Electromagnetic Theory
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Lecture - 39
Radiation

We have completed our discussion on wave guides and resonating cavities. What you want to do now, in the remaining lectures that we are left with in this course. We will be discussing the theory of radiation, now when electromagnetic waves are sent through a wave guide. Now, it is necessary that we convert them into freely propagating electromagnetic waves, which can propagate in free space and so that they can be picked up by a receiving mechanism and the things are the devices, which do this they are known as antenna.

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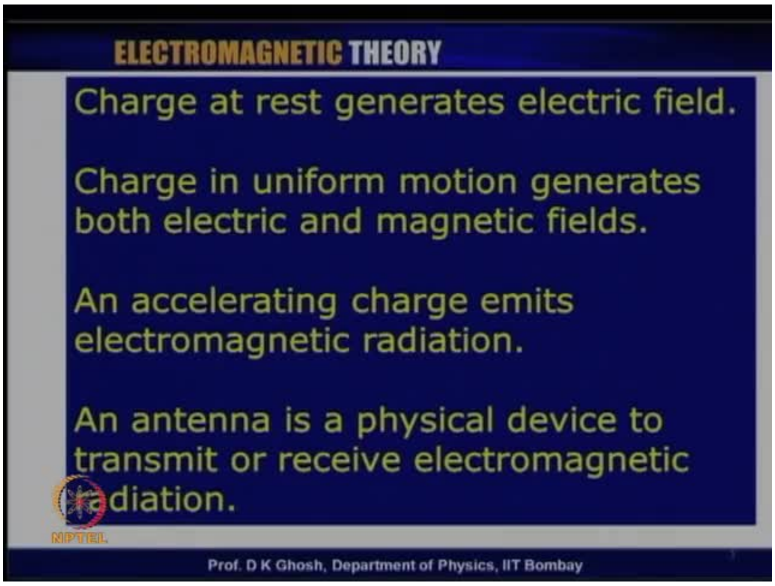

So, basically the way it works is the following that we know that we have studied electrostatics and we have seen that when we have charge at rest it generates an electric field, we called them electro statics field. However, if you look at charge in uniform motion, then in addition to the electric field, because this is a current and we have seen that the current is a source of the magnetic field. So, this will give rise to both electric and magnetic fields.

However, if we have charges which are accelerating then this will emit radiation. Now, an antenna is a physical device to either, transmit or receive electromagnetic waves or electromagnetic radiation. So, this is what we are going to be doing in today's lecture and the next, but before we do that as usual, let us recollect our bible namely the Maxwell's equation and we had seen that the in this case, we will be only involved with the two of the Curl equations.

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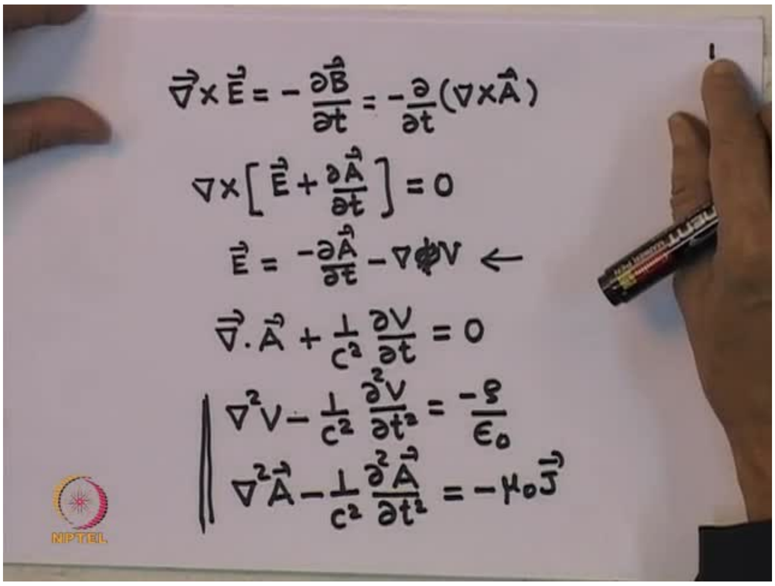

Namely I have got del cross E is equal to minus d B by d t and if I notice that, we know that we can define a vector potential curl of which is the magnetic field. So, this is minus d by d t of del cross A and now if you bring this to that side, what you get is that del cross of this quantity namely E plus d A by d t that becomes equal to 0 this is an equation, which we had seen earlier and we have seen that if del cross of A quantity becomes equal to 0, we should be able to A express that quantity as gradient of a scalar function.

So, that tells me that the E can be written as minus d A by d t and as has been our practice. We will take minus of a scalar function let me write it as grad V, we had seen that it is possible to formulate the Maxwell's equation, in terms of the potentials rather than in terms of the fields and if we take what we called as the Lorentz gauge, in which we had del dot of A plus one over c square d V by d t was equal to 0, then we had seen that the scalar and the vector potential satisfied, a very similar looking equation namely del square V minus 1 over c square d square V by d t square was given by minus rho over epsilon 1 and if we looked at the vector potential, it satisfied a wave equation of this type .

So, this is the pair of equation which essentially replaces the Maxwell's equations, which are in terms of magnetic and electric fields. So, we have instead written them in terms of three components of the vector potential and a single component of this scalar potential subject to the condition of course, that the Lorentz condition, Lorentz gauge is mandatory now. So, we need to solve these equations. Let me now notice that, these equations are very similar and therefore, I need to only worry about one of them and once I have solved one the other, one can be written down primarily by analogy.

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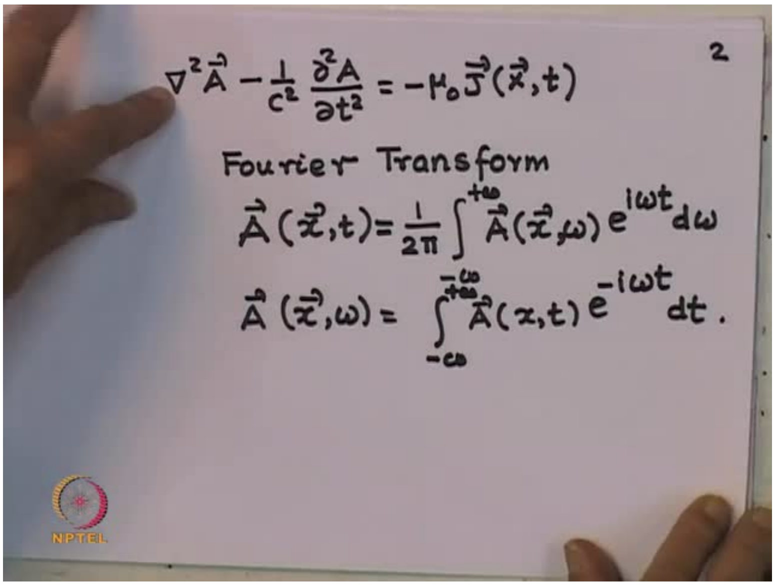

So, this is the equation I have got, I have got del square A minus 1 over c square d square A over d t square equal to minus mu 0 J. So, in principle these are all functions of the position vector x and t. Now, what we are going to do is, how does one solve these equations. Now, to get this we use what are known as the Green's function technique, but let us look at this, first we define what is known as a Fourier transform. So, this is written like A which is a function of x and t that is given by A factor 1 over 2 pi which, we have been taking in our definitions of Fourier transform minus infinity to plus infinity, this is time Fourier transform.

