Electromagnetic Theory
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Lecture - 38
Resonating Cavity

In the last lecture, we have talked about a Resonating Cavity which is a rectangular parallely piped of dimensions a by b by d, in which I have trapped electromagnetic waves, and the walls of the cavity are metallic, good conductors. And we had seen that it is possible to classify, the modes as before in terms of whether the electric field or the magnetic field is perpendicular to the perceived direction of propagation; in this case we have taken to be along the z axis, which is along the length, along which the length is d. And so we will continue to talk about resonating cavity and the q factor of the resonating cavity today, later on we will talk about a circular wave guide, that is one having cylindrical symmetry.

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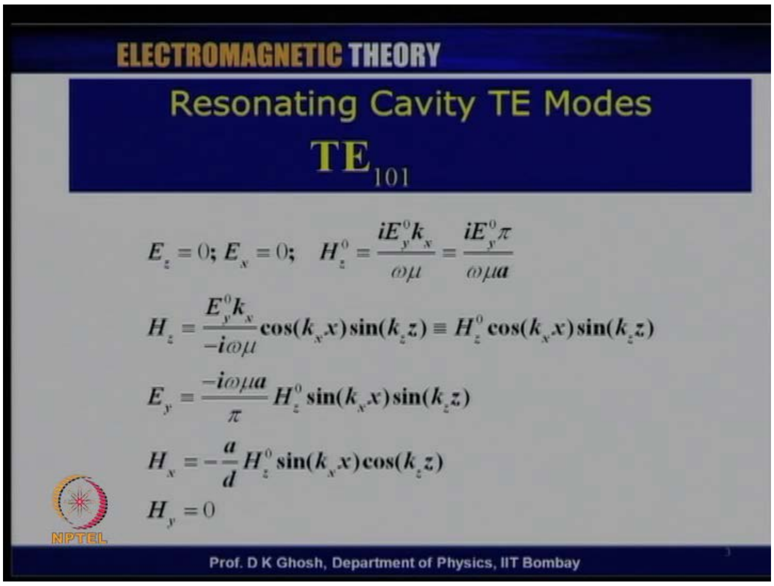

So, what we had seen is this, that the T E l m n mode, they are described by having E z is equal to 0 which automatically implies E x equal to 0, and so therefore, the entire field can be written in terms of H z. So, since we are writing things in terms of H z, so I will change my notation slightly, that is I will define H z 0, what I had done earlier is to write each one of the E components for example, I wrote E x as E x 0 times sin cosine function. But, instead of doing that since, it is going to be determined by H z so let us have this H z 0 as the multiplying factor.

So, H z 0, I define as equal to i E 0 k x by omega mu, and we know that k x is l pi over a, but in this case 1 0 1,so l is equal to 1soI have got i is 0 pi over omega mu a. So, in terms of this I have my H z works out to a simple expression, H z 0 cos k x sin k z z, and this is this was the coefficient that was there in front of the H z, which I have now call it, called as H z 0.

In terms of this I can redefine, the coefficient that appears in front of E y and H x, and it turns out that E y is given by this quantity times H z 0 sin into sin, and H x is simply given by minus a by d H z 0 sin k x x cos k z z. And of course, H y turns out to be equal to 0, and this is because, I have taken the mode which is the middle one, which is m equal to 0, and as a result E x and H y have become equal to 0.

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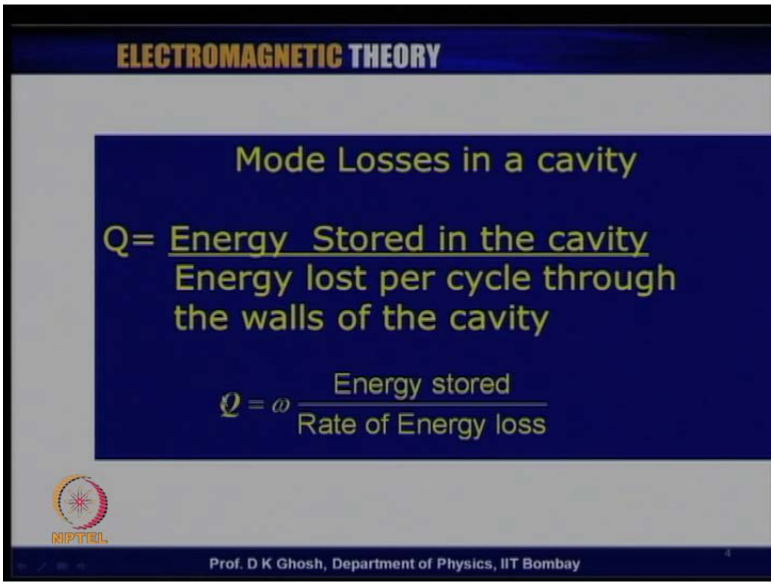

So, what we do next is to define what is called as the Q of the cavity, the Q is defined formally as the amount of energy that is stored in the cavity, by energy lost per cycle, through the walls of the cavity; and the formal definition is omega times, energy stored in the cavity divided by the rate of energy loss. So, what do we do now is to take a specific geometry, we will still be talking about T E 1 0 1 mode, and try to see how the Q value is calculated, remember till now we had said, that the walls are perfect conductor.

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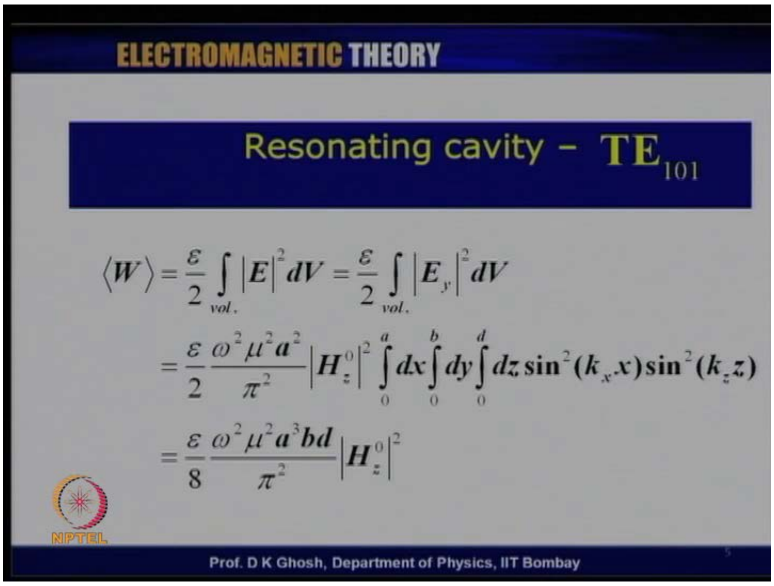

But, so let me first calculate the numerator, which is simply the total energy density, and that is given by epsilon by 2 integral E square d v, and since the only component of E that is non 0 is E y, so it is E y square d v. And if you recall my E y is given by this expression, there is a constant, and there is a sin k x and sin k z, so therefore, my integration will be over sin, well since it is E y square over sin square k x x, and sin square k z z. And the integration over y is still here, because that function, there is no function which depends upon y here, so therefore I get 0 to b d y which simply gives me a b, and each one these things, sin square k x recall that k x is given by l pi by a.

