Electromagnetic Theory
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Module - 5
Lecture - 36
Waveguides

Till now we have talked about electromagnetic waves rather in particular uniform electromagnetic waves, its prorogation in free space and in isotopic homogenous medium. We have also seen that electromagnetic waves carry energy and momentum. Therefore, the energy of the electromagnetic waves, it should be possible to transfer one place to another. When you are working with low frequencies, typically less than about 200 megahertz, sending electric signals through either parallel transmission lines or coaxial cable is fairly common place.

However, once the frequencies become higher, one need special convicts to send electromagnetic waves from one place to another. And the method by which these are sent they are known as wave guides, and in this lecture we will be talking about the waveguides, which transfer electromagnetic energy or power from one place to another.

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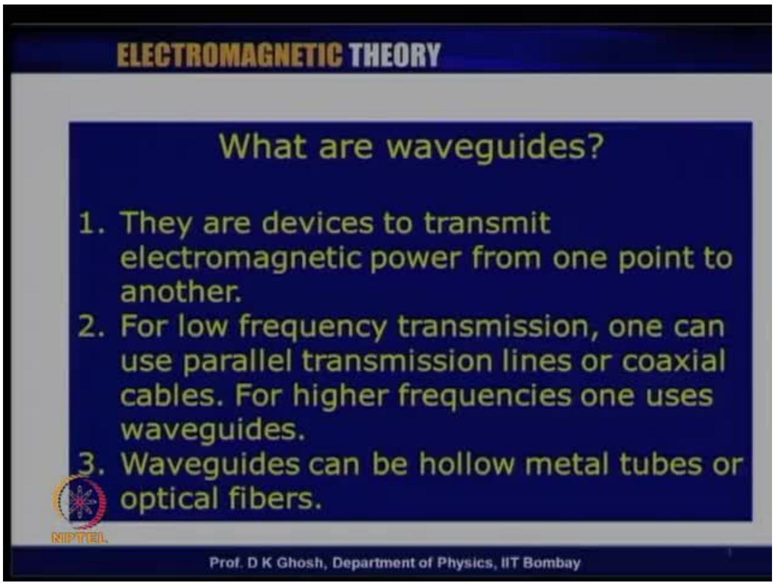

So, as we have pointed out a typical waveguide consists of a hallow metal tube, which is one of the ways of having transferring power. The other more common place today for example, optical fibers, which are widely used today to carry signals, the light signals for example, from one place to another.

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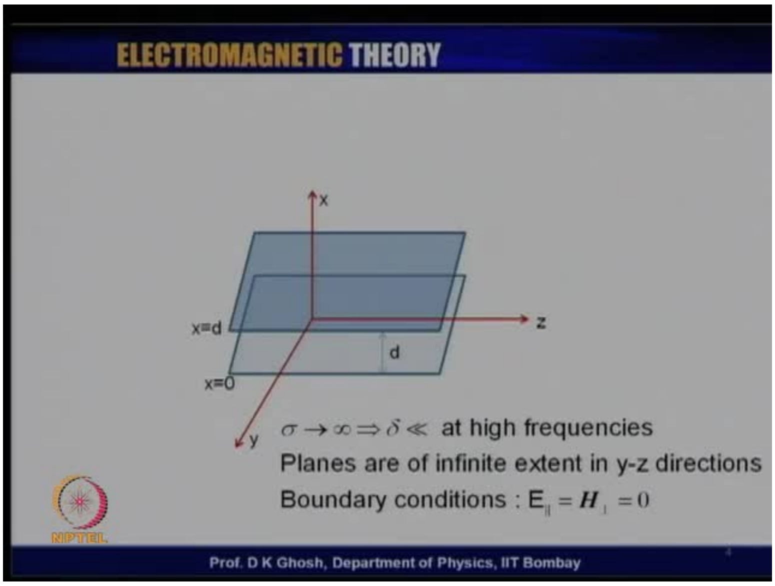

We will be in this lecture talking primarily about the first part and there in order to explain the basic principle what I have taken are two infinite metal plates. And basically what we have done is to, have one plate over the other separated by a distance d along the x axis, and the, well I have not shown the transmission section, but the power is being sent in the hallow space between these two planes. And it propagates along the z axis. The, between the metal plates there is basically empty space or here and the, so far as the metal plates are concerned, I will take their conductivity to be infinite, theoretically or in practice. What we require is that this skin depth in the material should be much lower, at very high frequencies.

Now, notice that what we have done is that the planes are of infinite extent in the y z direction. And I need to specify the boundary conditions. As we know that the parallel component of electric field and the normal component of the magnetic field, they must be 0 on the metal plates. So, basically we will start with equations governing the 6 components and we will put in these boundary conditions as you go along.

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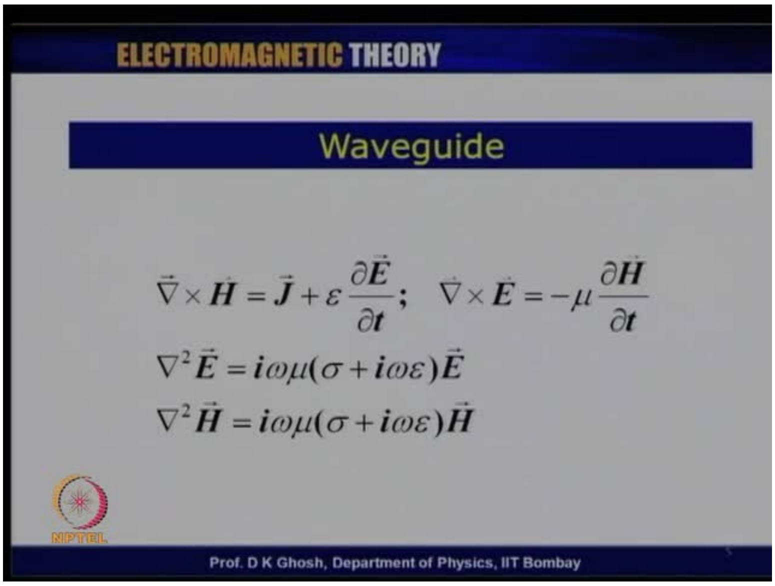

So, let us let us repeat the standard equations, that we have been taking about namely the Maxwell's equation. We will be primarily taking about the two Curl equation, so one is del cross H equal to J plus epsilon d E by d t and del cross E, that is the Faraday's law is minus mu d H by d t of course, del dot of H equal to del dot of E equal to 0.