So, nothing happens to this face so we write it as a A function of x and omega e to the power i omega t d t omega of course, you could invert this relationship and write this as A of x omega is equal to there is no 1 over 2 pi there because of the wave, we have defined our delta function. So, this will be A x t e to the power minus i omega t d t. So, this is the Fourier transform and what I want to now, do is this using this Fourier transform, we want to write down this equation.

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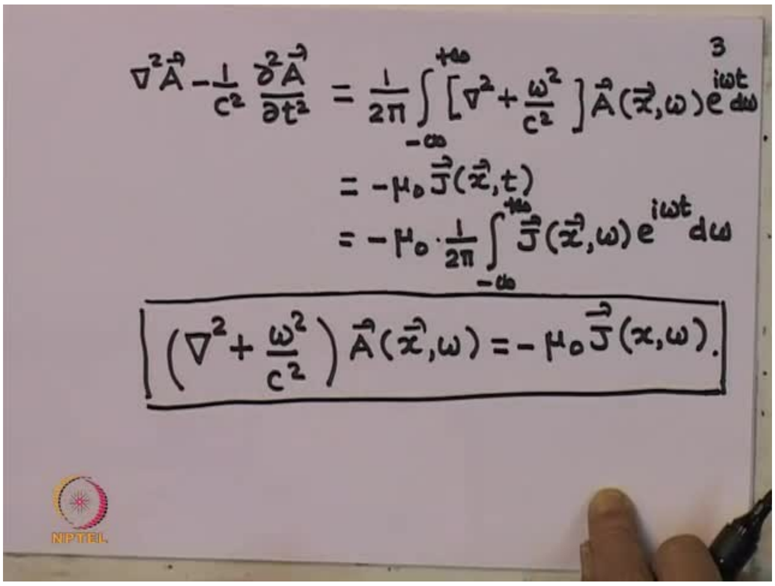

So, let us do that. So, we need to write down, how much is del square A minus 1 over c square d square A over d t square. So, substitute remember that del square is a derivative with respect to space and when, we wrote down the Fourier transform, this was with respect to time therefore, what do we do is that, we write down the expression for A x t and in terms of the Fourier transform equation and take the del square inside because the integration is with respect to time or frequency. So, this is minus infinity to plus infinity. So, I have got del square, which remains as it is.

Now, now since this is going to be A is going to be proportional e to the power i omega t. So, this d square A by d t square is equivalent to multiplying these with omega square over c square, this times A x omega x is and e to the power i omega t d omega. So, this is this is the way my left hand side looks like and this is equal to minus mu zero J of x t once again, let us write this down in terms of Fourier component of J itself. So, which is

1 over 2 pi integral minus infinity to plus infinity J of x omega e to the power i omega t d omega.

So, notice one thing that this essentially is an integration over various Fourier components, that is there and if you compare this equation with this, it is obvious that we can write down an equation for the Fourier component of a itself, which will be del square plus omega square over c square acting on A of x omega is equal to minus mu 0 J of x omega.

So, this is the equation, which we have so what we will do is we will try to solve this equation there by obtaining an expression for A x omega, which is the Fourier transform of A x t having obtained an expression for A x omega, we will take the inverse Fourier transform and find out, what is A x t and now notice that an identical operation can be done for the scalar potential V and the equations, will be obviously identical because the equations satisfied by A and the potential V are identical. So, let us look at what is to be done, the way to solve these equations. So, this is the equation that I have got so just for convenience let me write define omega over c as equal to k which is of course, the vector. So, let us write down omega over c equal to k.

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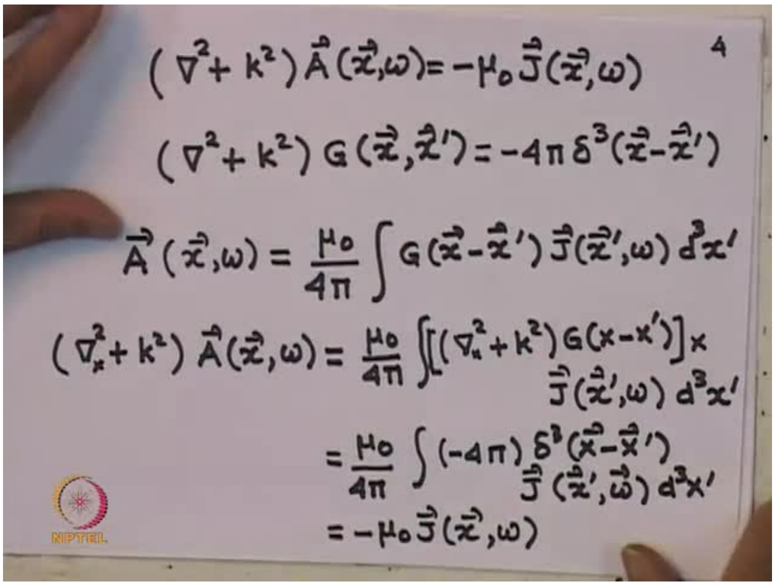

And in terms of that, what we get is del square plus k square A of x omega is equal to minus mu 0, J of x omega, we need to solve this the the way this types of differential equations are solved is to obtain solution of a subsidiary equation and that quantity which you get is known as the Green’s function. So, let us look at the definition of what is Green’s function and what is it that is there in the Green’s function that makes the solutions of these equations simple.

So, the Green’s function will be a function. So, let us write it down I write down del square plus k square the Green’s function is a function of x and x prime and that is equal to that is defined as this quantity is equal to minus four pi, this is this is in three dimension. A three dimensional delta function of x minus x prime so notice that what is happened is the in homogenous term, on the right has been replaced by a delta function and so what does it actually mean this tells me, will lets write it down and see how it works out that A x omega can be written as equal to mu 0 by 4 pi.

I will come back to the proof of this and an integral of a combination which is G of x minus x prime, J of x prime omega and we integrate it, over this intermediate variable that we obtain namely d cube x prime. Now, this is obviously not obvious as yet, but let us look at how does it go. So, if you take del square plus k square operate it on you are A x omega. So, first thing is to notice this is mu 1 by 4 pi this integration is with respect to x prime whereas, the del square is with respect to x.