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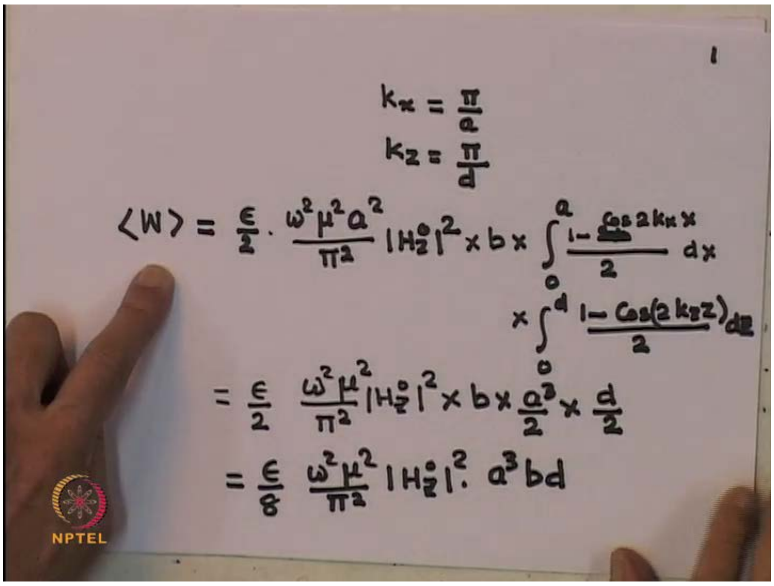

So, since l is equal to 1 it is pi by a, and k z is given by n pi by d, so it is pi by d so my total energy that is stored which is epsilon by 2 E y square modulus which gives me omega square mu square a square divided by pi square, then I have of course, H z 0 square integral over b simply gives me a b. Now, this is an integral from 0 to a sin square k x x d x, so now the sin square k x x is 1 minus sin sorry, 1 minus cos 2 k x x divided by 2 d x; and we have to sort or realize that k x is pi by a, and the other 1 is 0 to d 1 minus cosine 2 k z z divided by 2 d z.

So, notice that 1 by 2 gives me a by 2, this gives me a d by 2, this cosine if you integrate will give you a sin to a 2 2 k x x and so therefore, at 0 limit it is 0, and when you put x is equal to a, so you get sin 2 k x a, but then since k x is pi by a, so this integral and similarly this integral works out to 0 (Refer Slide Time: 07:12). So, therefore, this is equal to epsilon by 2 omega square mu square by pi square H z square H z 0 square I had a b there, I had an a square there, and I am getting a a by 2 there, so I get a a cube by 2 into I have a d by 2 there, so this is this is the expression, which works out epsilon by 8 omega square mu square by pi square H z 0 square a cube times b times d, so this is this is the expression for the amount of energy that is stored.

Now, what we are going to do now is to calculate the amount of energy that is lost, through the walls of the cavity, in order to do that we have to realize that. I have a rectangular parallelepiped with what we earlier assumed as perfect conductors, what do we have our finite conductivity of the at the walls, but since we can assume the conductivity to be large this, can depth will be fairly small.

Now, since the electromagnetic wave does not penetrate substantially into the plates, into the depth of the plates, we can assume that field in this particular case, we are only interested in that tangential component of the magnetic field. Remember, the tangential component, the normal component of the magnetic field was 0, but tangential component of the magnetic field to be more or less confined surface. Now, if it is confined to the surface, I have a surface current which I will designate by J s, and this J s is essentially related to the tangential component of the magnetic field by this relationship J s is equal to n cross H.

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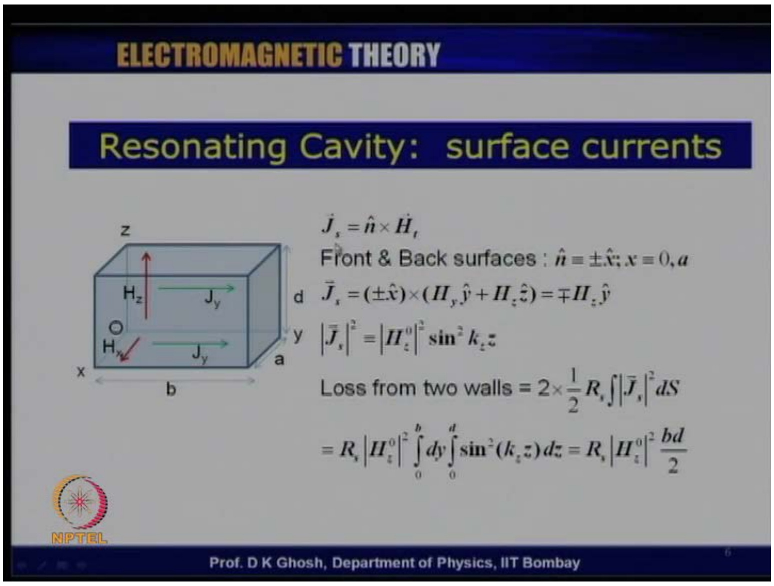

So, what we will need to is to compute the surface current at each of the six surfaces, and then calculate how much half R n J square which is my loss. So, let us illustrate this with some specific examples, so let us look at this, parallelepiped here, and you notice that there is a front face, and there is a back face, the front face is essentially an y z plane, and at x equal to 0; and the back face is also parallel to as a y z plane at x is equal to a. And the normal to each of the faces is parallel to the x direction, it is either plus x for the you know one of the walls and that is the back, wall and minus x for the front wall, because the normal has to be taken of the inside face. So, therefore, if I look at any one of this whatever I say will be valid for the, so let us look at the front wall, now front wall is an y z plane, so I write down J s as plus or minus x depending upon whether it is front or back, time cross since it is y z plane my magnetic field is H y y plus H z z these are unit vectors. But, we had seen that H y is equal to 0 for t modes, so t 1 0 1 mode, so therefore, I get x cross z which is equal to y and is equal to minus or plus. Now, which tells me that the modular's square of J s, which is what is involved in the calculation of the losses is simply given by H z 0 square sin square k z, so let us look at what it is.

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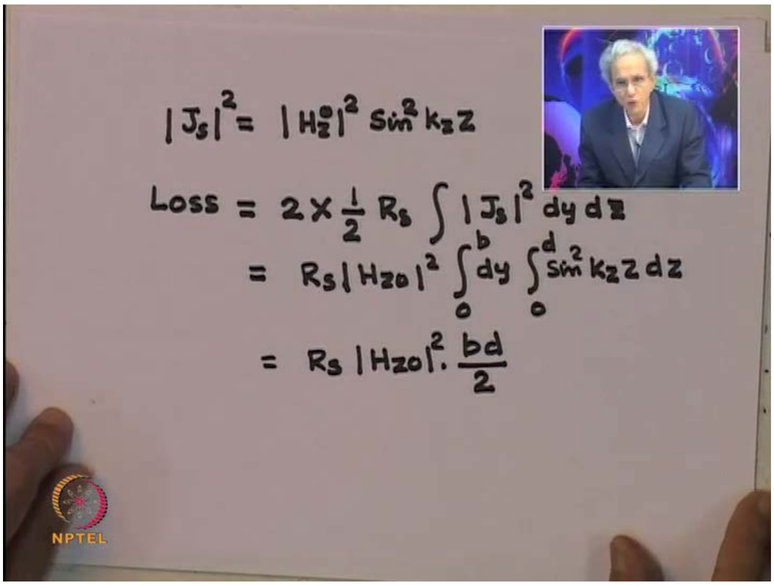

So, what we had shown is from the back and the front wall J s absolute square is given by H z 0 square times sin square of k z z that is because, all that I need is my H z z, and now I want to calculate how much is the loss from the two walls. So, the loss from the front and the back surface, factor of 2 because, each wall gives me the same value, half the surface resistance R s and I have to integrate J s square over the surface, and the surface is the d y d z. So, let us put it put the value 2 and half goes away I am left with R s H z 0 square integral d y there is no nothing to integrate actually, sin square k z z d z this is from 0 to b, and this is from 0 to d; well we have seen that this integral gives us a half, this integral half d, this integral gives us a b.

So, therefore, I am left with R s H z 0 square into b into d by 2, so this is the loss from the front and the back surface, well two more surfaces are there, let us look at the left and the right surface.