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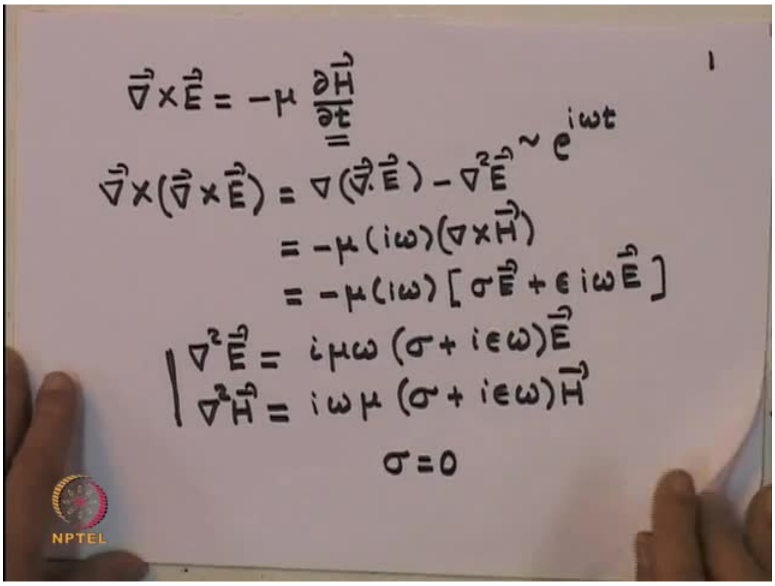

Now, the thing is that if I have this, let me start with any one of them. Del cross E is equal to minus mu d H by d t, the waves since I am interested in electromagnetic waves the time dependence is typically E to the power i omega t. Therefore, d by d t is the same as multiplying it with i omega. And let us take a del cross of this equation, we have seen that del cross del cross of a quantity is del of del dot of E, which of course, I know is 0 because of divergence of electric field is 0 minus del square of E and that is equal to minus mu d by d t which I have seen, shown is i omega times H.

So I got i omega times del cross H and that is equal to minus mu i omega and so I have got this del cross H is equal to J, which is sigma E and plus epsilon d E by d t is i omega times E. So, this is what I get for del square of E, so if you take care of this minus sign you get del square of E is equal to i mu omega times sigma plus epsilon i i epsilon omega times E. You can check that the magnetic field del square H also satisfies essential and identical equation, i omega mu sigma plus i epsilon omega times H.

So, these two equations and of course, these equations are are what we are going to be talking about. However, way you will be looking at the propagation between the planes and therefore, I will take this sigma to be equal to 0. So, between the planes sigma will be taken to be equal to 0, so let us then write down these equations, that we have. So, let us look at what we have got. So, let us start with for example, the equation on del cross H.

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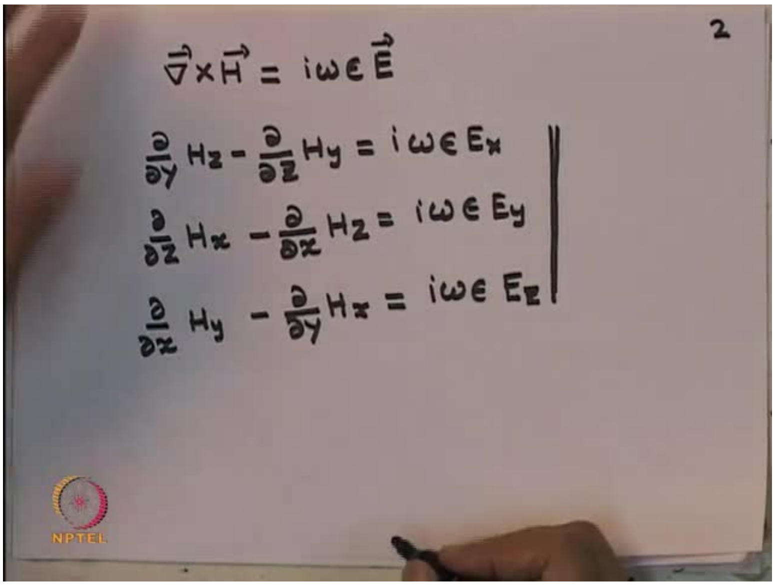

So, del cross H was sigma E, but then sigma E is 0 and I have got i omega epsilon E, this is my equation. And let me write this down in component form, so which means I will take x y and the z component. So, far as the x component is concerned, so I write del cross H s x component which is d by d y of H z minus d by d z of H y is equal to i omega epsilon E x, y component that is d by d z of H x minus d by d x of H z, that is equal to i omega epsilon E y. And z component is d by d x of H y minus d by d y of H x that is equal to i omega epsilon times E z. now, these are three equations and I have a set of parallel equations corresponding to the Faraday's law, namely del cross E is equal to minus mu d h by d t.