So, let us just put it in an x there so this equal to this into del square I again put this little x there plus k square g of x minus x prime. Now, notice that here J is with x prime therefore, this del square operates only on this k square is of course, A so this multiplied by J of x prime omega and d cube x prime. Now, we have seen that this quantity del x square del square plus k square G of x minus x prime is nothing but the delta function.

So, I get minus mu 0 by 4 pi integral so this is minus 4 pi delta function of x minus x prime and of course, J x prime omega d cube x, x prime. Now, notice that this delta function enables us to do this integration immediately. So, that J of x prime omega, what I am left with will be simply J of x omega four pi and four pi cancels out and I will be left with minus mu 0 J of x omega as I expected it to be therefore, once I will obtain the solution of the Green’s function, the solution for the corresponding equation for the vector potential or the scalar potential can be written like this. So, let us write down once again.

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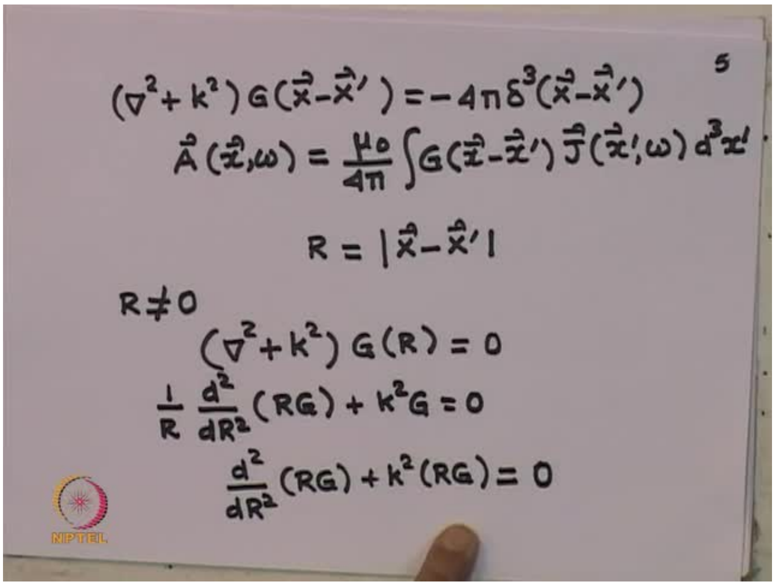

So, what we said is that my Green's function equation is del square plus k square G of x minus x prime is equal to minus four pi delta cube of x minus x prime and in terms of this, we have seen that A of x omega is given by mu 0 by 4 pi G of x minus x prime J of x prime omega and the integration is over the variable x prime. So, this is this is what we are interested in and this is the equation the Green's function must satisfy.

So, when we look for solving this equation I notice that, this equation must be spherically symmetric about the point x prime, remember x prime is a variable of the source and so as a result G of x minus x prime, as we have already assumed must be a function of x minus x prime. Let us denote R as equal to the modulus of x minus x prime and so this is to be satisfied everywhere, other than at x equal to x prime because that is where it is a delta function. So, let us first look at what happens excepting at R is equal to zero. So, if R is not equal to 0 then the delta function vanishes and I am left with del square plus k square of G, which is a function of R that is equal to 0, well actually I should have written x minus x prime explicitly, that is equal to 0 now.

Since, this is spherically symmetric I express the del square in spherical polar and since there is no R theta dependence. So, this tells me one over R, d square over d R square of R G plus k square G equal to 0, you multiply this R there, so I get d square by d R square of R G plus k square of R G is equal to 0, well this equation is very familiar to us because but this is an equation, which is for R times G therefore, let us write that down this will give me R G.

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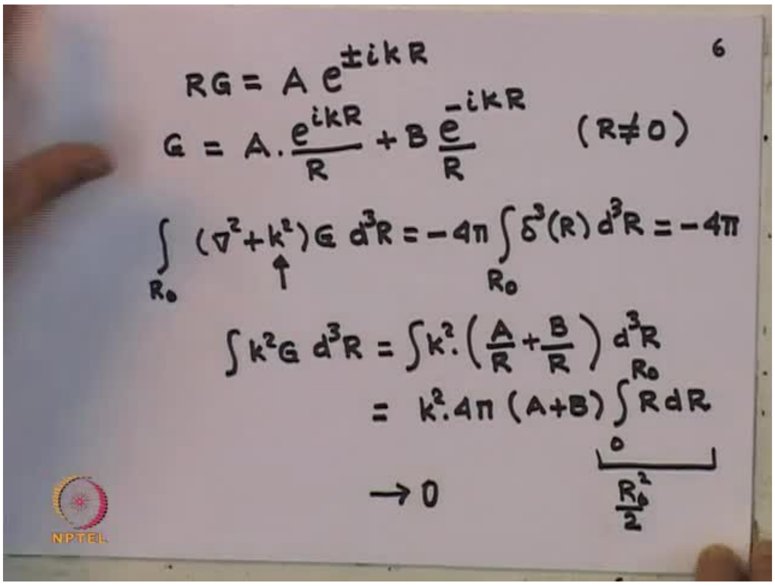

This will give me R G is equal to e to the power i k R, let's take it is a constant times e to the power i k R, it could be in principle plus or minus i k R. I will come back to this little later and therefore, my G in principle can be written as A times e to the power of i k R by R plus some other constant B times e to the power minus i k R by R, we need to evaluate this constant, but remember that this is a solution, which we have obtained excepting at the original.

So, let us write down R is not equal to 0 now close to the origin, if you want to the way the delta functions are handled you take a small sphere of radius R naught going to 0 and integrate, what we had so this is over a small sphere, this equation that we had written down here del square plus k square G R is a delta function, this is the equation and what we do is we integrate both sides of this equation by taking a sphere of radius small r 0. So, this is del square plus k square G d cube R that equal to minus 4 pi integral over R 0 delta cube R d cube R and since, the origin is included within this range of integration, this integration is straight forward it is simply gives me minus 4 pi.

Now, look at the left hand side, if you look at the left hand side I have a term here which is integral k square G d cube R. And we have seen that the solution for G because I have taken a sphere of radius R 0. So, on the I have to worry about, what happens on that sphere and this is k square times G, there is no discontinuity there. So, therefore this is k square A by R because it is close to the origin plus B by R, e to the power plus minus i k R go away and I have a d cube R. So, this is equal to since, there is no angle dependence I get k square 4 pi from the angles and there is A R in the denominator, I get R square d R. So, as a result I get A plus B integral zero to R 0 of R d R.

