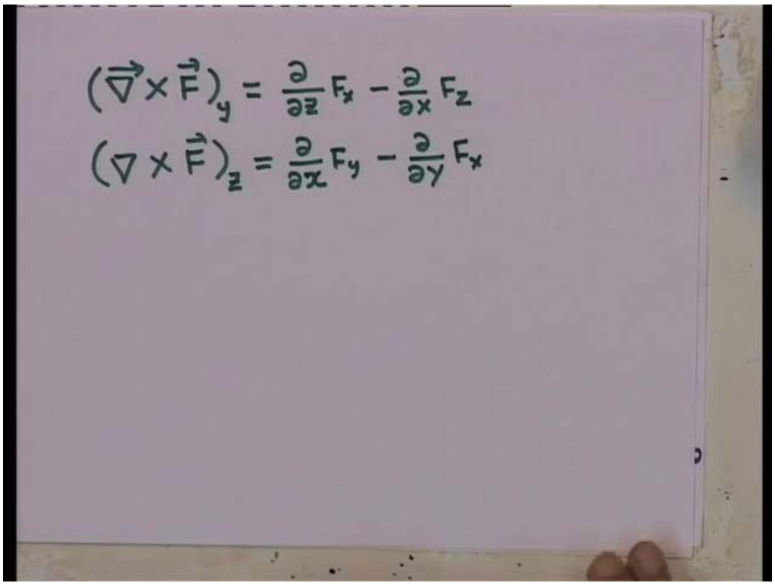

That can also be written down this del cross F. Supposing I want to write down the y component, the way to notice is that d by if it is a y component, first we get d by d z of F x. Notice that this index y does not appear on that side and minus d by d x of z component. Similarly, del cross F z component we have to go cyclically x y z x like that therefore, first d by d x y component minus d by d y of x component. So, this is the expression for curl of F.

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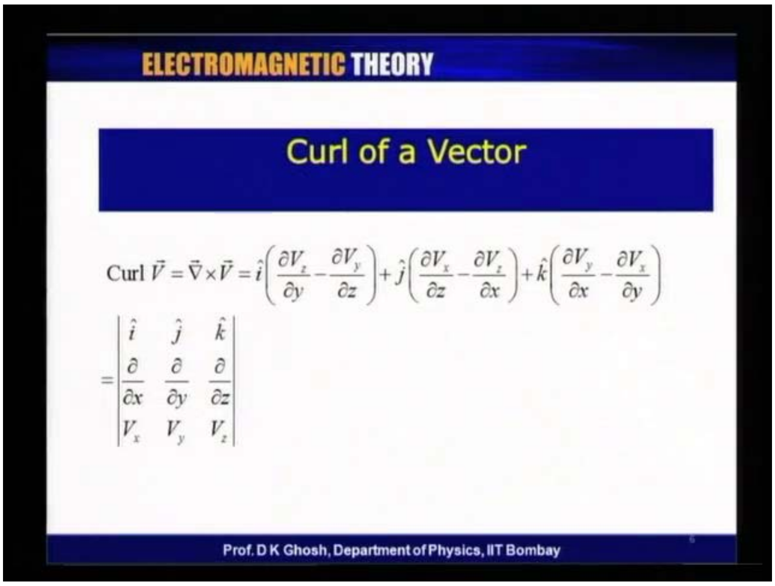

Now, you can write down like you write down A cross B in a determinantal form. I can write down this as I, j, k, d by d x, d by d y, d by d z and of course, F x, F y, F z.

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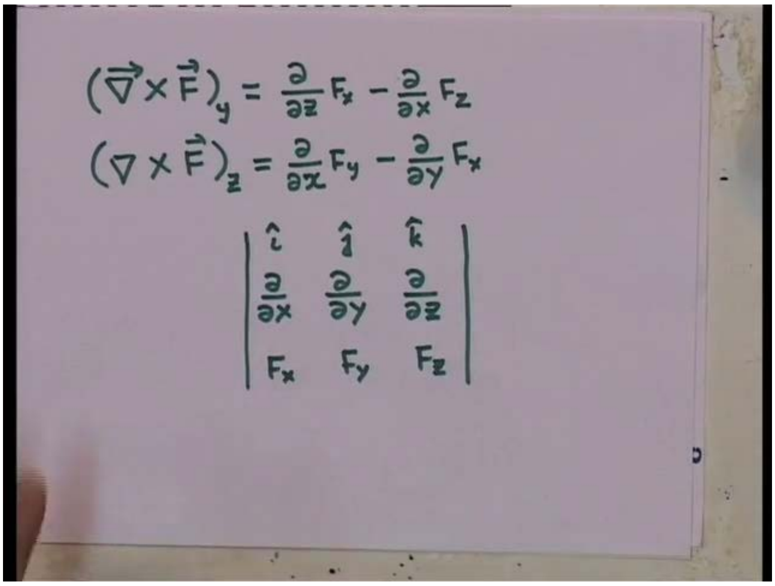

You have to, have a bit of a discipline in interpreting this, because this is not important when you normally evaluate a determinant, but in this case it is for example, if you want to write down the x component it is i times. d by d y of F z minus d by d z of F y. So, this determinant comes like this now, then minus j. Now, when you expand this coefficient of minus j, you have sort of realize that I have to go to the immediate right d by d z of F x minus d by d x of F z. So, this this is something which is to be, this is the notation.

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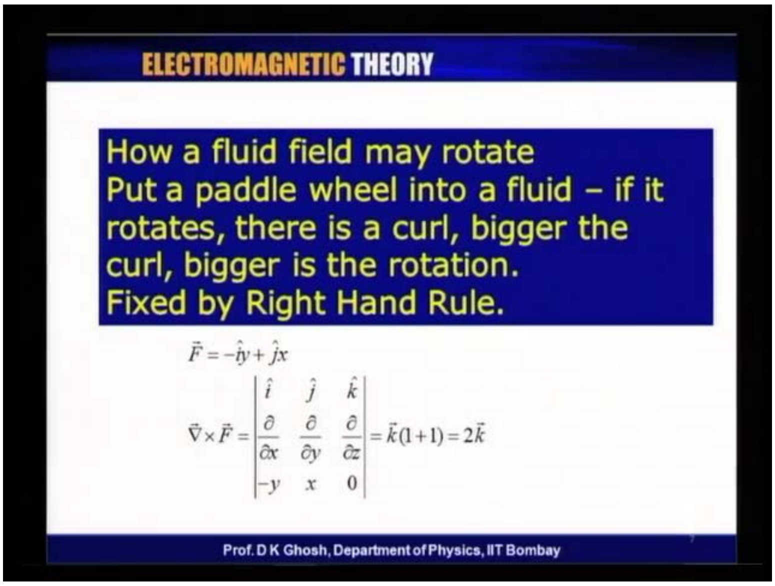