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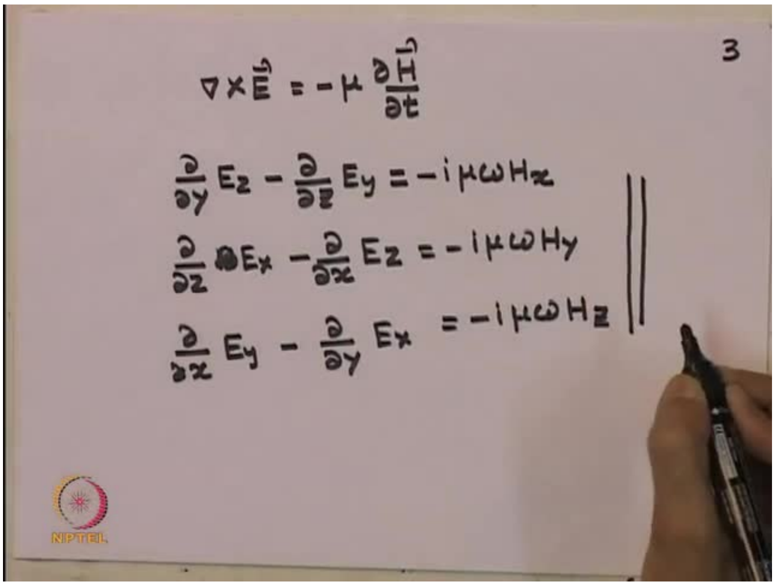

So, look at the corresponding x y and z component, I am going to write down three more equations. So, which is d by d y x components d by d y of E z minus d by d z of E y and this is i omega, so I get minus i mu omega H x d by d z of H x E z, sorry E x minus d by d x of E z equal to minus i mu omega H y. And finally, z component d by d x of E y minus d by d y of E x is equal minus i mu omega H z. So, I have got six equations, this is this is pair, first one pair and this is the second pair, and of course, I have the equation governing the del square of E and del square of H.

So, the what we need to do now, is to try to solve the set of equations, there are huge number of equations, that we will see what is the method of doing a you know systematic study of them. And then put in the boundary conditions namely the tangential component of the electric field and the normal component of the magnetic field, they should be 0 on the plates. Now, let me talk about some classifications. Now, what we will find, will show it later, that the set of equations that we have written down.

This is in terms of x y and z component of the electric field and the x y and z component of the magnetic field. We can express these or rather classify them into three distinct categories, they are done by defining, what are known as the longitudinal or the transverse modes. So, for example, in the present case my direction of propagation is the z direction. So, the z component of electric or the magnetic field is my longitudinal electric or the magnetic field components. The x and the y components will be known as the transverse components, in this case transverse meaning that that that directions are perpendicular to the direction of propagation.

So, let let us give some name, now what will see is this, that if you can define the z components. That is I can have a group of solutions, for which the longitudinal electric field component is 0, that is E z equal to 0. Now, what it means is electric field is transverse. Now, such a mode will be called transverse electric mode called t E mode now remember E z is equal to 0, will also imply H z not equal to 0. Now, if H z is not equal to 0, so t E mode is also known as the H mode, that is H, longitudinal H exits. Similarly, I can have a situation for which the longitudinal component of the magnetic field is 0, that is magnetic field is transferred and that is H z equal to 0, but E z not equal to 0. The corresponding mode is called the transverse magnetic mode, alternatively t m mode, alternatively.

Since, in this case the longitudinal electric field is not equal to 0, it is also called the E mode or the E wave. Now, in a in very special cases not always it is possible to have both E z and H z to be equal 0. This is a case which is very similar to propagation of uniform electromagnetic waves through space, were we have seen that the direction of propagation the electric field and the magnetic field that mutually perpendicular to each other. Now, remember because of the fact that you are confining or constraining, the electromagnetic wave to move in a particular direction, subject to certain bounded condition. A typical electromagnetic wave, which is been guided is not a transverse wave, in the sense that we have learnt about.

However, even in guided case, there is there are occasions, where you can have completely transverse wave that is both E z equal to 0 and H z equal to 0. Such a wave or a mode is known as t e m mode, namely transverse electromagnetic mode. Of course, there is a possibility that neither E z equal to 0 not H z equal to 0. This will be called a mixed mode, which you can write either as an E H mode or an H E mode, so these will be the basic way of classifying the modes, which are being guided.

Now, what will do is this, to illustrate our mathematical technique, we will take a simple problem first. And this is a parallel plate waveguide, the parallel plate waveguide meaning, that I have 2 semi-infinite plates, which are parallel to each other with a separation in the x direction equal to a, this is the picture that I had shown you earlier but let me write again.

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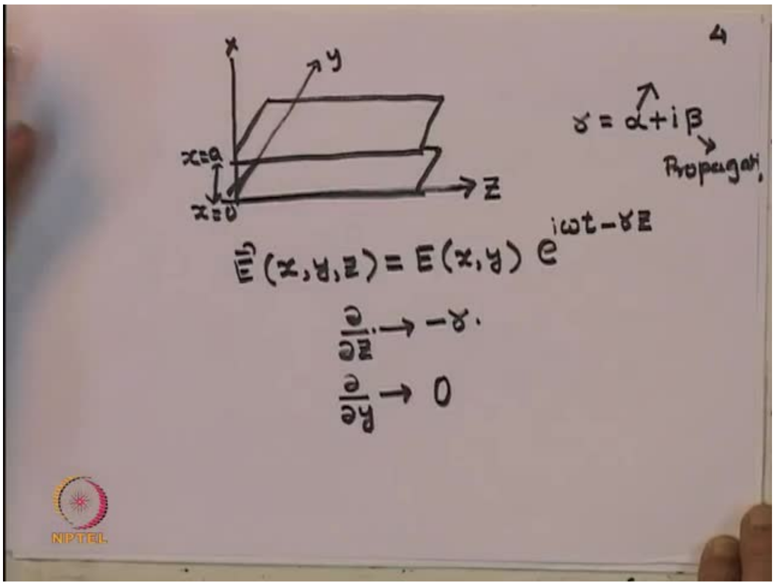

So, this is one plate and this the second plate and this distance is of course, this is the x direction. So, this is, I will take this to the x equal to 0, this as x equal to a, so this is my x axis, this should be my y axis and the direction of propagation, which is runs along the plates, these will be my z axis. Now, since the wave is being constrained to propagate along the z direction and the plates are infinite, so the only way the z dependence and come in in the electromagnetic field will be through a for example, take electric field which is a, x y z dependent and so there is a part which depends upon x and y, which we need to finalize, and there is a part which stands for the propagation as we have seen.