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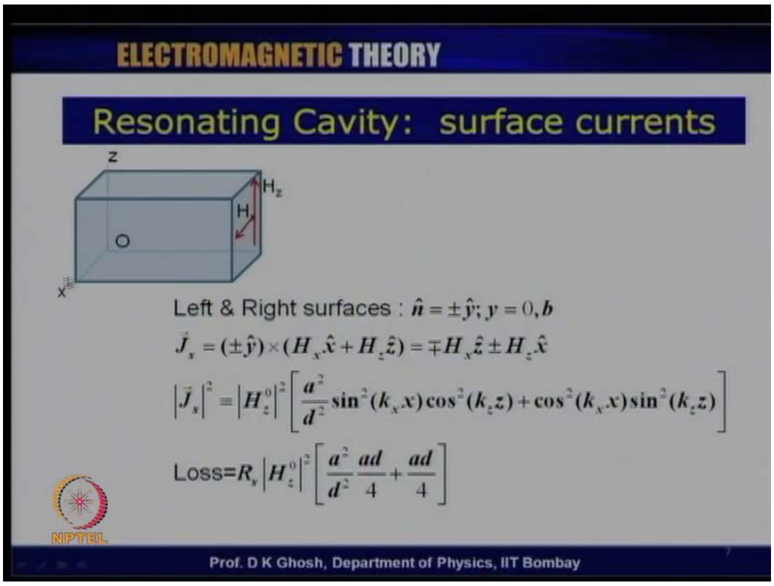

So, if we look at the left and the right surface, this is my left surface which is an x z plane, this is my right surface which is also x z plane; and the perpendicular to this is along the plus y direction, and on this is along the minus y direction, and the planes are located at y is equal to 0 and y is equal to b. So, my J x is plus or minus y, since it is x z plane, I write in general H x x plus H z z, remember that both of them are non 0, so therefore, I have got actually two terms coming out of there, so y cross x gives me a z, and of course, y cross z gives me an x.

So, I am left with plus or minus or minus or plus H H x z plus H z x, but fortunately I am interested only in the modules square of this, so therefore, what I get is, so this is side surfaces.

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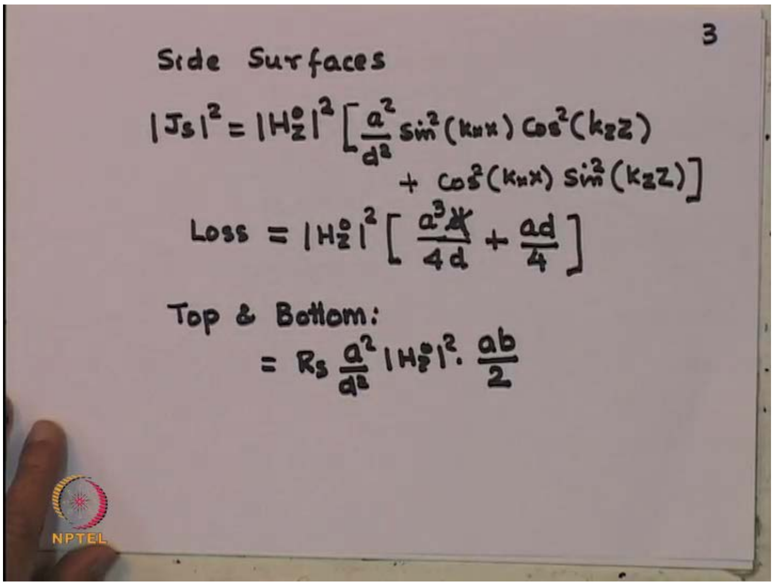

My J square then is given by H z 0 square, and look at what we did this was my J s, and I substitute for expressions that we had written down earlier for H x and H z; and this becomes a square by b square. Because, having a square, sin square k x x cos square k z z plus that is a z component, so which was simply at z square cos square k x x into sin square k z z. Well once again I can calculate the loss, so this is simply obtained by integrating over x and z, and each one of these integration will give you, well x integration will give you a by 2, z integration will give you d by 2.

So, I am left with H z 0 square here by a by 2, so that is a cube, and a so I have a d square there already, so therefore, it should be a cube d divided by 4, sorry a cube d on the top, but I have a d square there, so therefore a cube by 4 d and plus this is a and this d so I have got a d by 4, so this is the loss from the two side surfaces. I am still left with another pair of surfaces, I will not repeat this calculation, but this goes exactly the same way, and you can see that the from top and the bottom, you can take it up as an exercise, it works out to R s a square over d square H z 0 square into a b by 2. So, the job now is to add up the three factors, the factor of 2 has been already taken into account when we talked about two surfaces.

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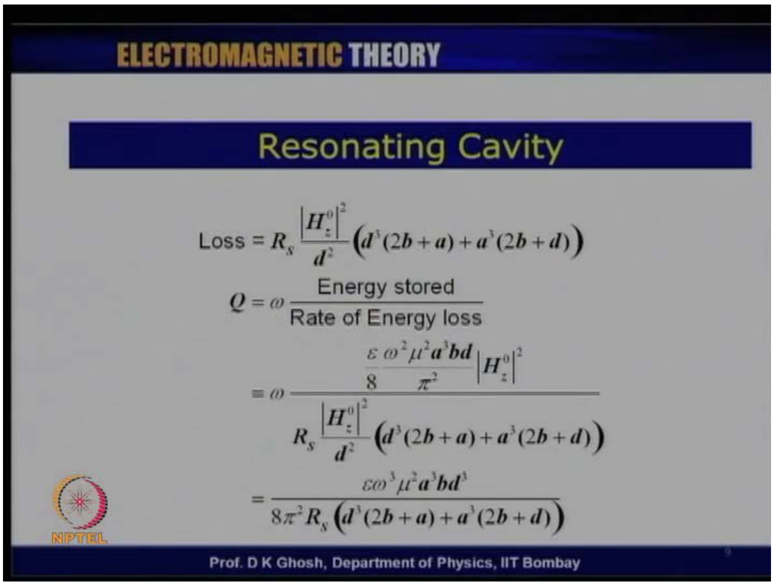

And so therefore, this gives rather complicated looking expression, but it is just nothing but, addition d cube 2 b plus a cube 2 b plus d, the Q value is omega times energy stored for which we had obtained this expression (()), divided by the rate of energy loss and this is what we have just now calculated. So, this divided by this and if you simplify then, this is what you will get, so this is, so in other words, if I know the frequency a and of course, the dimensions of the cavity give him the surface resistance, I can calculate how much is the loss.

As we have pointed out earlier, the cavity resonators are very useful, because they can be made to work at higher frequencies, and the amount of loss that you have there is much less than what you have, either in coaxial cable or in transmission lines. So, therefore, their good ways of storing energy, with this I come to a conclusion on the rectangular geometry, and I will not yet go over to since cylindrical or rather circular wave guides as they are called, because it is essentially a cross section is a circular, and the guide direction is as before is z direction.

The only thing that, I would like to point out is that what is generally of great use or what are known as optical fibers, these are actually dielectric cylindrical wave guides, but what I am discussing here is the situation where I have, the dielectric or vacuum or air in within, but on the (()) wall are still perfect conductors. So, therefore, I am still talking about wave guides in the same way as we talked about earlier, and we are simply assuming that the geometry is cylindrical. And the, this is not discussion of the optical fibers, because if you are discussing optical fibers what you require is the a dielectric wave guide.

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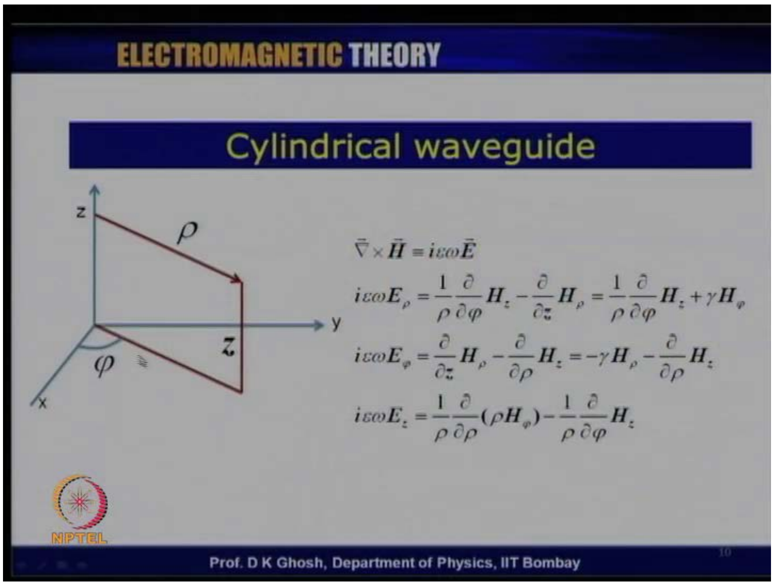

So, let us let us look at the geometry of this, so this I recall for you my cylindrical geometry I have x axis here, y axis there, so it is basically a polar coordinate, given by a rho and phi and of course, the z axis the z direction remains exactly the same. So, rho and phi are along the two dimensional cross circular cross section, and z is of course, along the right direction. So, what we will now do is this, we will write down the Maxwell's equations in the cylindrical geometry.