Now, this is of course, R square, R 0 square by 2 and when R 0 goes to 0 that is as if I make the radius of the sphere very small, this quantity goes to 0. So, you notice that on the left hand side, this term k square G d cube R integrated is actually 0. So, what I am left with is this is simply a del square operating on G and integrate it so let us do that.

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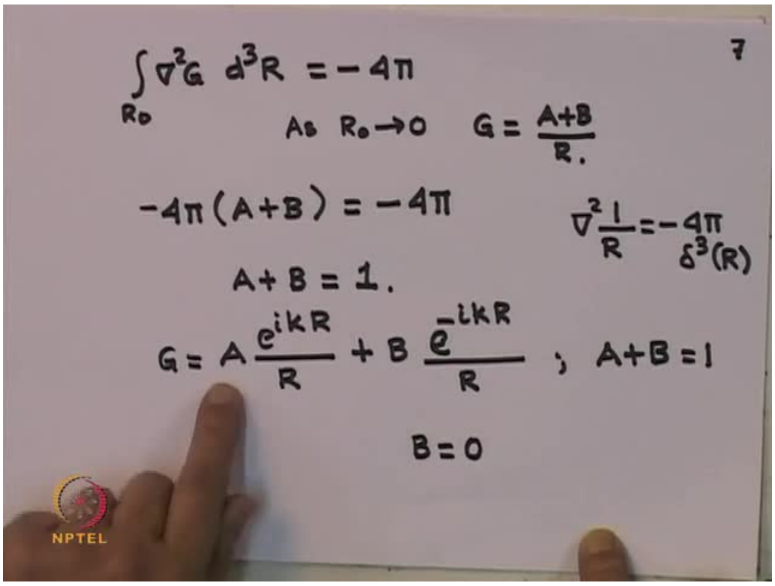

So, integral over R 0 del square of G d cube R must be equal to minus 4 pi. So, we have seen that as R 0 goes to 0, my structure of G has been G is equal to A plus B by R. So, therefore what I get here is this notice, that I have a del square A plus B is constant and I know del square of 1 over R is minus 4 pi times a delta function. So, as a result my left hand side gives me minus 4 pi into A plus B and right hand already has minus 4 pi that is simply, using del square of 1 over R is equal to minus 4 pi times delta cube of R.

So, this tells me that the constants A and B must be such that A plus B must be equal to 1, at this stage I need to look at what my structure is so my general Green's function is a e to the power i k R, remember capital R is x minus x prime by R plus B e to the power minus i k R by R subject to the condition that A plus B must be equal to 1. Now, I do the following look at the structure the what are we looking for our solutions, which are basically out going from the source. So, in principle I can take B is equal to 0.

Now, this is mainly this is not mathematics, this is purely on physical reasons. I could take B is equal to 0 because my outgoing wave must go as e to the power i k R by R, this is spherical wave and therefore, I will assume that B is equal to 0 it is also possible to talk about a equal to 0, in which case this becomes an incoming wave. So, this is the solution that you get.

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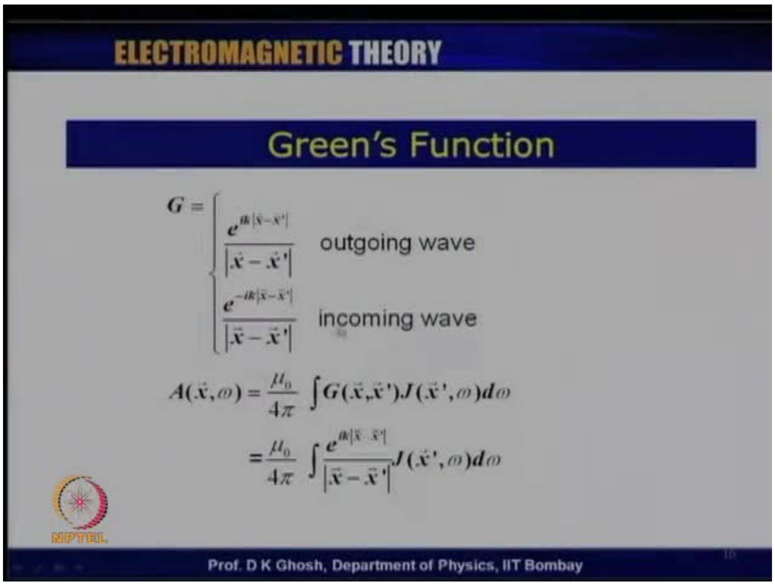

The Green's function is given by I have written explicitly instead of capital R e to the power i k x minus x prime modulus by x minus x prime modulus, which is an outgoing wave and e to the power minus i k x minus x prime by x minus x prime, that is my incoming wave and as we have seen that A of x omega is given by mu 0 by 4 pi G of x x prime J x prime omega d omega and here I have simply substituted, the expression for G of x minus x prime corresponding to the outgoing wave. So, let me rewrite the final expression once.

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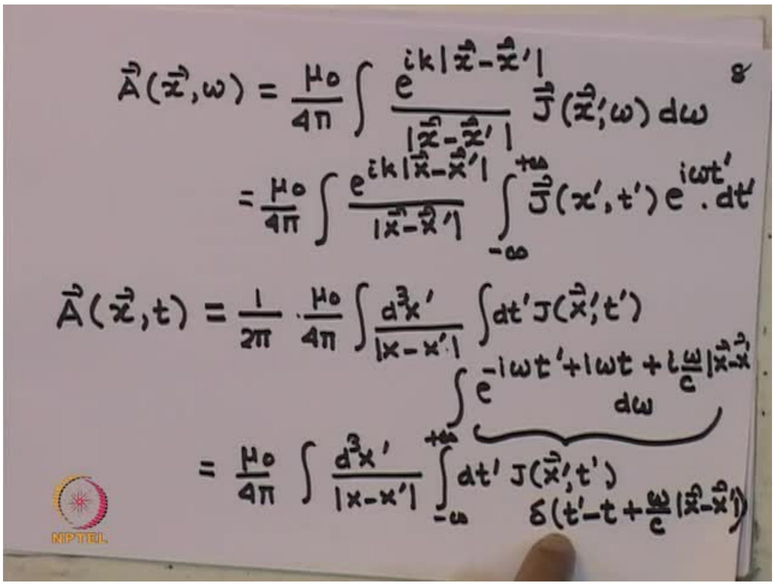

So, I get A of x omega is equal to mu 0 by 4 pi integral e to the power i k modulus x minus x prime divided x minus x prime J of x prime omega d omega. So, this is, this is what I get for the structure of the vector potential and as we have seen that, if you want to instead solve for the potential function v then of course, this one will change e to the power four pi epsilon 0 instead of J i will have a rho, but other than that because the homogenous equation, the Green's function definition is identical. So, I can use the same Green's function to solve the both vector potential and scalar potential. Now, what good does it do?