So, this I will take as i omega t as we have talked about earlier, minus gamma times z. This gamma is actually in general, if in the space between the media is a general space then this will be essential equal to alpha plus i beta. So, this is that attenuation part and this my propagation part, but since we will assume that this space between the two plates is essentially vacuum or sigma equal to 0, we will see that this is either propagating namely beta not equal to 0 alpha equal to 0 or attenuating that beta equal to 0 alpha not equal to 0. So, this is the structure, that is the z component of the wave is a propagating solution and of course, that leaves us with x y.

Now, this next thing that we do is this, This is basically done by observing that whenever we have a d by d z, we replaces this with a minus gamma times whatever you are differentiating on. So, d by d z of any component of E for instance will be minus gamma times that component of E. The second thing is that since the plates are of infinite extent in the y direction, there is no boundary condition to be met in the y direction. So, all y must be the same therefore, d by d y of anything is equal to 0. So, there is no variation with respect to y direction because of the nature of infinite extent of the plates.

If this is so then let me come back to the set of equations that I had and write down what does it actually imply? So, let us look at the first set if equations that we had for the electric field, these these were the set of equations that we had here. So, this d by d y of H z that is equal to 0 because I do not have a derivative with respect to y, I do not take. d by d z of H y will be minus minus gamma, so I will a plus gamma times H y.

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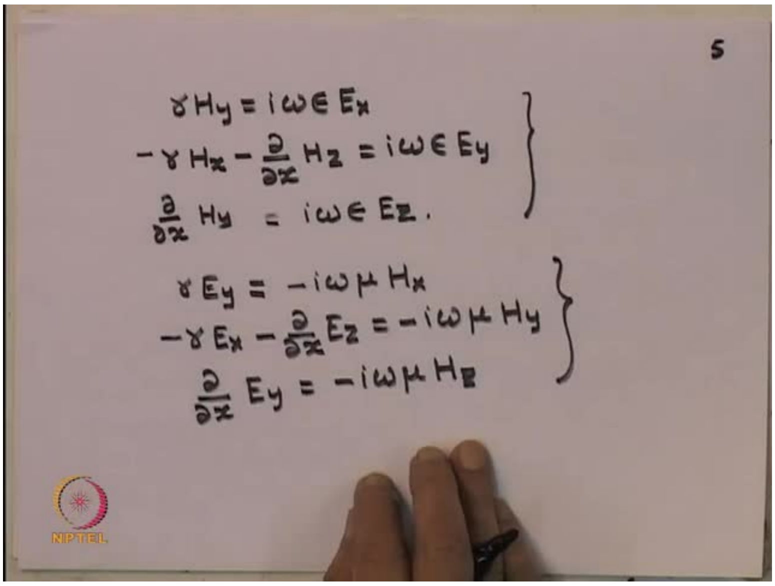

So, this equation gives me plus gamma times H y is equal to i omega epsilon times E x. Likewise I have d by d z of H x, so which gives me a minus gamma times H x minus d by d x of H z equal to i omega epsilon E y. The third one is d by d x of H y minus d by d y that term is 0. So, the this is equal to i omega epsilon E z, so this is one set of equation that I have. The other set which came from the corresponding del cross E equation, I will write down with the very similar way, that is using d by d y is equal to 0 d by d z is minus gamma.

So, that gives me gamma times E y, the equations are very symmetrical, excepting that you have to replace epsilon with a minus mu, that is because the Faradays law has a minus sign there. So, minus i omega mu and E with an H, H x minus gamma times E x minus d by d x of E z equal to minus i omega mu H y and finally, d by d x of E y is equal to minus i omega mu of H z. So, this is the second set of equations that I have got. Finally, the wave equation that I have written down, which was del square of E here, this was del square of E is i mu omega sigma is zero.

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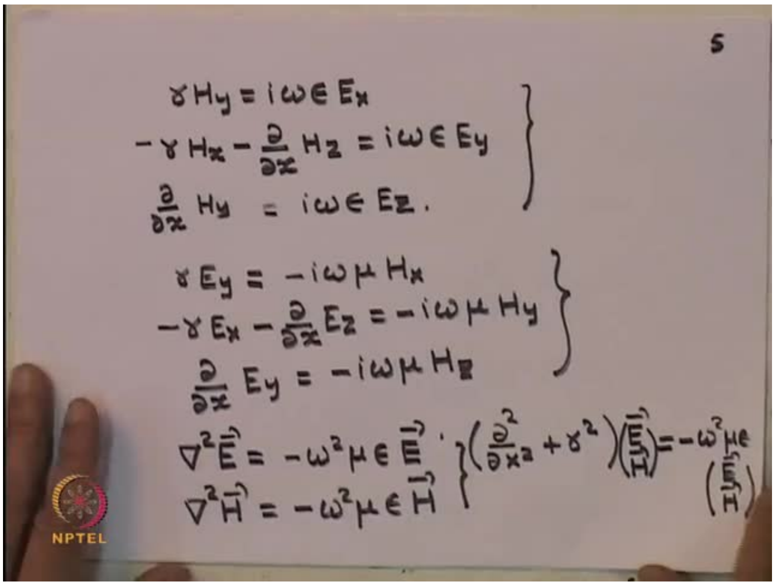

So, let us write it down, del square of E or del square of H does not matter, is i well… So, this is del square of E, so I get i mu epsilon, so there is no omega square there. So, i and i is minus 1, so that I am left with a minus omega square mu epsilon E. And of course, a very similar equation for the del square of H as well, so del square of H is minus omega square mu epsilon H. Now, remember that these are actually equations each one of is actually three equations because this is similar equation for E x, E y and E z and this for H x H y and H z.