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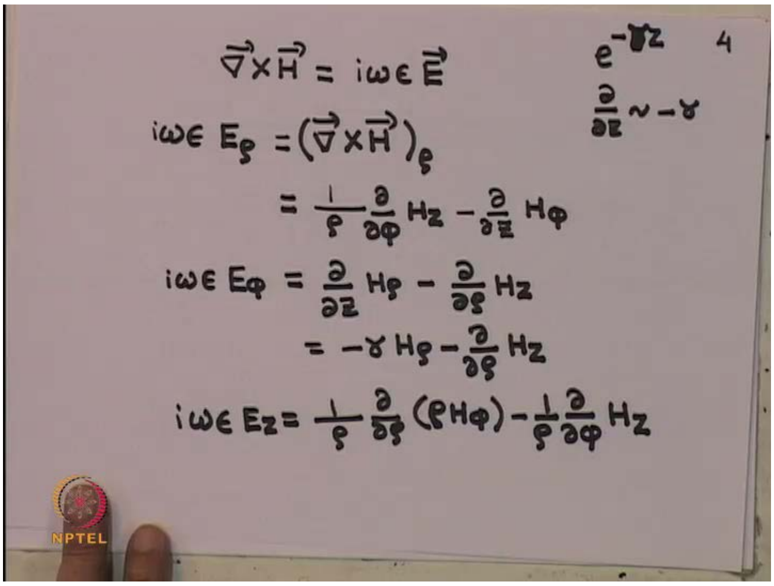

So, let us look at that, so let me let me take the Ampere's Law del cross H is equal to epsilon d by d t, so epsilon d by d t and we have seen that we are working with E to the power I omega t, so therefore, I have got I omega epsilon E. The writing down cross product is of course, fairly straight forward in the rectangular coordinates, the Cartesian coordinates, but we have to be slightly careful when you write down the cross product in cylindrical coordinates.

And it is primarily, because the one of the variables namely phi does not have the dimension of length, and so therefore, it is actually rho phi which has a dimension of length, and that is what makes things slightly different. Any way let us let us look at the rho component of this, I omega epsilon E rho, so this quantity is del cross H I will just illustrate one or two of them, and then you can do that, so this is del cross H s rho component. So, you realize that rho phi and z they form a trait, so therefore, I get d by d phi I got up, so it is actually d by 1 over rho d by d phi of H z minus d by d z of H phi.

Now, we will as before assume that d by d z goes as is the same as multiplying it with a minus gamma, that is because, we are taking the propagation to be given by E to the power minus beta z, E to the power minus gamma z. If gamma happens to be imaginary, then of course, there will be propagation, this is something which you will keep in mind. And now, look at what is I omega epsilon E phi, so that is d by d z of H rho minus d by d rho of H z, and this is d by d z, so I get minus gamma H rho minus d by d rho of H z

And likewise, the third equation I omega epsilon E z is 1 over rho d by d rho of a rho H phi you have to simply look up, the expression for del cross in cylindrical coordinate minus 1 over rho d by d phi of H z. The Faraday's Law equation, which gives me del cross E is minus d b by d t or minus mu times d H by d t is very similar in structure, but I will have del cross E is equal to minus i omega mu times H. And so writing down these equations will be exactly similar, the variables differentiations, they all remain the same, interchange E with H and remember, epsilon the omega goes to minus omega, because there is a minus sign in front of in the, Faraday's law, so this these are the set of equations that you have.

Now, our next job is exactly the what we have been doing, that is try to classify these modes, these in terms of modes. And we will again assume T E and T M mod, I will take one pair of equation, and illustrate that it can be done, and then you can similarly do it for the remaining.

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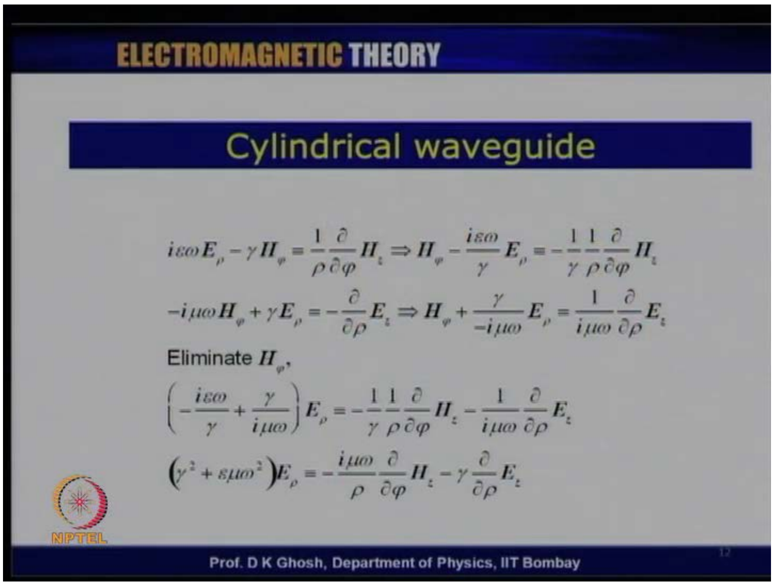

So, take for example, the the equation that we had written down are these equations, let me just go back.

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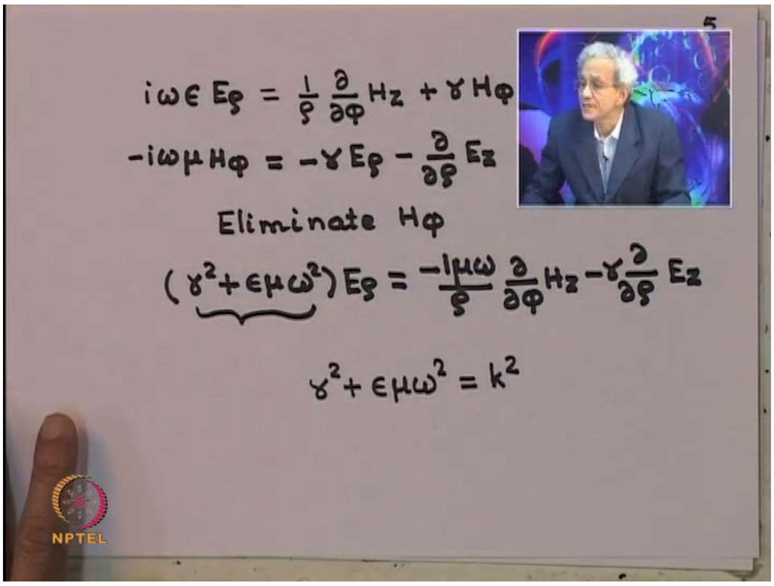

So, for instance the first equation here, I got i omega epsilon E rho, this is equal to 1 over rho d by d phi of H z plus gamma times H phi, this is one equation I will take; the other one is again the equation involving the same rho component, and the phi component, so I got H phi part here. So, I get minus i omega mu H phi is equal to minus gamma E rho minus d by d rho of E z, so I will I will simply play around with these two equations, and show and eliminate H phi from here. So, this is what is shown here, that if I want to eliminate H phi, I multiply this equation with minus i omega mu, and that that equation with a gamma, and usual subtraction.

And you can see that, this is the way the equation looks like, that is do a eliminate H phi, so I will be left with simply E rho E z and H z, E z and H z I want, so therefore, I will write this as gamma square, this is fairly straight forward algebra. Gamma square plus epsilon mu omega square E rho is given by minus i mu omega by rho d by d phi of H z minus gamma d by d rho of E z; and similarly, you can write down the remaining equations, which I have simply plugged it in here.