So, let us then little bit worry about this name that so, why it is called a curl. As you realize that curl is as the name suggests something is curling or rotating. Now, what you do is this, that as I told you last time, many of this nomenclatures come from their first applications in fluid dynamics. So, suppose you have a fluid which is moving. Now, if you put a paddle wheel, I will show you a paddle wheel on it or for instance if you do not understand, what is a paddle wheel, just think in terms of a door, which is on a hinge. Now, supposing I have to put in a door with a hinge inside water Now, if the so…

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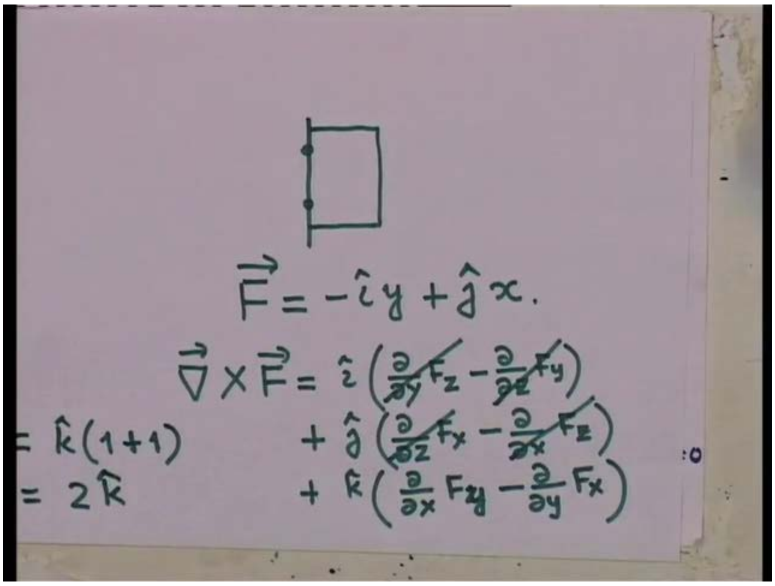

Let me let me try to figure it out. Supposing this is a hinge, I have a door like this. Now, notice if this door is in the plane of the paper and if I have a component of a force that is acting on the door, which is now, this door is attached to the hinge. So, which is such that a a force acts on the door surface, then this will make this door rotate or swing. On the other hand if the liquid is flowing past on this plane then of course, there will be no rotating effect. The first one is the curling effect. So, if a paddle wheel is very similar we will just see a picture of paddle wheel little later.

So, if a thing rotates in a velocity field, it means that there is a curl which is non zero that is a curl which is not zero. Bigger the curl is more is the rotation and the direction of the curl the direction of the curl of a vector is fixed by the right hand rule as we have explained earlier. So, notice I am taking you back to a force field which we have discussed several times. Supposing, my force field is given by minus i y plus j x. Now, you can easily compute the curl of this vector. Remember how it is done i times d by d y of F z minus d by d z of F y plus j times d by d z of F x minus d by d x of F z plus k times d by d x of F y minus d by d y of F x.

So, let us compute this. Notice there is no F z in this expression it is a only i and j. Therefore, these will not be there. There is no dependence on z in this function. So, anything connected with z derivative with respect to z goes away that leaves us with only the z component of the curl d by d x of F y d by d x of F y is F y is j x. So, that is actually 1 minus d by d y of F x which is minus y. So, that gives me another minus 1, but this minus and that minus so, I get 1. So, i get 2 k. So, notice that this field, the force field is in x y plane, but the curl we have calculated is along the z direction.

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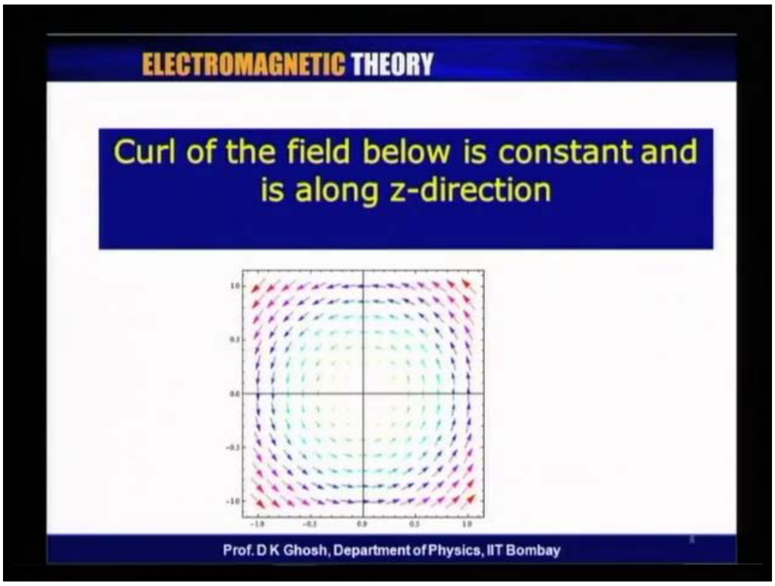

This is a picture which is familiar to you. That, this is actually a mathematic a plot of this force field that I showed to you some time back. This is the way the force field looks like, it has a constant curl which is along the z direction.

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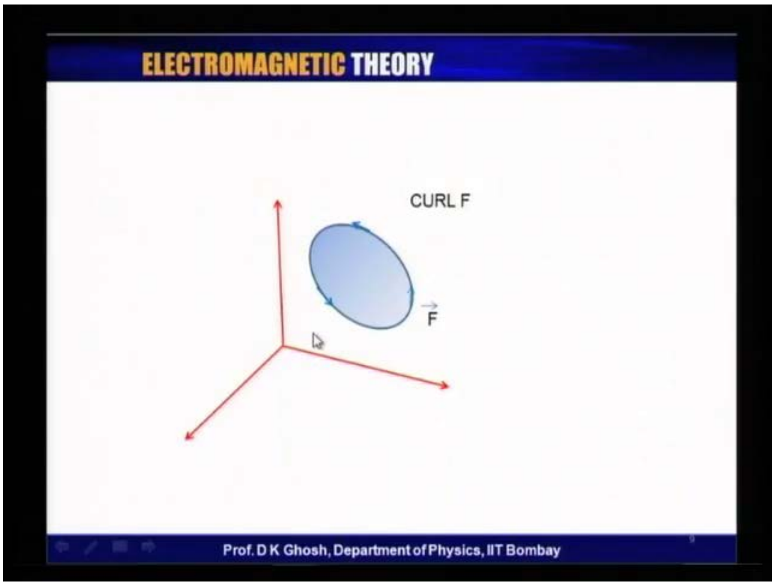

So, these are just other pictures which is sort of telling you, how to use right hand rule that so, this is in this case the direction of the force field is like this. Using a right hand rule you will find out that the direction is perpendicular to the screen still.