So, basically this whole set of equations that we have got are our workhorses. And what we will do now is to see, how to symmetrically or systematically solve this separate equations. So, here again since del square of E is d square by d x square, let me just write down for one of them, plus d square d y square which is equal to 0 plus d square by d z square, since d by d z is minus gamma. So, I get gamma square either E or H, I do not care, that is equal to minus omega square mu epsilon times E or H.

So, the this is among the set of equations, so let us now try to find out, how to solve this set of equations? So, once we have said that we have a classification and the classification that will work out while we for example, I will work it out in detail for that T E mode.

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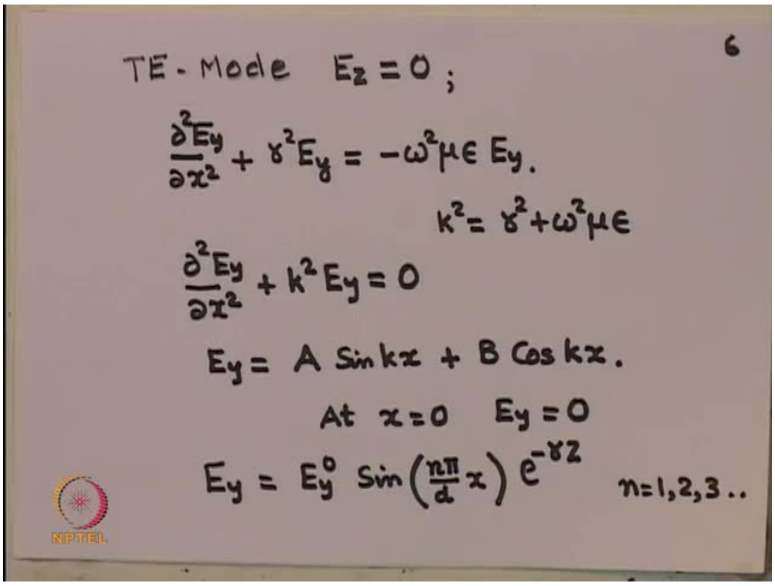

Remember T E mode, implies that the electric field is transfers the direction of propagation, which implies E z equal to 0. And what will now do is to write down, the equations corresponding to the six equations; that we have wrote down here. So, I have said E z equal to 0 and let us look at what it implies? This implies, let us look at here. So, if E z equal to 0, you can talk about this or this equation also has an E z. So, what we will do is this, that we will write down first the solution for E y, so notice that if E z is equal to 0, then I get H y equal to 0.

It is actually constant because d by d x of H y is equal to 0, but this is a constant and since we have said that it is independent therefore, this is, this will turn out to be constant. So, let us look at what it implies? So, we will write down d square E y by d x square. If E E x is equal to 0, then I get… So, we will this is equal this plus gamma square E y is equal to minus omega square mu epsilon E y. So, I can now define a new constant. For example, I will define k square equal to gamma square plus omega square mu epsilon.

So, this equation gives me d square E y by d x square plus k square E y equal to 0. This of course a very well know equation. So, I write down E y as equal to some A sine k x plus B cos k x. So, this is, but but you remember that, what we actually mean when we do this, is that the actually y is this quantity, multiplied by E to the power minus gamma z. Now, I also know that when x is equal to 0, that is on the plate my E y is equal to 0. So, at x equal to 0 E y must be equal to 0 because it is tangential component of the electric field.

Therefore, this constant B is equal to 0 and as a result my solution then becomes, full solution let me write down, is some constant which is E y 0 sine of k x, but let me keep it tentative for the moment times E to power minus gamma z. Now, this sin of k x I have to determine by realizing that at x equal to d, this field must also be 0 because x equal to d is the upper plate. Now, which means sine of k d equal to 0. Sine of k d equal to 0 means, k is given by some number n pi by d times x. So, n is a number which can be 1, 2, 3, 4 etcetera. It cannot be 0 because that would make the E y also identically vanish.

So, this is my solution for E y and using these equations, which we have written down earlier. I can now write down for example, what H x is, so H x is given by this is what I have got here. So, H x is gamma by minus i omega mu times E y and so therefore, I can write down how much is H x.

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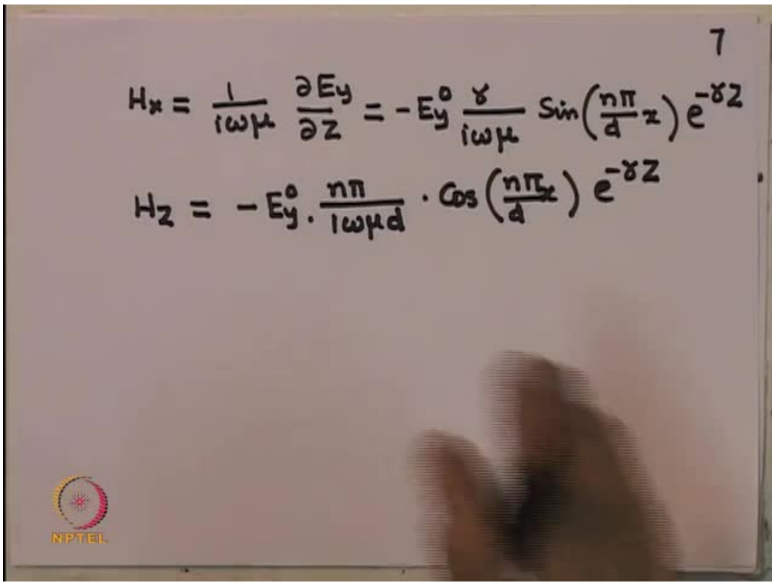

So, 1 over i omega mu and d by d E y, this is what I have determined by d z. And this is simply equal to minus E y 0 remember d by d z is just gamma. So, gamma by i omega mu times sine of n pi by d x times e to the power minus gamma z. I can write down H ,z remember H z is not equal to 0, this is very trivially. This is E y 0 n pi by i omega mu d times cosine of n pi by d x E to the power minus gamma z. So, these are the components the of H and of course, E i have determined. So, let us look at a specific case.