This quantity gamma square plus epsilon mu omega square is what I would designate as k square, so this is k square E rho equal to this and you can immediately see that this is the way the four quantities, E rho, E phi, H rho, and H phi will be expressed in terms of H z and E z, rather in terms of their directives. So, let us for instance talk about the T E mode for which E z is equal to 0, notice immediately this drops out, that drops out, that drops out, that drops out; so, you have things written in terms of the derivatives of H z only.

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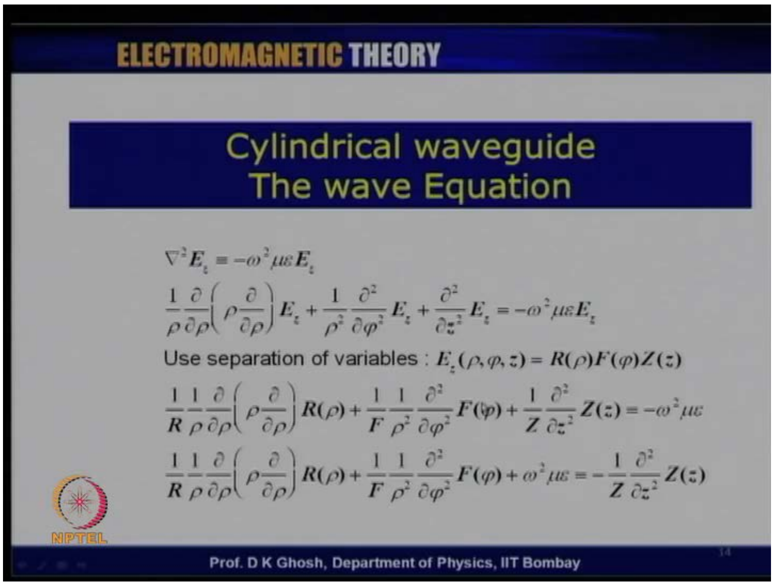

Now, the next job is exactly what you did earlier, namely I need to find out what my E z or H z happen to be. So, does not matter I have written it for E z, but since the equation is identical for H z, the same equation will be valid for H z as well.

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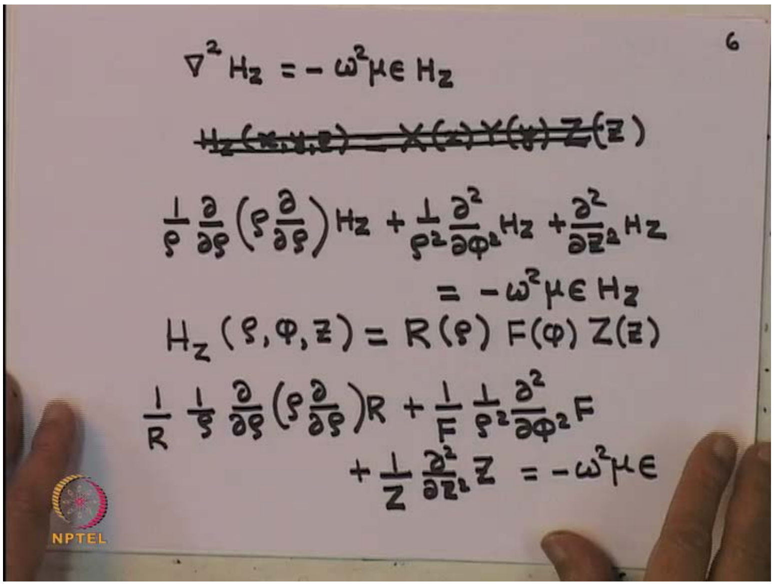

So, this is given by del square of any component of e, so in this case I have written down E z, but I could write down H z, because that is what I have to substitute there. So, this quantity let us write down H z this equal to minus omega square mu epsilon times H z, this equation is solved by a technique which we have been talking about earlier, namely separation of variables. So, what is done in that case is H z which is a function of x, y, z is written as a function of as x which a function of x, capital Y which is a function of y, and a capital Z which is a function of z.

So, if you substitute this here, remember del square is let us write down that in cylindrical coordinate, the del square is 1 over rho d by d rho of rho d rho by d by d rho of H z plus 1 over rho square d square by d phi square H z, and d square by d z square of H z, this is equal to minus omega square mu epsilon H z. So, substitute H z equal to this, and then divide all through by x, y and z, and then you get 1 over I am sorry, I made slight (( )) because, I am a working in not in rectangular system, but in cylindrical coordinate system.

So, let me write this as H z of rho phi and z is equal to capital R which is a function of rho, then I will use Q which is a, or F which is a function of phi, and a capital Z which is a function of z. So, having written this, I divide it by R, phi and Z, and you can see easily that this must be 1 over R the same thing, 1 by rho d by d rho of rho d by d rho of R plus 1 over F 1 over rho square d square by d phi square of F plus 1 over z d square by d z square of Z, that is equal to minus omega square mu epsilon since, I have divided both sides by R, F and Z, so on the right side I have nothing.

Now, then I argue the same way, that here I have a term which is a function of R, phi, z etcetera or at least these two terms, depend upon R and phi, and this term depends only on Z, and I want both these terms when they added together to give me a constant. So, that as a we have seen is the rather tall order, so that can be achieved if this term is a constant and pair of terms is also a constant, so let us do that.

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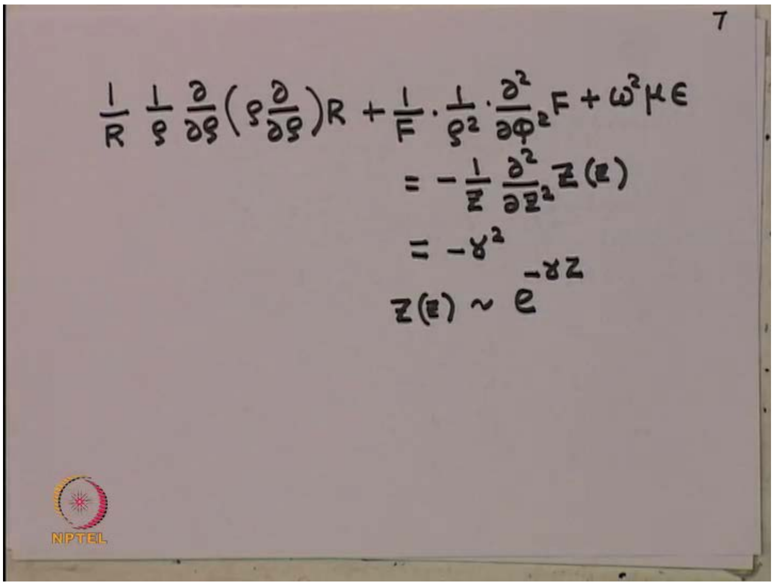

So, we rewrite 1 over R 1 over rho d by d rho of rho d by d rho of R, which is a function of rho plus 1 over F 1 over rho square d square by d phi square of f, let me bring the omega square mu epsilon term to this side, that is equal to minus 1 over z d square by d z square of Z; so each one of these terms must be a constant. So, let us do that, so firstly it implies that if this is a constant, let us call it as minus gamma square, I will tell you why I have done that, because I know the Z dependence of this equation, because we have seen that Z of z should go as E to the power minus gamma z.

So, therefore, this quantity, if you assume this Z dependence has to be equal to minus gamma square, so this is solved, but then this gamma square will be taken to the other side, and omega square mu epsilon plus gamma square will me a new constant. So, I will be left with if you refer to this I get 1 over R 1 over rho d by d rho this quantity plus omega square mu epsilon plus gamma square equal to 0, so this is the equation that I need to solve.