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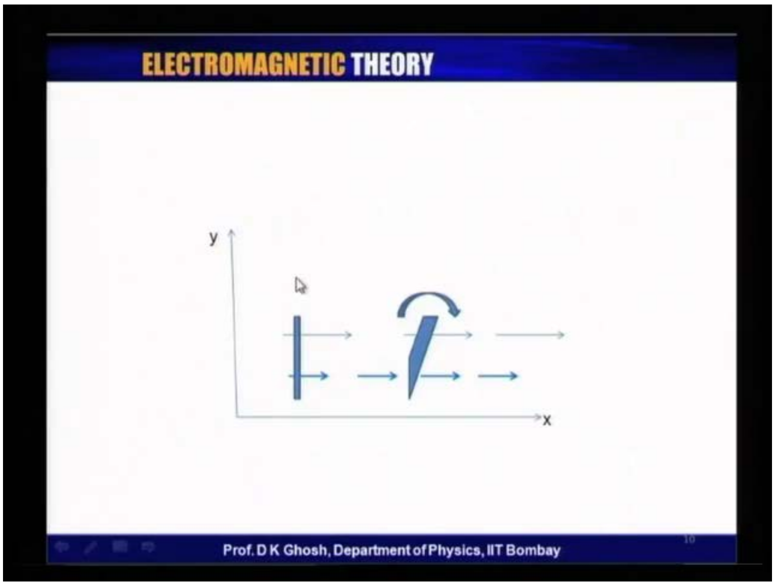

On the same topic for instance suppose, I have a fluid flow in this direction, but let us suppose the velocity of the fluid here is higher than the velocity of fluid there. Supposing, I have put a plank on such a situation. So, if I put a plank the lower part of this plank experiences a smaller force, because of the velocity the force of the fluid, than the upper part. Therefore, the upper part will be tilted much more than the lower part and so this will start rotating, this will start rotating.

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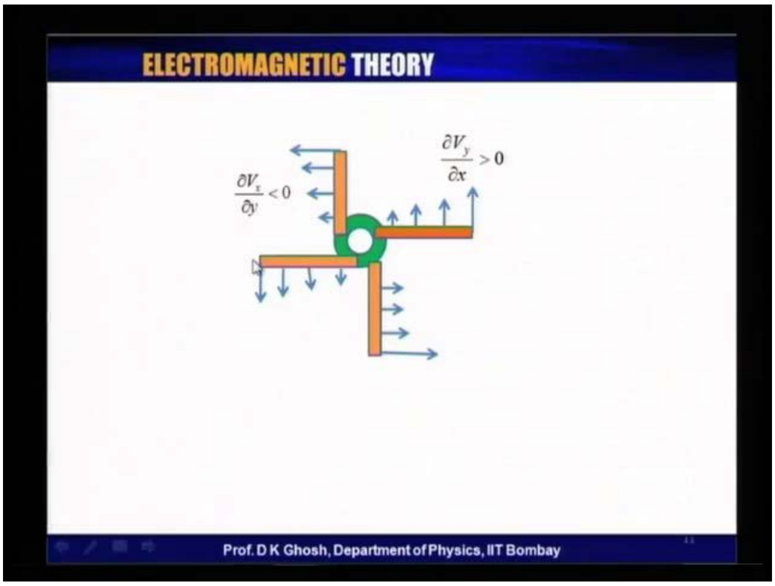

This is also, this is what is meant by curl this is a paddle wheel. This is a paddle wheel and these are the force field directions. So, you notice that in this on this arm of the paddle wheel. So, it will rotate paddle wheel rotates about this axel on this arm of the paddle wheel, what we have found is that the field is along the y direction, but as x increases the field increases. In other words d V y velocity field by d x is greater than 0. On the other hand here the velocity field is along x direction, but is d V x by d y is less than 0. So, therefore such a paddle wheel in force field shown here, will rotate in the anti clock wise fashion.

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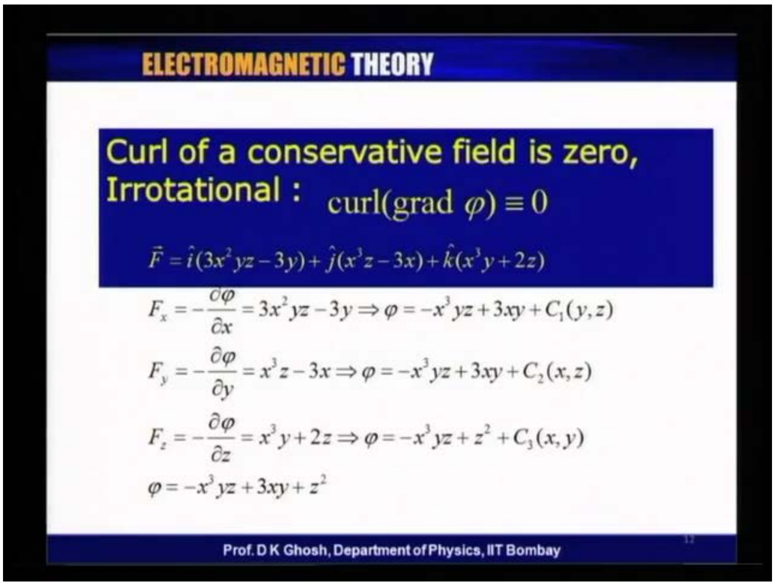

Now, one of the things that you can easily calculate. Now, remember that we said curl of a vector is a vector. One of the vectors, which we have been familiar earlier is to take a scalar field and compute its gradient. So, supposing I have a scalar field phi the gradient of the scalar field is a vector field.

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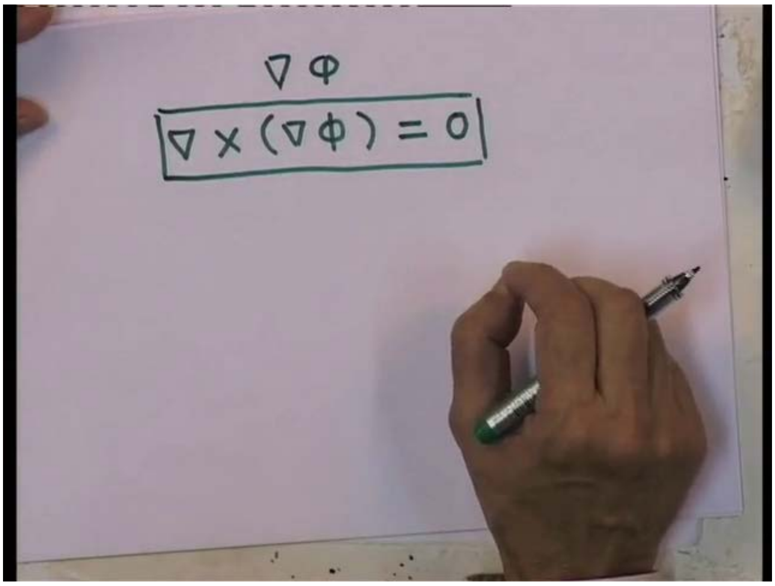

Now, you know how to compute its components in the Cartesian coordinate system, do that. Calculate the curl of such a gradient, you will find that it identically vanishes. Curl of a gradient is 0. Now, this relation is used as a test of a conservative field. We have seen that in case of a conservative force field, we can think of a potential file whose gradient actually sometimes you put a negative sign in front of the gradient whose gradient gives me the force. So, if in this equation I interpret phi as a potential function a scalar function its gradient give or take a minus sign gives me the force field which is a vector field.