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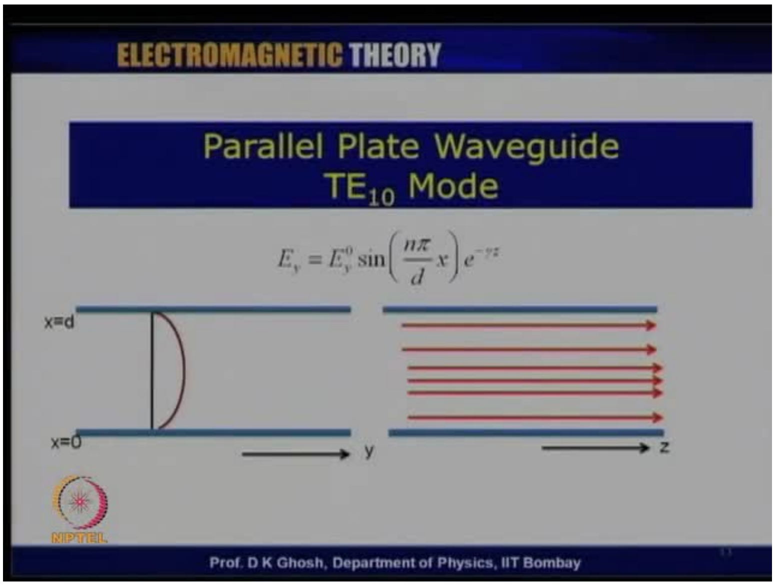

So, look at in this picture, I, we are showing E y equal to E y 0 sine n pi by d x E to the power minus gamma z. now, this is the cross section, this is the upper plate at x equal to d, this is the lower plate at x equal to 0. Let me look at, what is the distribution of E y? So, notice that it says E y must be 0 at both the plates and supposing I take n is equal to 1, then this is the way the field looks like, this is the way the field looks like that E y is this. Now, if your to look at the propagation direction, right?

Remember that this is E y and so if you are looking at this section, then of course, the fields sort of get crowed at the center and the fields sort of becomes, because they get crowed at this center of because the strength is much more here, and then they becomes past towards them. And this is the way T e 1 0 mode look like. A word about the nomenclature, this number n its number, I have told you it cannot be 0, so it can be 1, 2, 3, 4 etcetera. So, T e 1 0 is the lowest transverse electric mode.

The 0 is a written without any specific meaning at this movement, but that is because this is a notation, which is also used for the rectangular waveguides, that will be talking shortly. So, this 0 is to be essentially ignored and this is the one which stands for the value of n.

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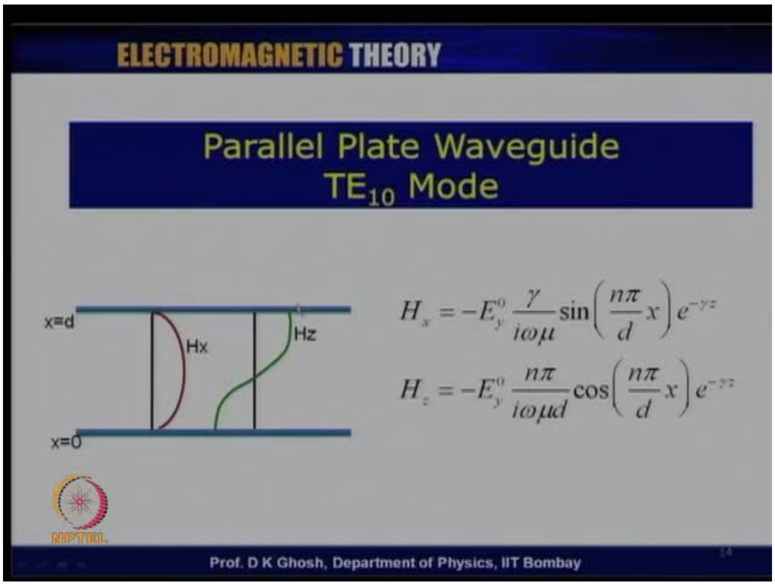

If you looked that, so we had drawn H x and if you looked at H z it is sort of looks like this, remember it was a cosine function. So, this is, these were the quantities, which we had derived jut now. So, this is the wave the H x and H z look like.

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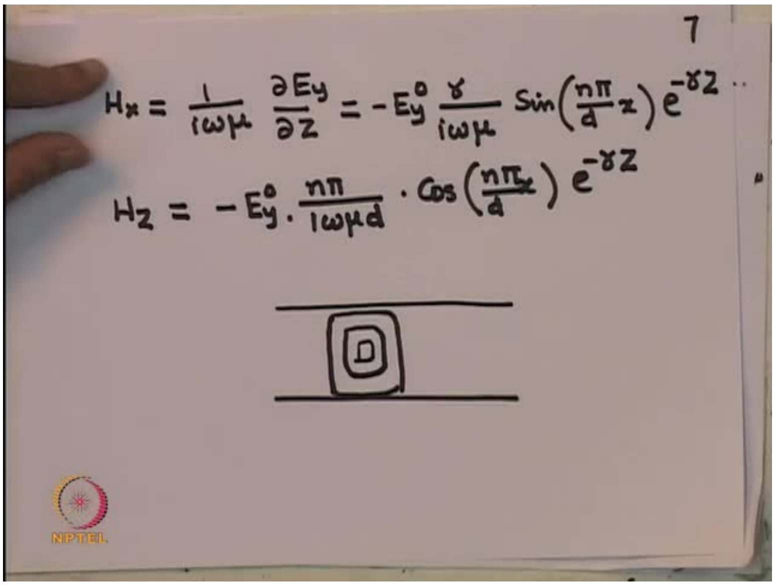

If you want to sort of plot along the axis, so what you will find will be that, this will sort of become… Remember that these are H field, so they become normal near this and out of this you will be closed to in its section like this.