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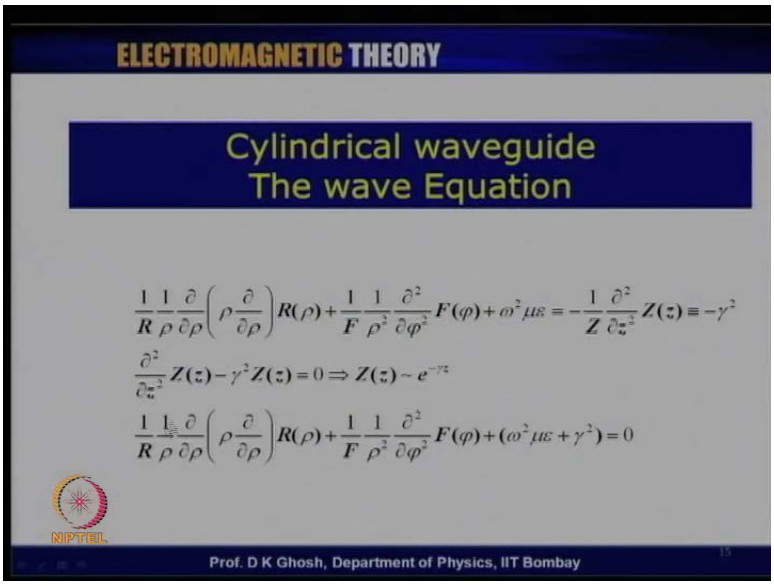

I have now what is to be done is this, I need to now separate this, equation into a function of F and a function of R, and to do that I do the following, the so you notice that, I had this was my equation 1 over R 1 over rho d by d rho, and here if you look there is a 1 over rho square there. So, clearly if I multiply this equation with a rho square all over, I will get 1 over F d square by d phi square F phi, and this term will then depend only on phi, and rho square will be multiplied there, and rho square will be multiplied here as well. Now, that will give me an equation of this type, I am not rewriting it, but us a just look at that equation again.

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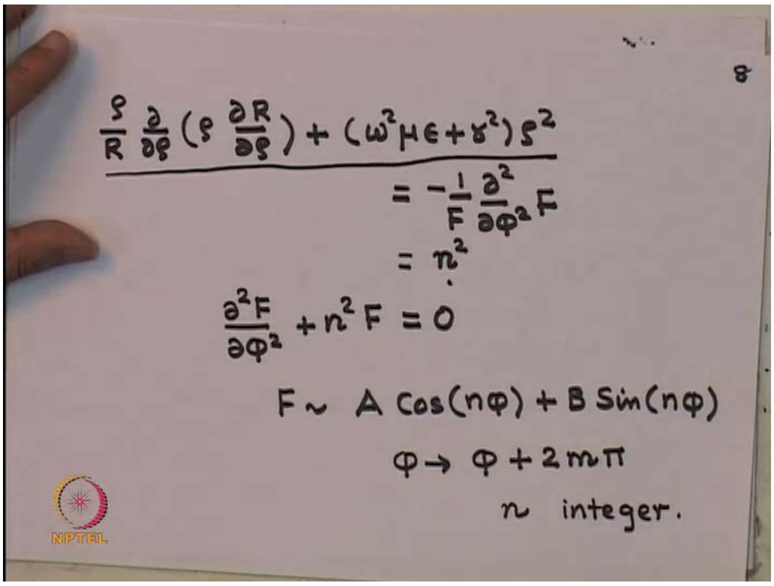

So, we have saying that rho by R d by d rho, I am multiplied the former equation by a rho square, rho d R by d rho plus this constant which was there omega square mu epsilon plus gamma square rho square that is equal to minus 1 over F d square by d phi square of F. The argument is identical this term is equal, to this term, so each one of them must be equal to constant, anticipating that this is a function of phi, I put it this constant to be equal to n square, what is n, I will talk about it later, but but that makes me solve this equation fairly literally.

So, which gives me d square F over d phi square plus n square F is equal to 0 which of course, has the solution that F goes as A cosine n phi plus B sin n phi. Now, since I know that if phi goes to, phi changes by 2 pi for example, if phi is equal to pi plus let us say two times some integer, m times pi, then the solution must be the same, because that is like you know coming circling the cross section once or twice. So, this tells me that n must be an integer in order that, this function is single valid, so we have said n is an integer, now once you know, n is an integer you need to solve this equation, that is this quantity is equal to n square, this is the equation I am interested in solving.

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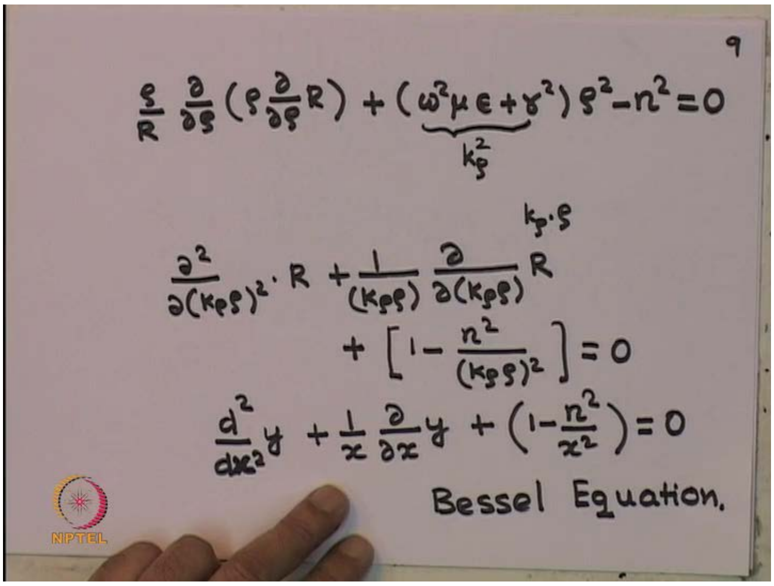

So, this equation is written like rho by R d by d rho of rho d by d rho of R plus omega square mu epsilon plus gamma square, this has been multiplied with a rho square minus n square is equal to 0, this looks a bad equation, but this equation has been known to us. So, what we do is this that firstly, this constant is appearing to often, so let us just call it k rho or rather k rho square, now if you now do a slight change in the variable, that is instead of rho being the variable you take the variable to be k rho times rho. So, what you can do is this, you notice that then I will get, this equation you can simply split it into two terms d by d rho of this.

So, I will get d square by d k rho whole square of R plus 1 over k rho rho d by d k rho rho of R plus, since I have divided everywhere by k rho square, I get 1 minus, the k rho into rho what I have taken as my variable, so I get 1 minus n square by k rho rho square equal to 0. Supposing, I put k rho is equal to some x, then this equation becomes a difference equation of this type, d square by d x square, this x as is not to be confused with the x axis, some variable x supposing R is written as y, then i get plus 1 over x d by d x of y plus 1 minus n square over x square is equal to 0, this equation is known as Bessel’s equation, this is the solutions of this are known as Bessel function, so this is this is, the nature of Bessel’s equation.

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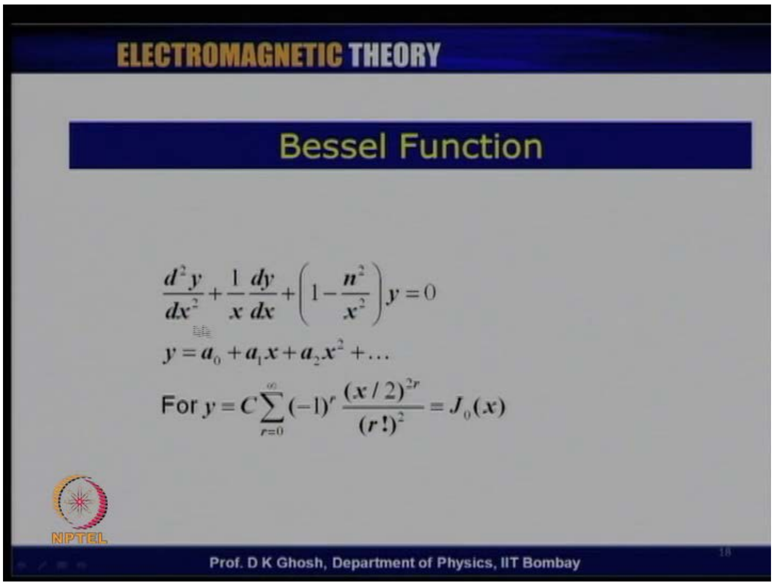

And for instance that you could take for instance n is equal to 0 just for convenience, you can see that this solution must be a power series in x, and the solutions of Bessel equations are given in terms of J n, that actually since, it is a second order differential equation. I have two solutions, one is called J n or the Bessel function, the other one is written as a N n or sometimes y n, this is called Neumann function, this is normal Bessel function; this is also known as Bessel function of first kind, and this is known as the

Bessel function of second kind. So, that the solution for this R is a linear combination of Bessel function of first kind and the second kind in general.