Starting from this I would go back, step after step and my things will be like this I will convert this by a inverse Fourier transform to J of x prime t. Now, if I do that x prime, let us call it t prime after this I will use that expression to get A x t by taking the Fourier transform, the whole thing so let us look at how does it go. So, this is equal to mu 0 by 4 pi integral, I am not doing anything here right now, i k x minus x prime by x minus x prime.

Now, this is J x prime omega which is and I am going to write down in terms of J x prime and a time variable, which I will call as the t prime, minus infinity to plus infinity J of x prime t prime, e to the power i omega t prime and d t prime notice that, there is still a dependence on omega therefore, I need now to write down what is A x t which is the inverse Fourier transform on this quantity and so this is equal to 1 over 2 pi and I had a mu 0 by 4 pi integral of d cube x prime by x minus x prime. Now, notice that I am right now not writing, the explanation part because if you recall my k was equal to omega by c therefore, since this is omega by c times a distance.

So, we will use that so this is d cube x prime, this integral d t prime J of x prime t prime and an integral of e to the power minus i omega t prime plus i omega t plus i times, this is k therefore, this is omega by c into x minus x prime d omega. So, what I have actually done is to combine all the terms, which have omega dependence and this of course, integral over omega is durable, it will give you 2 pi times a delta function and therefore, let us write that down that will be mu 0 by 4 pi d cube x prime by x minus x prime integral d t prime.

I am not being careful write all the integration values d t prime J of x prime t prime minus infinity to plus infinity and this is a delta function 2 pi times delta function that is takes care of this t prime minus t plus omega by c, x minus x prime this will enable us to do this time integration, this will enable us to do time integration so that t prime will be written as t minus this. So, let us write it down.

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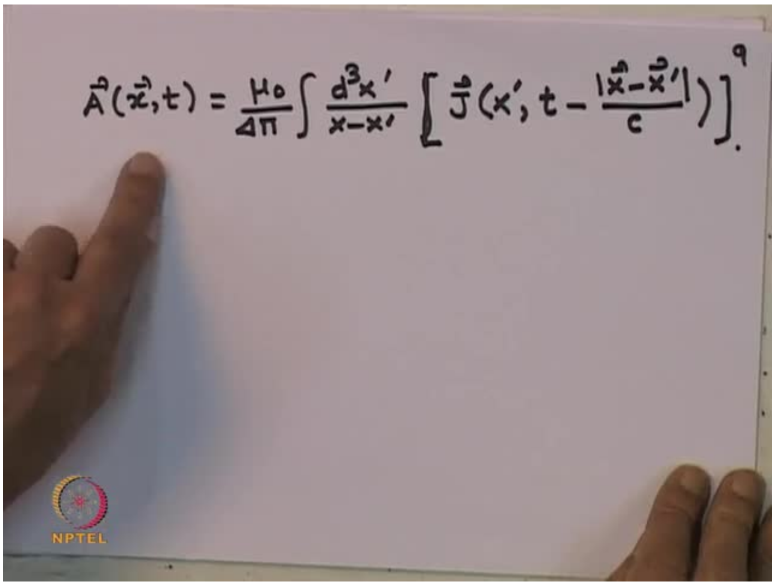

So, by A of x t will be mu 0 by 4 pi integral d cube x prime by x minus x prime integral of there that, the delta function enables us to do this integration therefore, there is no integral actually times J of x prime and t equal is equal to rather t minus x minus x prime by c and in a very similar way, you could write down the expression for the scalar potential which will be V of x t, which is given here in this screen.

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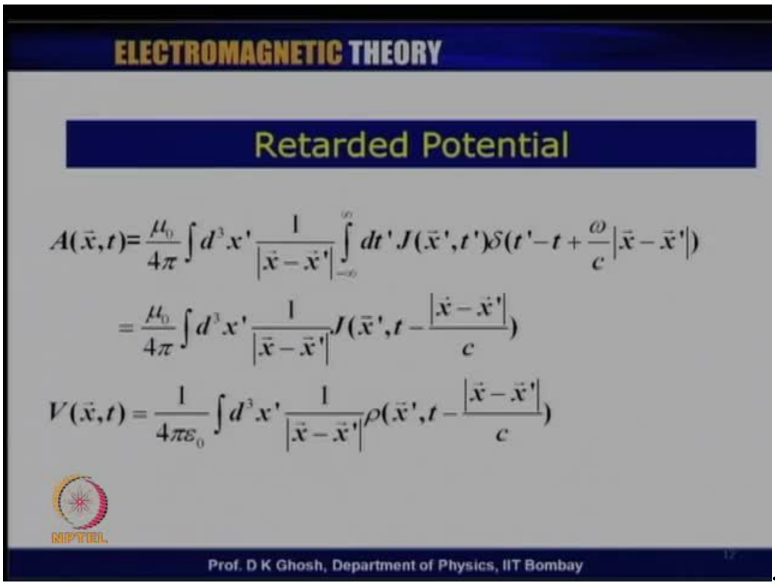

And so this is it, but let us look at what, what does it imply this remember that x prime is basically A, the A coordinate of the source because this is integrating, the all the points where the current density or the charge density is there. So, therefore these are source points. So, x prime is the variable corresponding to the source and x is the field variable that is in principle, any point where I am calculating A of x t. So, this tells me that the time at which I calculate, the current density is not quite the time that which I calculate the potential function, but is an earlier time and this is, this time is earlier by an amount x minus x prime by c.

Now, this simply tells me that supposing I have a disturbance generated at certain time or I am interested in calculating the potential at a given time t, then this potential is influenced by a wave which came from the source at time earlier, than the time t by an by an amount that by a time duration, that is taken by the wave travelling with a velocity of light c to cover the distance and that is your x minus x prime divided by 0. So, these are what are known as the retarded potential.

The retarded because the potential at a given time t is determined by the, by the disturbance generated or the current density or the charge density at an earlier time and hence the phrase retarded, remember the other solution of the Green's function, that we had which will be e to the power plus i k R over R minus i k R over R. Now, you could if you did that, that is an incoming solution. Now, that is something which is not very clear to us at this stage, the that will simply give me a t plus here, which can be then what we are trying to say is that the disturbance is given by A. I mean the potential is determined by a disturbance, which is generated later.

So, that is the incoming solution, we will for this moment not discuss that aspect, it has these are the retarded solutions that we are interested in, they are fairly simple this is nothing but the way you normally, expect a solution for x t and the only difference is that the time instead being the same time, that would corresponding to x and at a distance, but or a instantaneous change, but we know that disturbance takes a finite time and this disturbance, which travels with the velocity of light would take A time x minus x prime by c to reach the point, where we are interested in calculating the potential.