Now, since the curl of such a force field is 0. Calculating the curl of a force field if you that the result is 0, you can conclude that such a force field is a conservative field like electrostatic field the gravitational field etc. Now, here is a little more tricky example or little as an exercise I have taken. The exercise is absolutely straight forward, this is given a force field. Given a force field, I am computing this is known to me to be a conservative field. Now, how does one goes back and calculate the scalar function phi? So, you notice here that I have a force F. I know its x component y component and z component. So, F x is minus d phi by d x. I have taken F to be equal to minus grad phi, is a partial derivative so d by d x of minus d phi by d x.

So, that gives me calculating so, d phi by d x is 3 x square y z minus 3 y. So, integrate that because the integration with only with respect to x. So, I get minus x cube y z plus 3 x y. Now, I know there is a plus a integration constant, but since the integration is with respect to x, the constant of integration can in principle depend upon y and z. Repeat the same thing for the y component and you get phi equal to minus x cube y z plus 3 x y and a second constant now, which since the differentiation or integration is respect to y depends upon x z. Likewise F z is given by this. Now, notice these are the three equations which I have. I can immediately get an expression for the phi because I have x cube y z minus x cube y z in each of the expression.

So, I keep it, I have a 3 x y here and a 3 x y there. There is no 3 x y, but I knew that there could be a constant depending upon x y. So, I identify that constant C 3 as 3 x y and keep it here in first and second expressions. C 1 and C 2 could depend upon y z and x z here. So, therefore since z square is not appearing in either of them, I identify z square to be equal to C 1 and C 2 and this other than for an overall constant is my potential function. If you compute the curl so, the gradient is given if you compute the curl of this function you are guaranteed to get 0 because I have been able to find a potential function corresponding to this force field. So, this field is a conservative field. So, we had said earlier that if a divergence of a vector field is 0 it is called a solenoidal solenoidal field. If curl of a vector field is 0, it is a known as irrotational. The name is obvious because we have identified curl with rotation.

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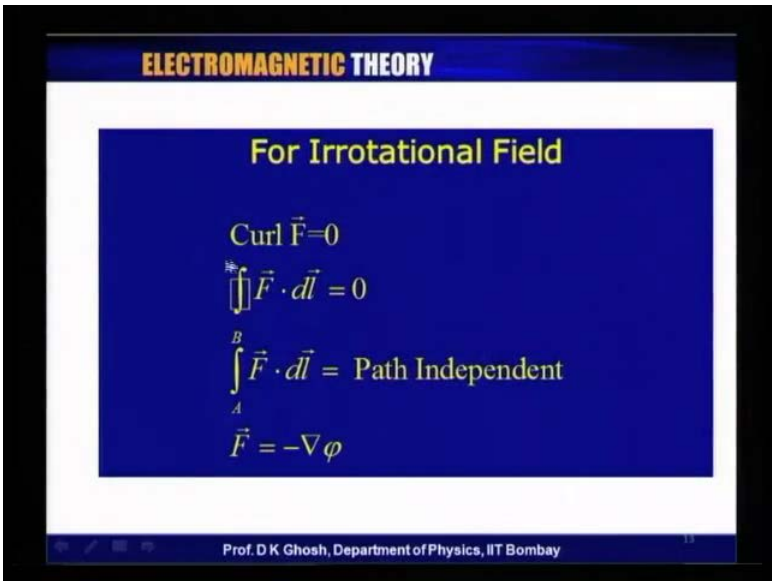

So, identically equivalent statements to curl f being equal to 0. Now, if curl f is 0, it means integral of curl F dot d s is 0, which means line integral of the force F dot d l over a closed contour is 0. This also implies that, if you take the line integral from A to B of F dot d l, that turns out to depend only on the points A and B that is it is path independent. Equivalent statement is that the force field can be written as minus or just plus, gradient of a scalar function. So, all these are equivalent statements.

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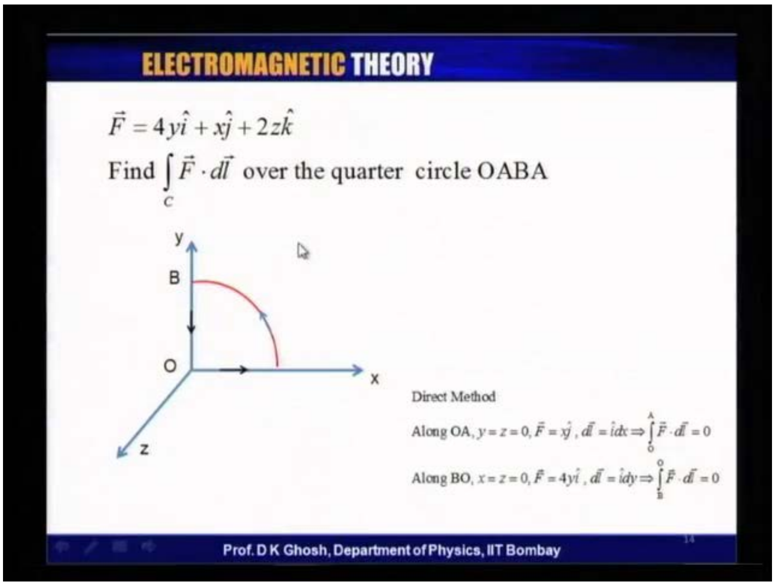

Well let us go ahead and look at a few applications of the Stoke's theorem. So, I will give you, start with some examples. So, consider a force field consider a force field given by 4 y i plus x z plus 2 z k and I am interested in calculating computing.