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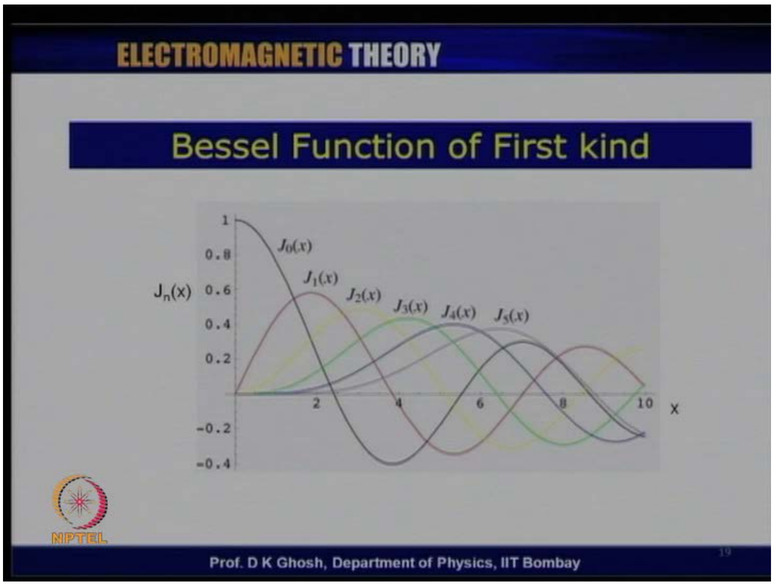

Now, if you look at the way the Bessel equations, Bessel functions look like, you will notice like Bessel function of first kind is well defined at the origin, and actually is an oscillating function.

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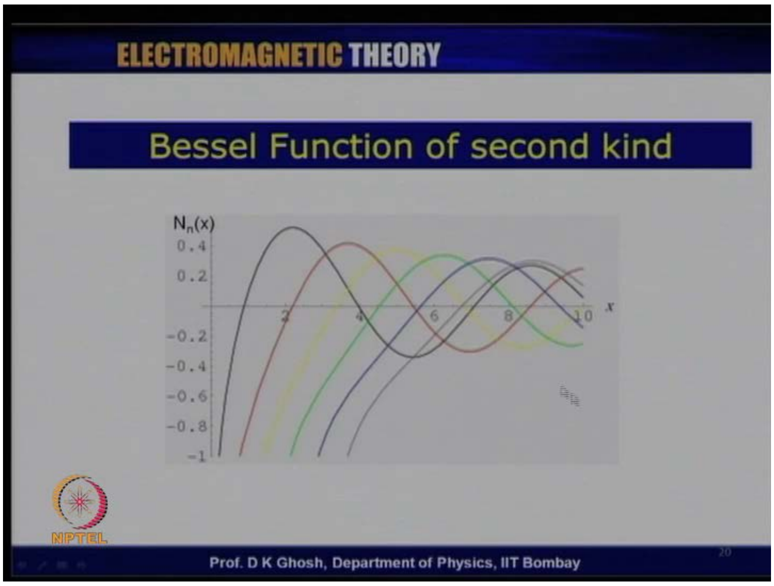

On the other hand, the Bessel function of the second kind diverges at the origin, and is also an oscillating function.

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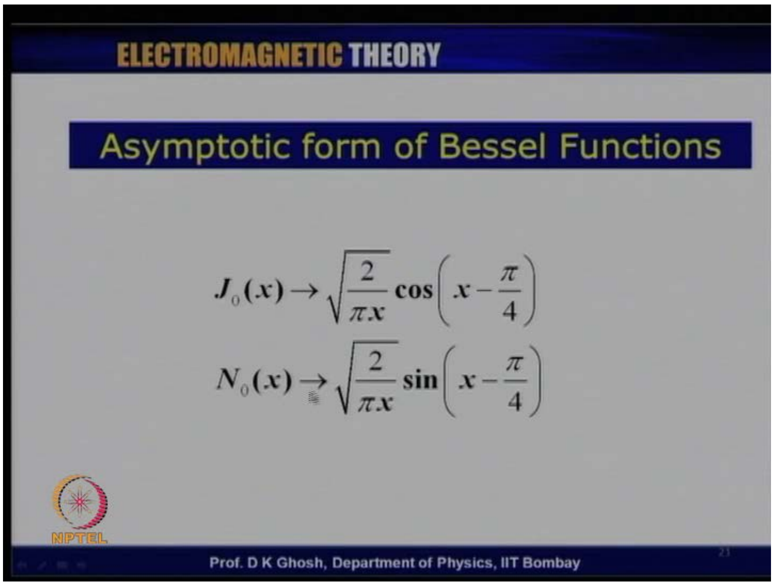

And we can sort of find out for instance, if you look at J 0 and n 0 their asymptotic form is this, these are oscillating forms. But I am looking for solutions which are finite at the origin, so therefore, I do not have this n contributing to my solution.

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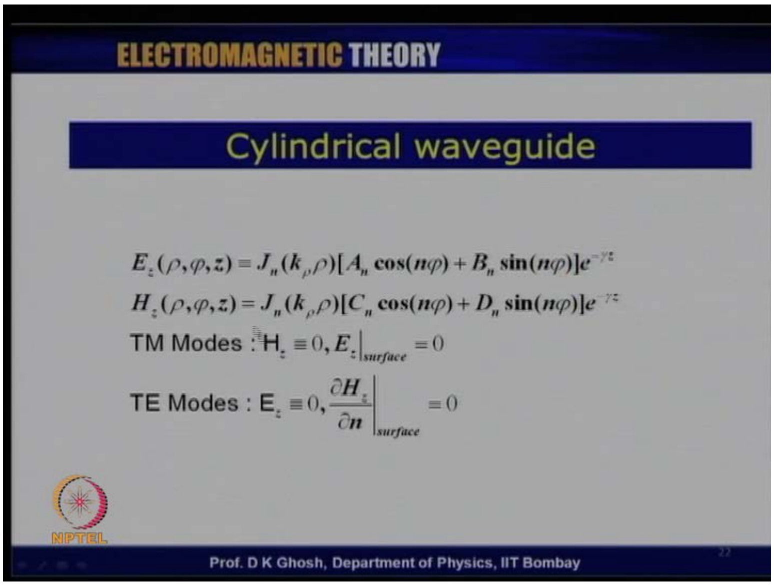

So, therefore, my solution for either E z or H z depend upon which mode you want, will be J n, n is an integer, a function of k rho rho, times this sin cosine which came as a solution on the phi equation, times E to the power minus gamma z. If I am looking at T E mode my E z is equal to 0, but the normal component of the magnetic field d H z by d n on the surface will be equal to 0; if I am looking for T M mode H z equal to 0, E z on the surface will be 0.