So, with that let me now take some specific examples, what we all assume is suppose I have a localized source, localized oscillating source remember I made a statement that, if there is a an accelerating charge, it will give raise to a radiation well this simplest accelerating charge that we take is a charge, which is oscillating with a time frequency omega.

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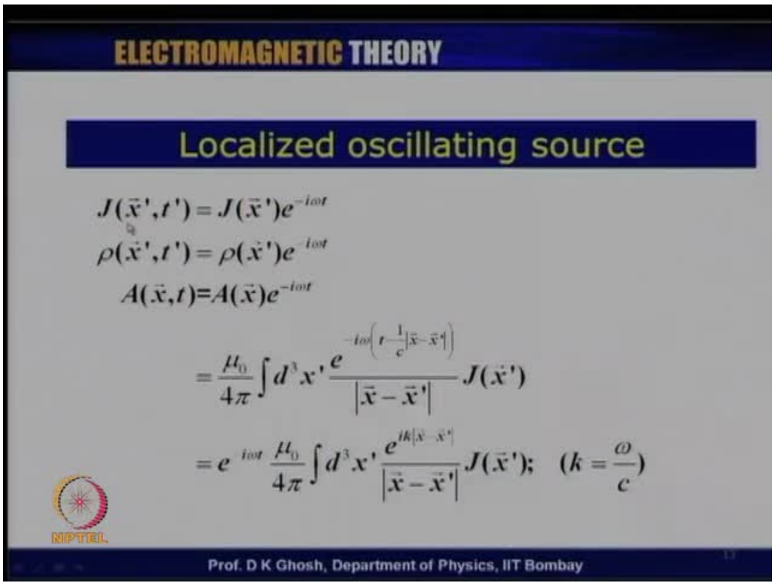

So, let us take this J of x prime t prime, which is I want as J of x prime e to the power minus i omega t time prime, I am sorry that t prime is not written here both of them should be t prime and likewise A x t is given by A x e to the power minus i omega t. So, let us see, what is it that we get?

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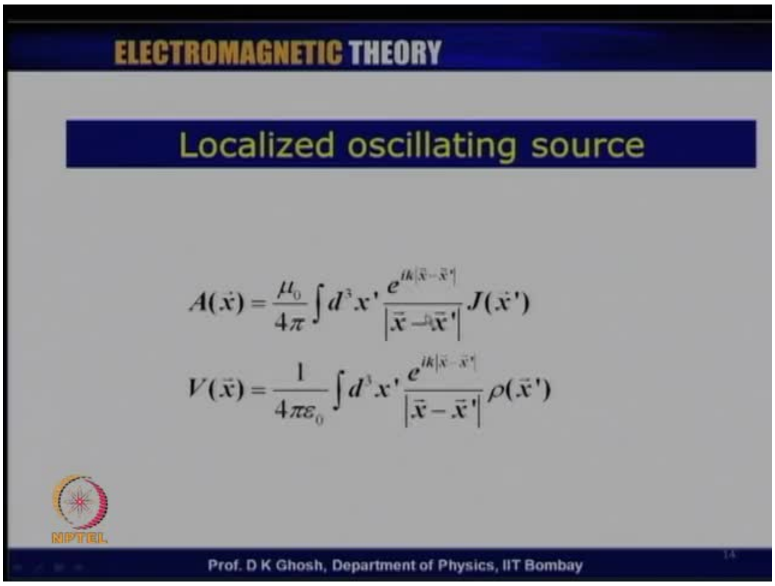

Going back a little bit this is my A. so this is, this is what we obtained and we are now saying that we will be only confining ourselves to the time variation of A x t, which go as e to the power i omega t. So, this is what we want to do now, what I have done is to write down on the right hand side, the expression for the A that we wrote down just now, one and if I am saying that the variation in A goes as e to the power minus i omega t and this was the expression that I had obtained for the vector potential A of x t.

So, this immediately gives me a an expression for A of x and remember all that I have done is to say my time variation must be simply A of x into e to the power minus i omega t. So, this is the same expression which we wrote down and therefore, what I do is this I pull out the e to the power minus i omega t there and I have left with e to the power minus i omega t into this expression where, if you recall k is equal to omega by c. So, this is what I have written down so this is my A x t and the corresponding A x or V x will be given by mu 0 by 4 pi integral d cube x prime e to the power i k x minus x prime by x minus x prime into J x prime. So, this is the integral any one of them that I need to inaugurate.

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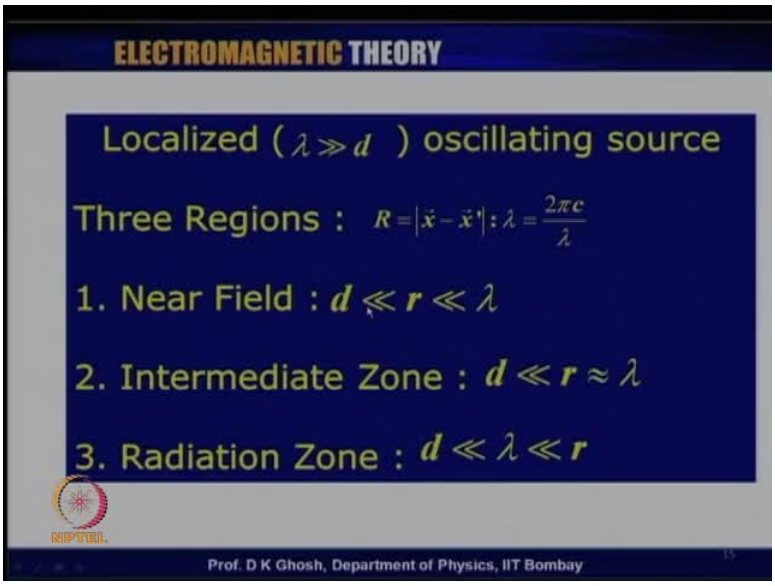

Now, before we proceed with the evaluation of these integrals, let me tell you that there are three distinct regions that one considers, the firstly we assume that the variable x prime is confined within a small distance. Now, this is like saying what is the extent over which your charge distribution or the current distribution takes place, so this essentially my size of the antenna; for example if it is a transmitting antenna. So, the x prime is the quantity, which is whose magnitude in some scale is small. So, let me say that quantity is given by d.

So, d is the essentially the dimension of the source now, when we say something is a localized source, we compare this dimension with the other scale that I have in the problem namely, the wavelength and so I assume that the wavelength of the emitted radiation is much greater than d. So, d is the region over which the current density or the charge density varies and is non zero and I am taking oscillating source, as we have said now.