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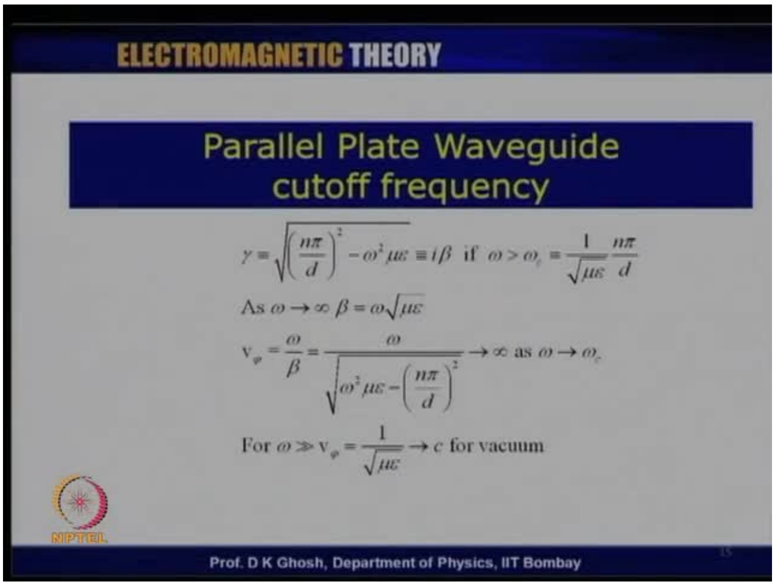

So, let us look at is such a transmission always possible. The first thing that to understand is this, that we had shown that gamma square, when we define k square in terms of gamma square.

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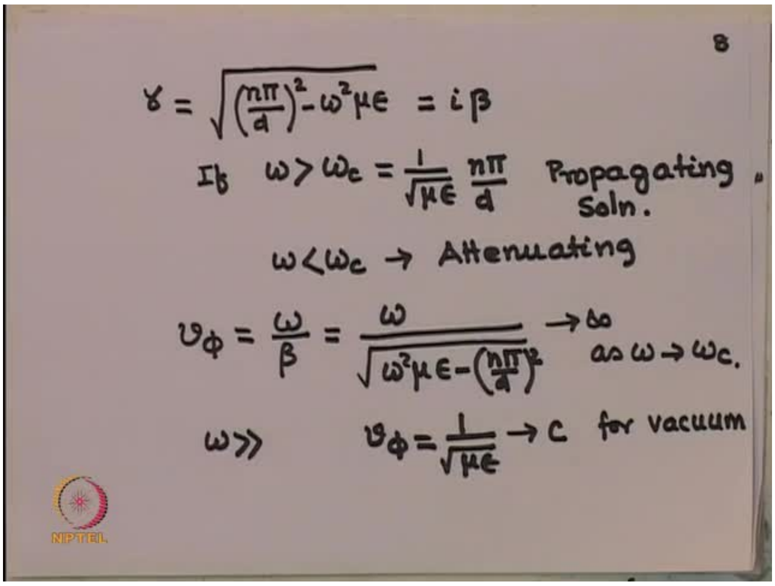

So, we had shown that gamma square or gamma square root of k square, which we have shown just now as n pi by d whole square minus omega square mu epsilon. Now, I want this to be a propagating solution. Now, if I wanted it to be probating solution, it tells me that the omega square mu epsilon term, must be greater than this term. So, this can be written as equal to i beta. Beta is the propagation constant, if omega is greater than some critical omega and this critical omega I define, when this quantity equals that quantity. So, in other words this is equal to 1 over square root of mu epsilon n pi by d.

So, if omega becomes greater than this quantity propagation takes place, so I have got propagating solution. If omega falls below that, I have attenuating solution, that is because in this case my gamma becomes real and E to the power minus gamma z. So, I can say evanescent or a decaying solution. The phase velocity if you like is, for the propagating solution is omega over beta. The incidentally, if you just find out the corresponding normal frequencies f and that would, that is then called the cut off frequency. So, omega by beta is given by, omega divided by square root of omega square mu epsilon minus n pi by d whole square.

You notice that as cut off frequency is approached, this phase velocity approaches infinity, as omega goes to omega C. And if omega is very large approaching infinity, then you find the velocity phase velocity is 1 over square root of mu epsilon because in that case this term can be neglected, and I will get 1 over square root of mu epsilon. If the space between the plates is vacuum or air then this will simply go to the velocity of light , this is for vacuum. So, basically what we have seen in parallel plate capacitor are two things, one that the modes can be classified into transverse electric or transverse magnetic waves.

There is a special case of T e m, which is same as T e 0 0 or T e m 0 0 is possible. We have seen that the frequency of the electromagnetic waves must be greater than a cut of frequency in order that such a transmission can take place. I will not be working out the, T e m mode in detail because the method is identical. You have to take H z is equal to 0, impose this similar boundary condition and we have done, and you can write down these components.