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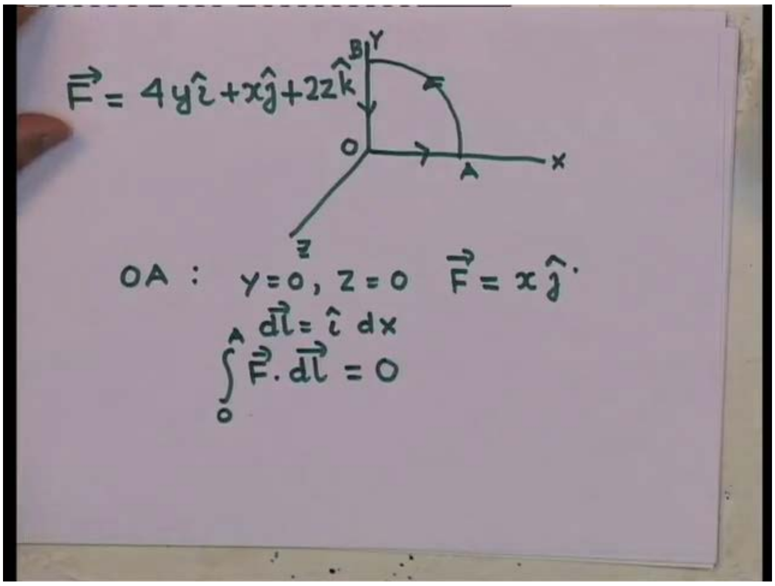

I have slightly changed the directions of the axes so, this my x y the paper plane is the x y plane and the z direction comes out of the paper. So, my force field is 4 y i plus x j plus 2 z k. Now, I am going to ask compute the line integral from O to let us say, A, A to B and back to O over this close contour. Now, I can do it directly, it is not a very difficult problem to be handled directly. So, what I do is like the way we have given other examples earlier. So, let us take along O A. Now, along O A notice y is equal to 0 and z is equal to 0, that gives me the force field to be very simple and simply I left with x along the y direction.

However, when you go from O to A you are going along x direction. So, your differential element is d x. So, this would be so, this is differential elements is i d x, that is why d l. So, integral O to A F dot d l is 0 because i dot j will come in. Now, I can repeat the argument for B to O and here also you notice that, along B to O my x is 0 z is 0. So, I will be left with only a force field along x direction, but the differential element is along the the d l vector is along the y direction. So, again contribution from B to O is 0.

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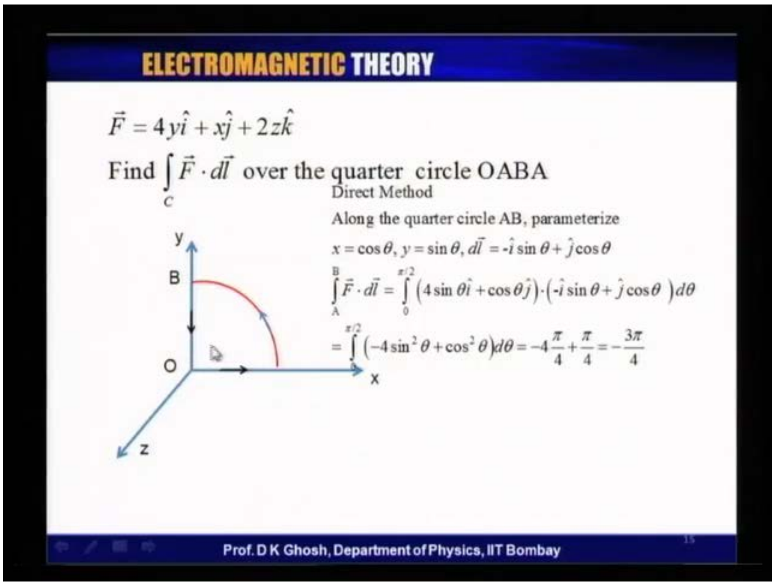

So, the only thing that I have to actually worry about is to compute the integral line integral over the circle.

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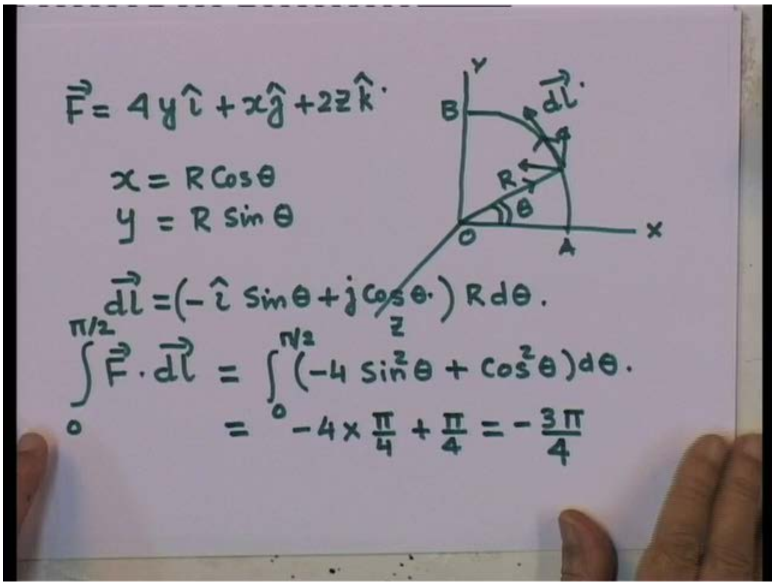

Let us take it a unit circle. So, this is O A B. Now, how do I calculate the line integral of F, which is 4 y i plus x j plus 2 z k. Well Cartesian coordinate is not a very comfortable co ordinate in this case, but since, this curve lies entirely in the x y plane I could parameterize it, using a polar coordinates system. You are all familiar with polar coordinates, remember in polar coordinates y x. So, in polar coordinates I take this is theta and is my radius R. So, x is equal to R cos theta y is equal to R sin theta and d L which is along this tangent. This is the direction of the tangent so, the d L vector unit vector along that is you can see it the d L's has a minus x component and a plus y component.

You can check it as equal to minus i sin theta plus j cos theta and the length element which is of course, I have to give the magnitude which is RA d theta. Now, what you can do now is to directly compute F dot d L from A to B the angle theta varies from 0 to pi by 2. I can write down these things in terms of the i j etc. So, you notice F dot d L, I get integral. I have 4 y 4 y So, when I take y is R sin theta, when I take the dot product i with I get sin theta 1 again, once again so, I get sin square theta. So, there is a 4, but there is a minus sign there. So, I get minus 4 sin square theta. This curve is entirely in the x y plane. So, z is equal to 0 and does not contribute to the integral.

Then I am left with x j. x is R cos theta. This already has a cos theta therefore, I add cos square theta d theta. You can compute this from 0 to pi by 2 you get minus 4 zero to pi by 2 of sin square theta. You compute you get a pi by 4 there and another pi by 4 here, which is equal to minus 3 pi by 4. Not, a very difficult way to do things, but you know takes a bit time.