So I have a scale which is lambda, I have a scale which is d. Now, I need a third scale which is the distance from the source, where you are making the observation. So, based on that I split the problem into three different regions of interest, the first region is what is known as the near field region. So, look at this my capital R which x minus x prime modulus a lambda as you remember is I am sorry, this is the a lambda is two pi c over omega. So, there is a an error there actually it is a 2 pi c by k 2 pi by k is lambda, but then k is omega by c. So, it is 2 pi c over omega. Let me write it down so there is no confusion.

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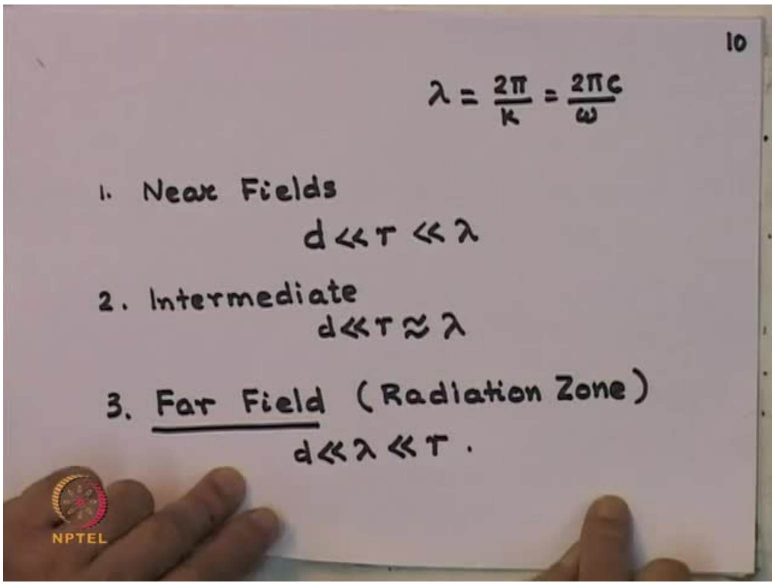

So, lambda is 2 pi by k which is 2 pi, this what we got so my three regions are a near field region, this is the you are looking at the fields near the source which means, now notice that I have d is much less than R. R is the distance where, I look at the potential and this R where I make an observation is much smaller than the wave length lambda. So, this is what I would call that is wave length is the largest dimension. Next is the distance at which I make my observation and d is of course, the source parameter, this we will see there is a region where, which is called an intermediate field region, this requires lot more rigorous solutions, we will not really be talking much about it.

So, this is d much less than R, but then R and lambda are of similar orders of magnitude. Finally, I talk about what is noticed a far field region in that most of my interest will be in this region, which is also known as the radiation zone or radiation field region and here d is of course, much less than lambda still, but then lambda is much less than the observation distance R. My primary interest will be to look at this region and this region I will make some comments, this is the region where it requires lot more rigorous, you know the solutions and we will almost not talk about that let us, let us look at what happens in the near field region.

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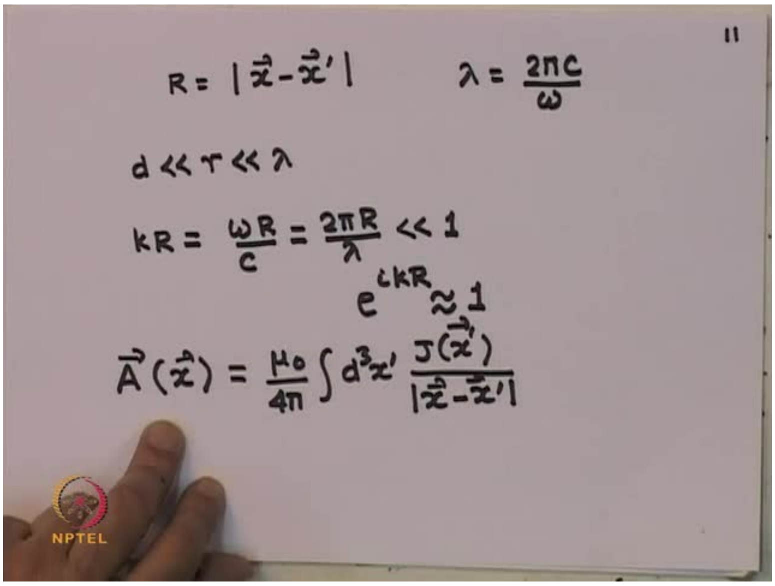

Remember that my R, which is the distance between the source and the point of observation namely x minus x prime and we had said that well once again, the same error perceives. So, lambda is 2 pi c over omega. Now, if I have d much less than R much less than lambda. Now, remember what I am trying to do is to look at that e to the power i k R by y R. So, what is k R so k is omega by c. So, this is equal to omega R by c and so that is equal to 2 pi R by lambda and this quantity since, lambda is much greater than this R and notice that capital R is just the modules of the distance from the source therefore, this is much less than 1. Now, if it is much less than 1 then I can write down e to the power of i k R, which was appearing in that as approximately equal to 1.

Now, if you substitute for that e to the power i k R, I get A of x is equal to remember that time dependence is only on e to the power i omega t is equal to mu 0 by 4 pi integral d cube x prime J of x prime e to the power i k R is taken as 1. So, this is x minus x prime, if you look at this expression, this is the expression which was familiar to us in when we studied, the magneto statics therefore, the I may say that the fields are Quasi stationary the only variation, the time variation in the vector potential comes because of the e to the power minus i omega t, the effects are the same as what one obtained.

I had a steady current, the expression will already the same I can borrow the entire mechanism of the magneto statics and simply multiply with that e to the power i omega t to get, what my time dependence vector potential is so this is a statement that I have been making that the near fields as essentially, quasi stationary as I said I would not be saying much about the intermediate field. So, let us go over to a discussion of the far field, which we also have been saying, as the radiation field and this is this is of great interest to us this will involve certain amount of complicated algebraic manipulation, but never the less they are all fairly straight forward.