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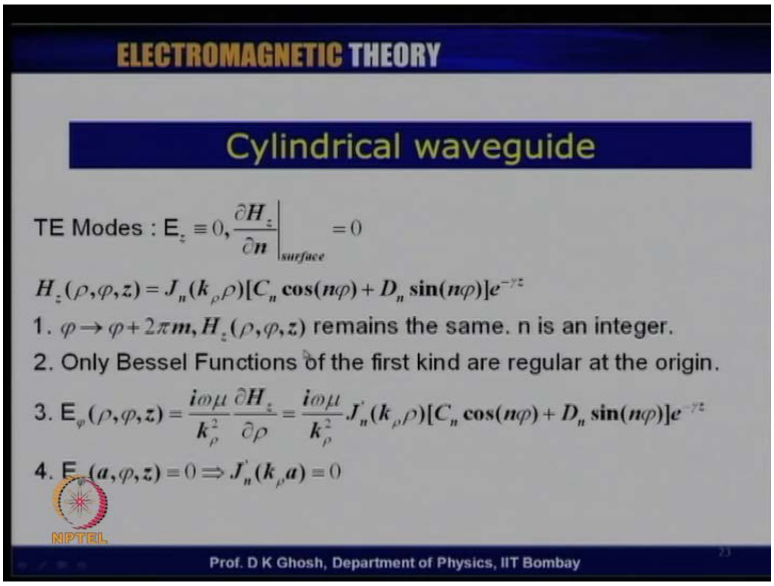

Let us look at the T M mode, so these are the conditions which I require, H z is this as I have earlier mentioned that I want H z must remain the same from phi becomes, you know changes by a multiple of 2 pi, so n is known to be an integer. Second point I have made is, only Bessel functions of the first kind are involved, so therefore, I have to only take J n, and that is why n I have removed. Now, this condition that if you take the normal component of H z, this is same as E phi, if you multiply with some constants, so you are differentiating with respect to rho that gives me a derivative of the Bessel function, and the remaining things are of course, already separated.

So, therefore, E phi at the surface, namely when rho is equal to a will be equal to 0 that is a requirement, because the normal component of the normal derivative on the surface must be 0. And if this is to be 0, it means my derivative of J n at k rho a must be equal to 0; now Bessel functions are some of the most well studied function, and so I have the incidentally I wish to point out that if you are looking at T M mode, the condition will not be on the normal derivative. But on E itself the tangential component of E in which, case you will not need the derivative of the, the zeroes of the derivative of the Bessel function, but you will need the zeroes of the Bessel functions itself.

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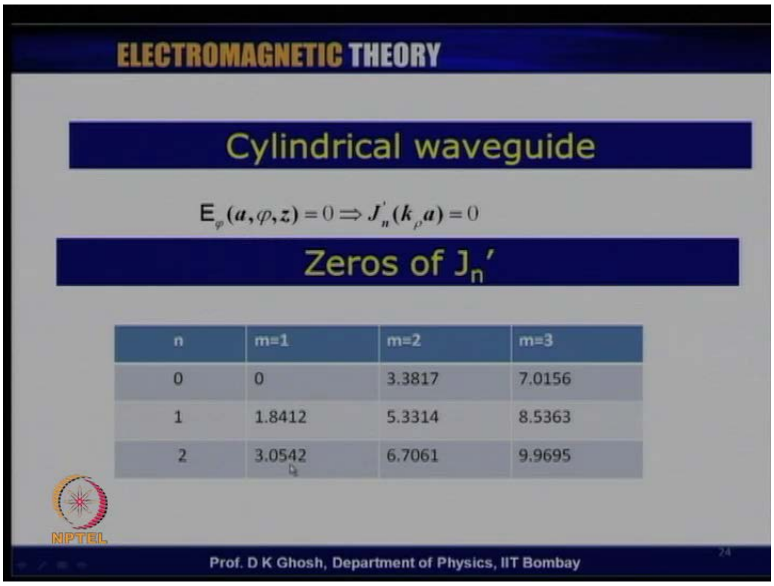

But, let us look at this, so I want E phi a phi z equal to 0, which requires me to take J n prime k rho a is equal to 0. Now, the zeroes of J n as well as J n prime are well tabulated, so remember n is an integer 0, 1, 2 this m here indicates the first 0, the second 0, the third 0, etcetera, actually this is only formally written down, this is not relevant. Because, if you take the first 0 of n is equal to 0, then you will find the entire field will be equal to 0, so therefore, this is actually the first 0 for J 0, and which occurs when the argument is 3.3817. And the, this is, so this will be called a T E 0 1 mode, and likewise this is T E 0 2, this is T E 1 1 2 etcetera, etcetera.

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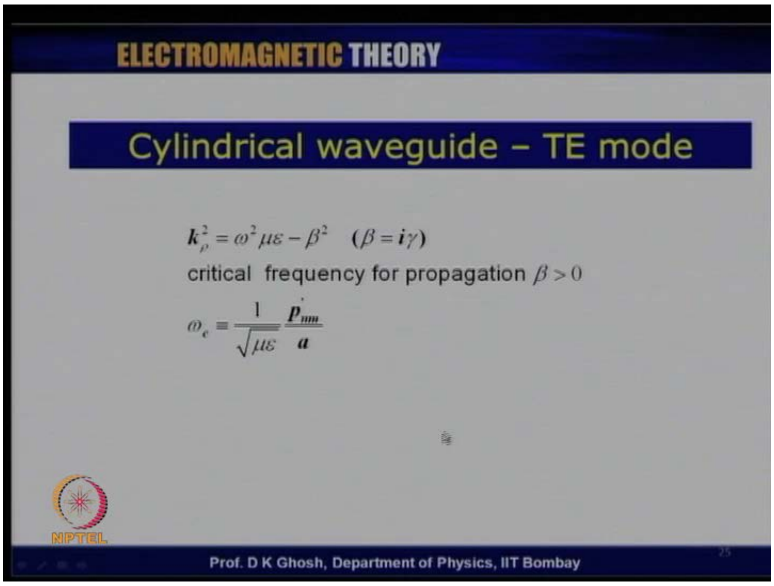

K rho square, now what I have done is instead of plus gamma square, because I am interested in the propagation, I have put it as equal to I beta gamma is equal to I beta, this is slightly reversed, so omega square mu epsilon minus beta square is what I have got. So, therefore, there is a critical frequency for propagation, because I want beta to be greater than 0, and clearly if I want this to be greater than 0, then there is a critical frequency, the which is given by because, beta square is omega square mu epsilon minus k rho square.

And I want that, then omega c should be well omega is written in terms of 1 over root mu epsilon times k rho, and we have seen that what the values k rho can take, which is P n m prime which is gives me the 0 of the Bessel functions of derivative divided by a. So, that gives you the critical frequency, above which there is transmission, this is simply taking those structure of various things.

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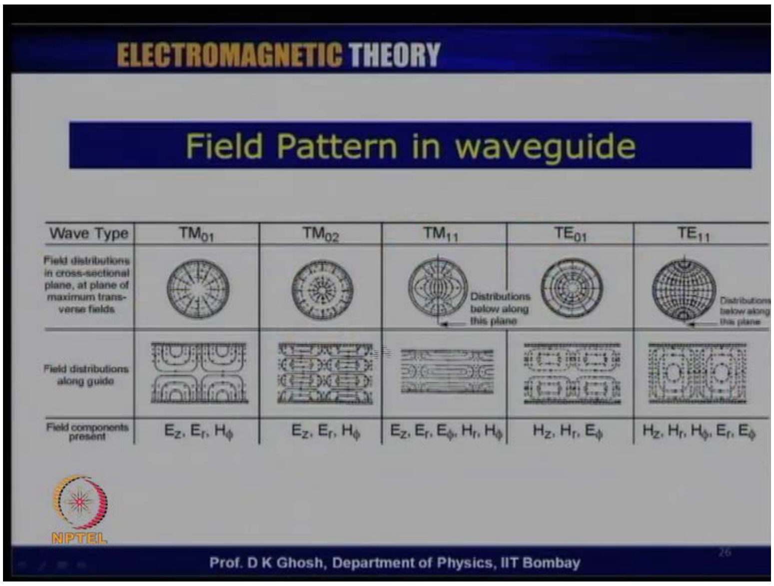

And trying to plot for example, one of these plots, let me take T is 0 1, this is a field distribution as you go along z, so notice that this sort of close back, and this is if you are on a cross section; theso these are these also tell you which components are non 0. So, with that we conclude our discussion of cylindrical wave guides, and we will spend the remaining two lectures in talking about elements of an antenna, which is used as a source for producing electromagnetic waves.