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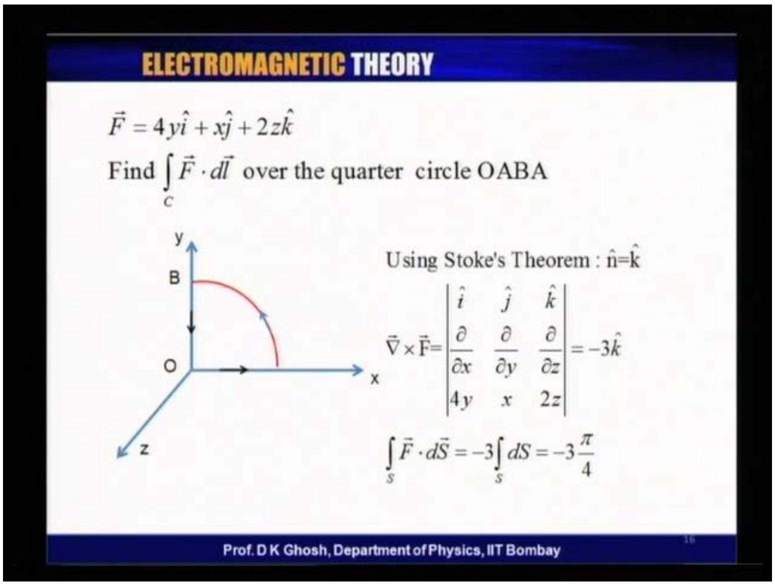

Let us look at it in a slightly different way. I want to apply a Stoke's theorem. So, I have this function F. The curl you can easily calculate and you can see that the direction is right because this curve is in the x y plane and the curl has to be along. So, del cross F is what I am calculating curl of this vector happens to be minus 3 k and the direction of d s the direction of d s is also along the positive k direction.

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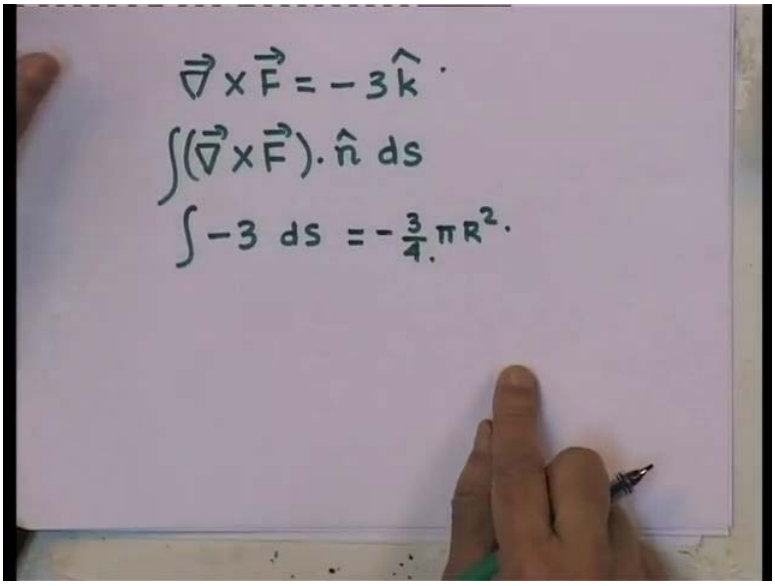

So. since curl of F computed directly is minus 3 k integral of curl of F dotted with n d s. So, n is along the curl direction so, this is nothing but minus 3 k dot k times d s and I know that since it is just a quarter circle the amount of area that is enclosed within this is just pi R square by 4. So, it is minus 3 by 4 pi R square which is exactly the result that I got earlier. I have taken R is equal to 1. Therefore, the R square; I did not write there explicitly. I could use the Stoke's theorem. Here is a another example where we use the same contour to define to different surfaces.

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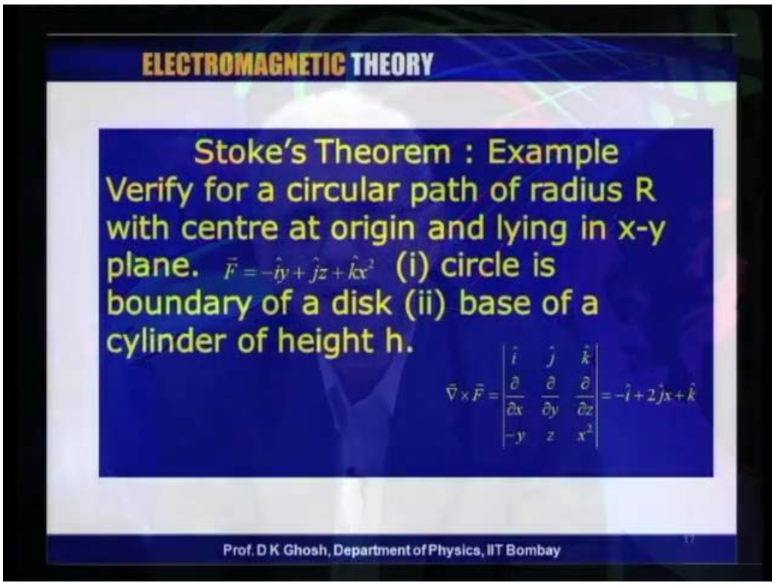

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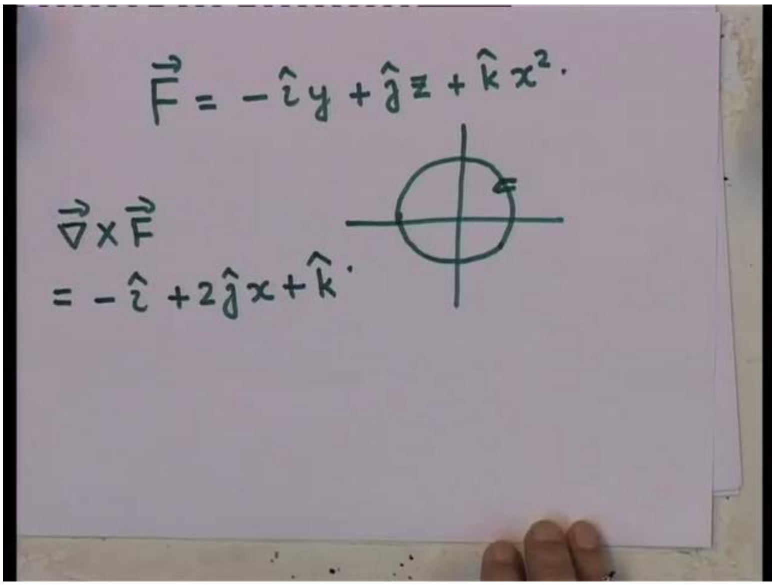

For example, I have taken a rather simple field F is equal to minus i y plus j z plus k x square and I wanted to, want to calculate F dot d L over a circle in the x y plane in the anti clockwise fashion. Now, I will do it in two ways. First let us compute what is my integral m dot d. Now, I could that by the same technique that I have given you last time. However, let us calculate curl of this function. Since I have an explicit expression for it you can check that the curl of this expression happens to be minus i plus 2 j x plus k. Now, what I am doing is thism that the the calculation is shown here.