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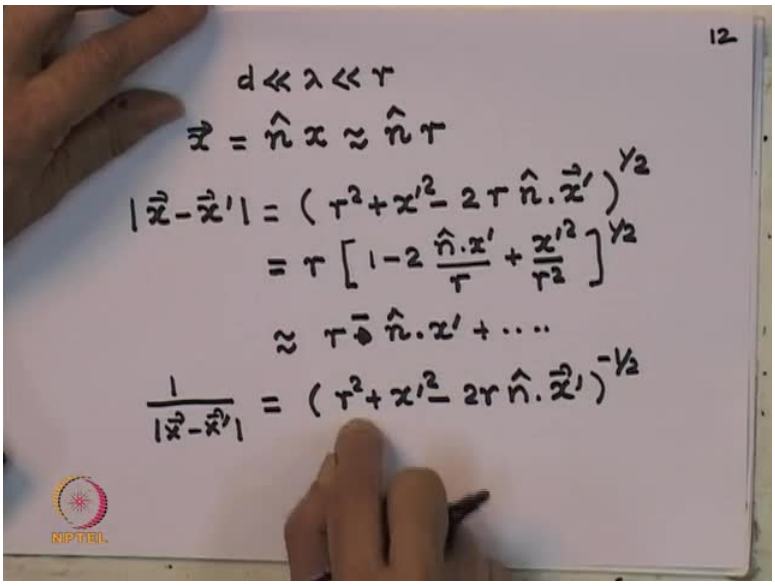

Recall that I am having d much less than lambda much less than r. So, this is my so let us write down x vector x, let n be the unit vector along x. So, that I write this as this and let us indicate the magnitude of x by r. So, that this is n times r I am writing this as a small r remember so capital R was vector x, the modules of vector x minus x prime. So, in my expression for the vector potential, so I had that e to the power i k R over R. So, let us write down what is x minus x prime, which appeared in the denominator so this is as we know is square root of x square, which is r square plus x prime square minus 2 well x dot x prime, but x magnitude is r therefore, write let me write it down r n dot x prime this raise to power half and that is equal to r.

I am just doing an binomial expansion so I pull out r I get 1 minus 2 n dot x prime by r then I have a plus x prime square by r square this raise to power half, do the binomial of this you get r plus this is half. So, this one goes away I am left with n dot x prime should be a minus sign there plus of course, this what I get actually, what I require is not quite this i require 1 over x minus x prime. So, I need this quantity here same r square plus x prime square minus 2 r n dot x prime raise to the power minus half and you can carry on this the standard binomial expansion, I am showing it.

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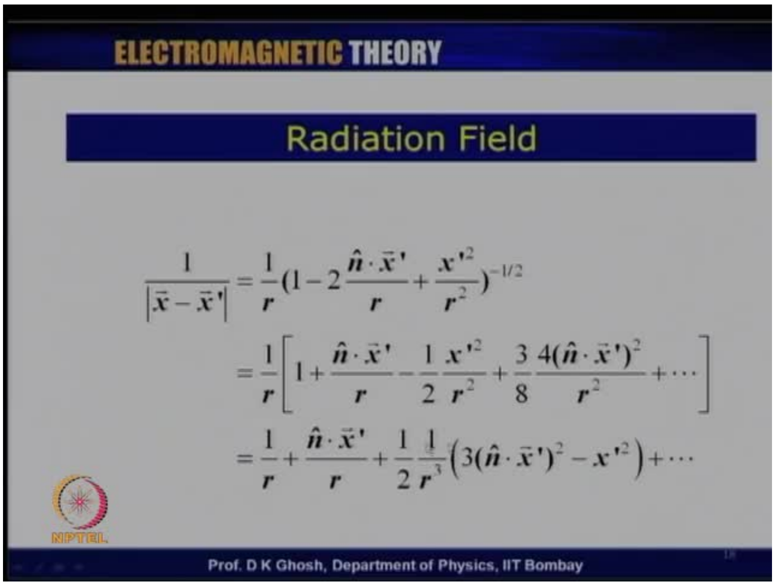

And you can see that there is no great mathematics here, I simply will do this binomial raise to the power minus half and I will get an expression like this, while I will complete this derivation next time. Let me tell you what I am going to do.

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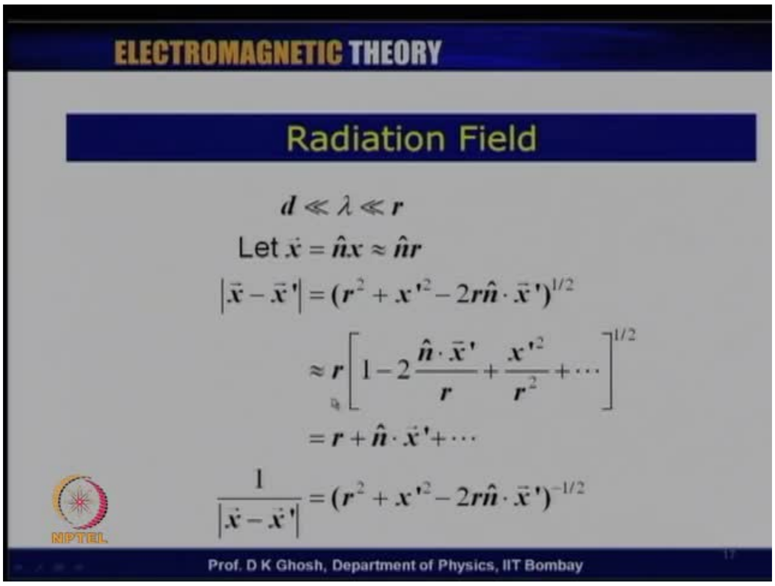

What I am going do is I am going to use this expression, x minus x prime expression in the argument of the exponential and this expression to divide it.

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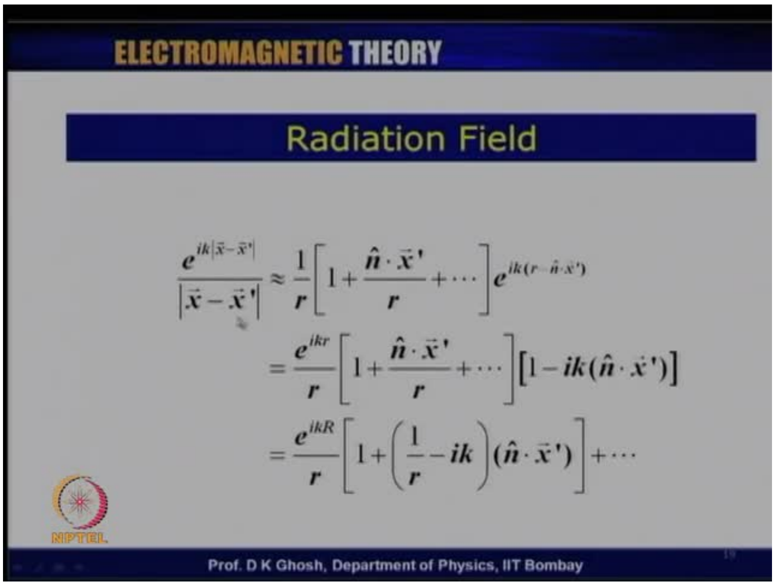

And then get an expression for what happens to e to the power i k R over r and then do an approximation, on the far field and from there we will have an expression for the vector potential and the scalar potential and then by calculating, the curl of the vector potential I will get the magnetic field and from the magnetic field, the electric field etcetera, etcetera so we will continue with this the next time.