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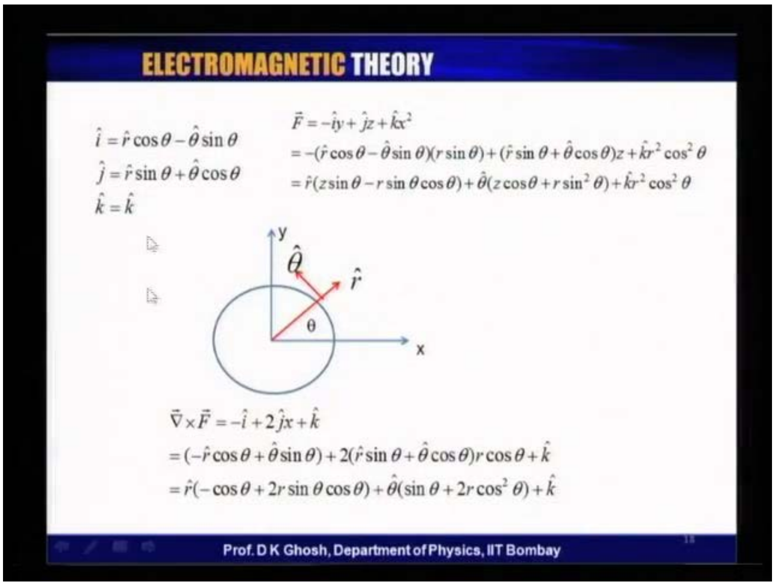

Now, I calculate this in the same parameterized form that I talked about earlier. This just a little bit of involved algebra, but there is nothing in it. I write down what is i in terms of the radial direction unit vector theta and the tangential direction unit vector theta sorry radial direction unit vector R and the tangential direction unit vector theta. The k direction remains the same this incidentally is known as the cylindrical coordinate system, about which we will be talking about in our next lecture. So, F you write down F and del cross F using this.

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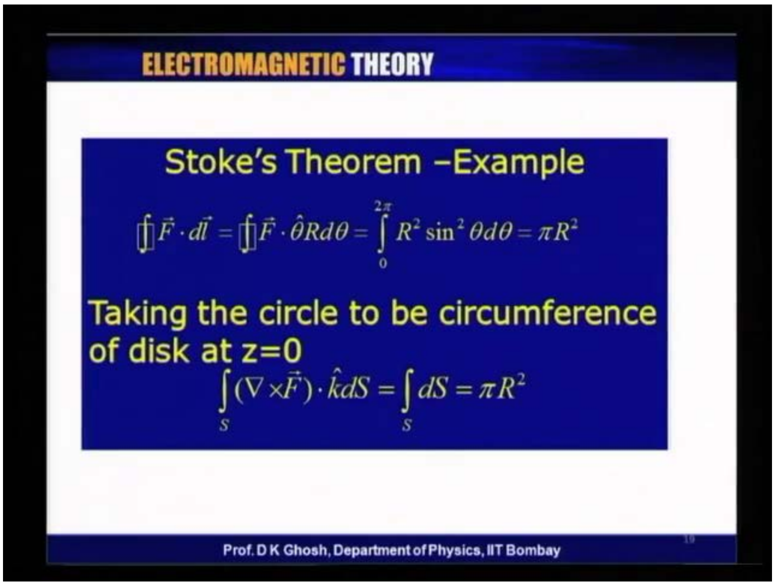

Now, let us look at the how does one calculate F dot d L using the Stoke's theorem. Now, first is that using these expression that I showed you you could directly calculate and find this F dot d L do it as an exercise at home to be pi R square. This is a as you have seen it is a lot of calculation, but if I use Stoke's theorem, I need only the curl del cross F and I need its z component only, because the contour over which I am doing the integration is in the x y plane. So, the area enclosed by it is directed along the z direction. Now, del cross F has a rather simple z component which is just 1.

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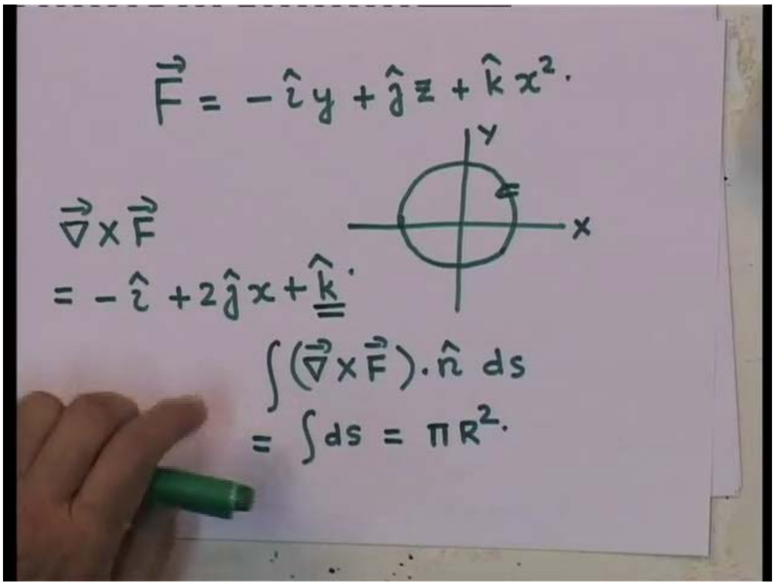

So, if I take my circle in the x y plane, my curl of F dot n d s is simply equal to k dot k, which is equal to 1 times d s and this is integral is a surface integral of a circle. So, this is pi R square.

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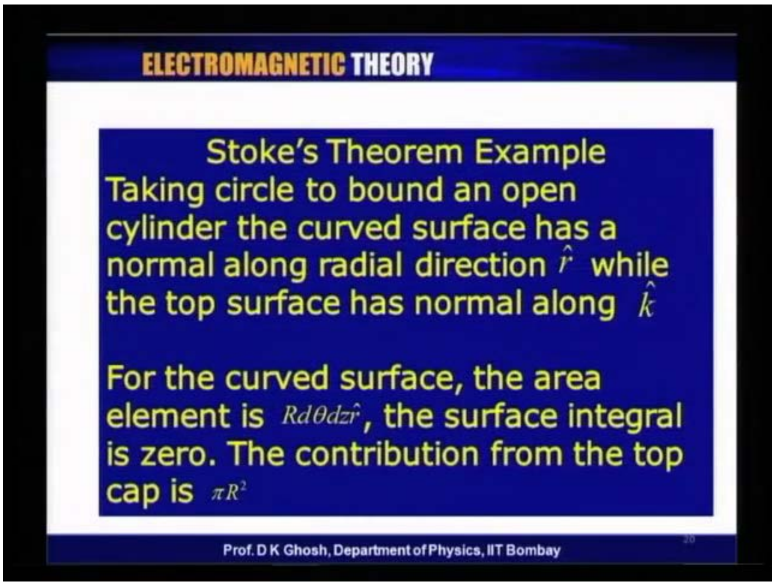

That is rather trivial, but take a bit of a time and try to repeat this exercise by taking that.

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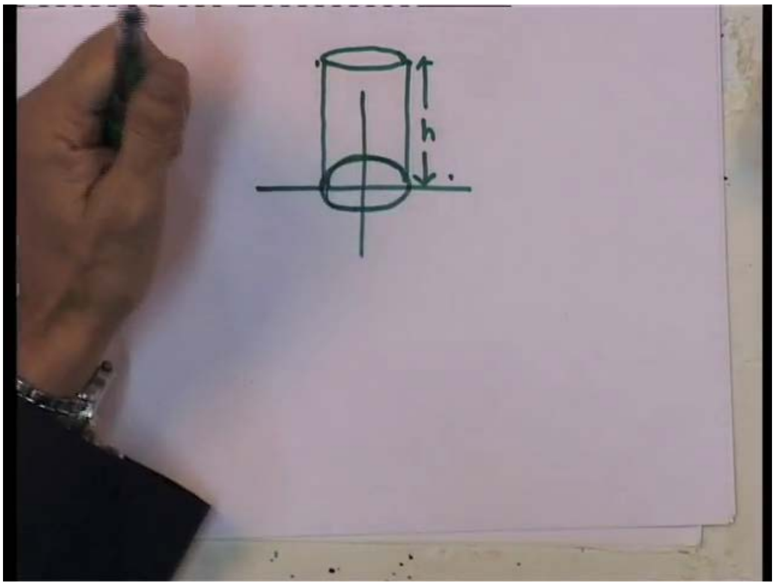

I have this circle of course, in the x y plane. Let me let me draw it slightly differently and let it be the base of a cylinder, which is in the, which is the height h along the z axis. Now, notice again that so, far as this is concerned. The, this boundary, the circular boundary, remember I told you that I am talking about an open surface. This circular boundary defines a cylinder which is on it, something like this. That I have a circular boundary which is just I am showing you the open rim of this. Now, so supposing this is in the x y plane now. So, if this is in the x y plane then I have a height h along the z direction.

Now, there is the stop top cap. The the perpendicular to the top cap is along these k direction and the perpendicular to the curved surface is along the radial direction. Now, I already have given you the expression for the force and its curl in the R theta that is my polar coordinates and the z direction. So ,you could calculate the curl F. Now, once again the top cap is trivial because top cap direction of n is along z direction. So, as a result I get the same result pi R square there. Stands to reason if you compute the integral of the curl along the or from the curl surface you will get it to be equal to 0. Little involved exercise, but do it it will give you the confidence on the Stoke's theorem.

Having completed a discussion of the curl of the vector and the divergence, let me now, take a slide breather and go through the things that we have learnt, the new operators we have learnt in the vector calculus. First, we talked about scalar field and vector field. We have seen that the scalar fields have gradients which I wrote as a del operator. So, del of phi, result is a vector field. Then I concentrated on the vector fields and we found that we could define a divergence which gives me a scalar field and a curl which gives me yet another vector field.

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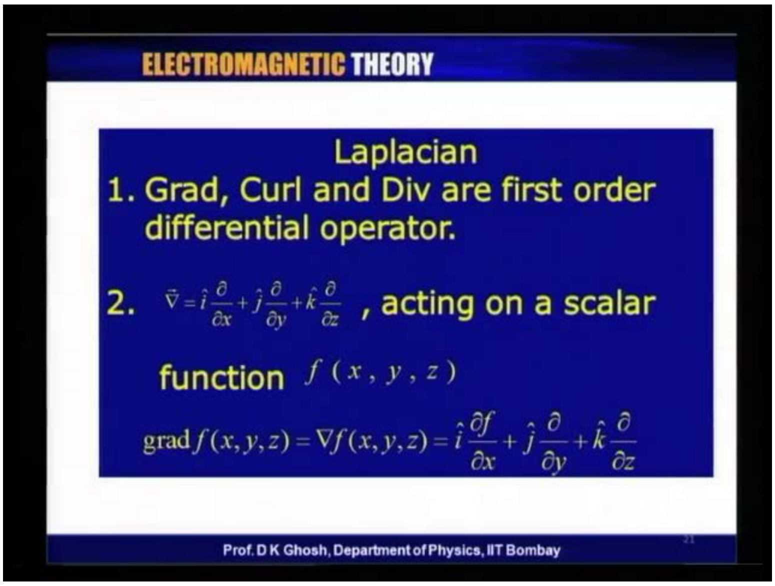

So, but notice that the differentiations involved in these processes where all first order differential operator del operator is an operator which in Cartesian coordinates is written as i partial d by d x plus j partial d by d y plus k partial d by d z. When it acts on a scalar function, we call it grad which gives me i d by d f by d x plus j d f by d y plus k d f by d z.

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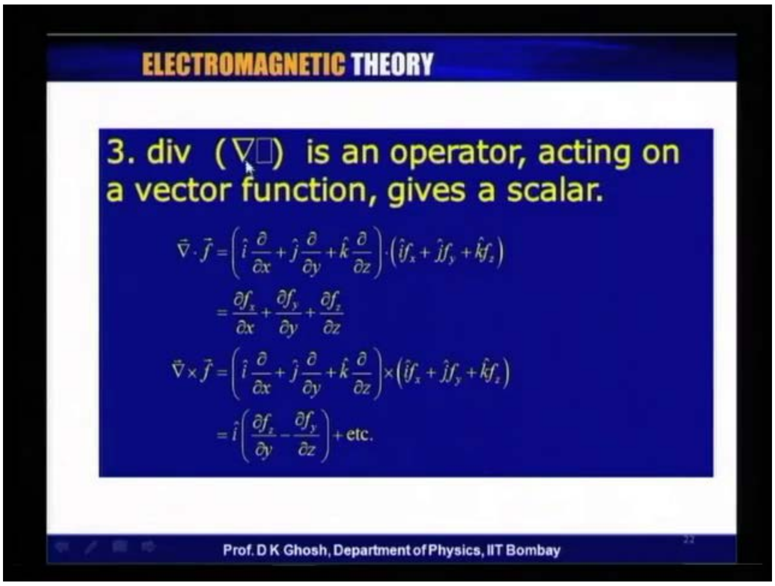

Divergence del dot does not come in the this thing, is an operator acting on a vector function gives me a scalar. We had shown del dot F is d F x by d x d F y by y d F z by d z. Finally, we will talk about del cross F. So, notice one thing that when you write this del operator what is we doing i vector is dotted with i vector and differential operator acts on the function. Similarly, i dot j is equal to 0, so you do not get any contribution from there. Exactly the same way write down del operator which is i d by d x j d by d y k d by d z cross i F x j F y k F z. Let these unit vector state the cross product i cross i is 0, i cross j is k i cross k is minus j like this and let the differential operator then act on the respective function that is coming in and you get the expression for the curl.

Next time I will be introducing you with the Laplacian operator which is written as del square,which is a differential operator, second order differential operator like in d square by d x square.