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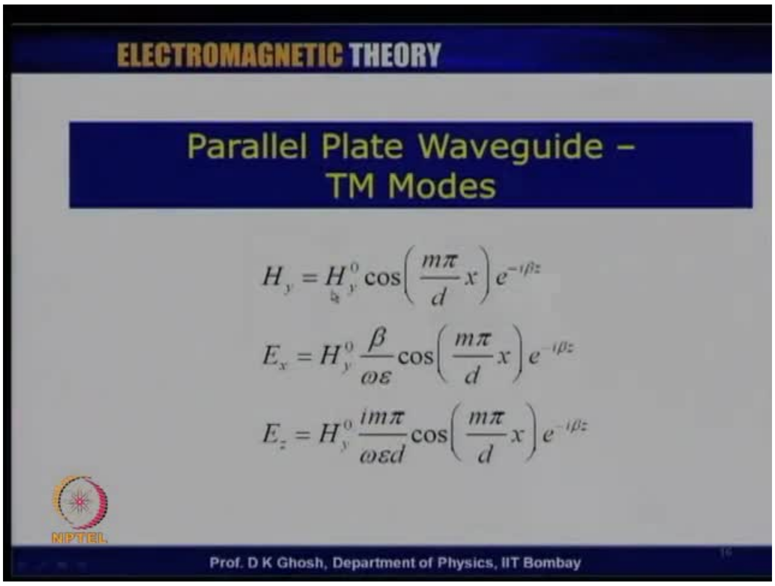

You can see that in this case your E z becomes a cosine function and in fact these are all cosine functions there.

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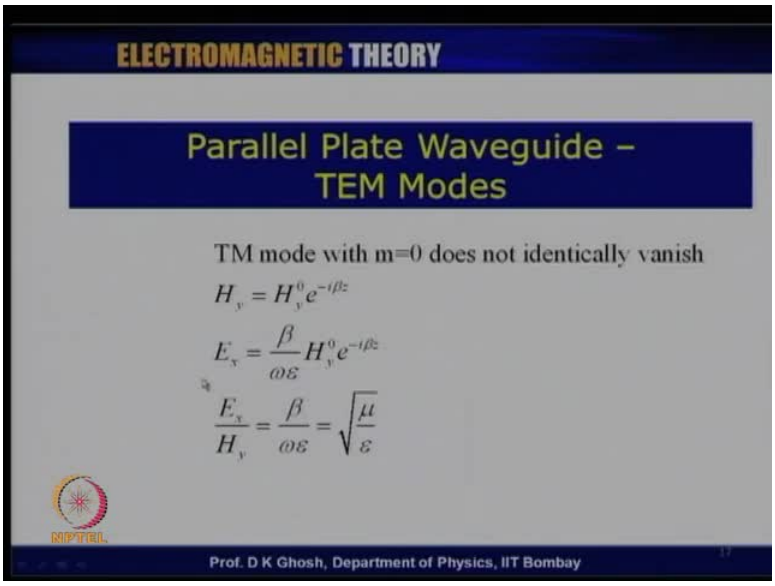

Incidentally we notice that in case of transverse electric wave because these were proportional to the sine function. The H z was proportional to the sine function, so I could not put n is equal to 0 there because that would have made all the field components vanish. Now, in this case I find that my solutions are in terms of the cosine function. Therefore, in principle I can have a m is equal to 0 here. Now, in such a case if m is equal to 0, then we look at this H y. So, H y becomes H y 0 and cosine is equal to 1, so H y 0 E to the power of minus i beta z.

E x becomes H y 0 into beta y omega epsilon and this 1 again and this is E to the power minus beta z. Now, if you take the ratio of the two and since m is equal to 0, E z is equal to 0. Now, E z is equal to 0, H z is equal to 0 because it is a T m mode, but not all components of the electric and the magnetic fields vanish. I notice that the magnetic field is along the y direction and the electric field is along the x direction. Now, if you take the two ratios, you find E x by H y become beta by omega epsilon, which is root of mu by epsilon. Root of mu by epsilon, if you remember is the intrinsic impedance, that we have talked about earlier.

Now, this mode the T m mode corresponding to m is equal to 0, which we would have normally written down as T m 0 0. This is identical to what we have been calling as T e m mode, namely transverse electromagnetic mode. Having done the parallel plate waveguide, this was done primarily to establish the notation and the method of doing.

Let me go over to a more practical waveguide. Here I have metal tube, but for convenience I have taken the cross section to be the…

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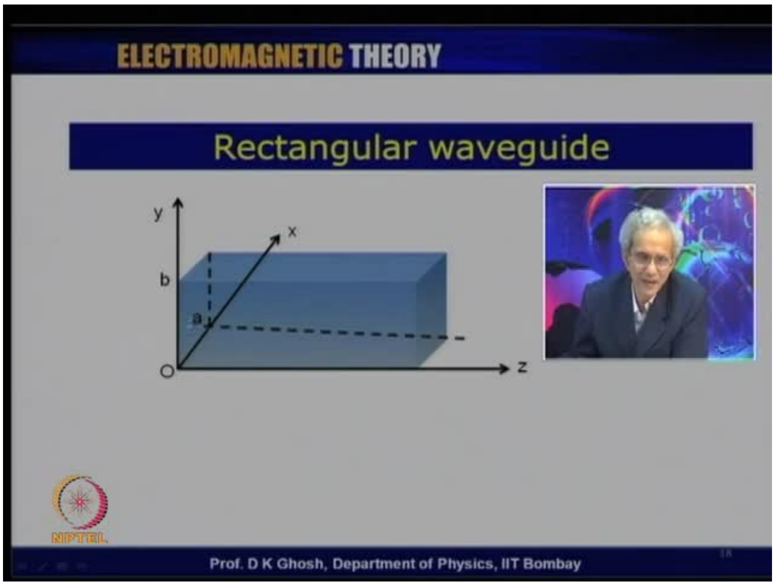

So, it is not a infinite waveguide. So, basically I have got the metal tube, but this propagation direction I will assume it to be very large. So, what I have got is this, that there are two plates there, separated along the x axis direction by an amount a separated along the y axis direction by an amount b. So, this is the origin and at x equal to a there is 1 and at y equal to b there is the other plate.

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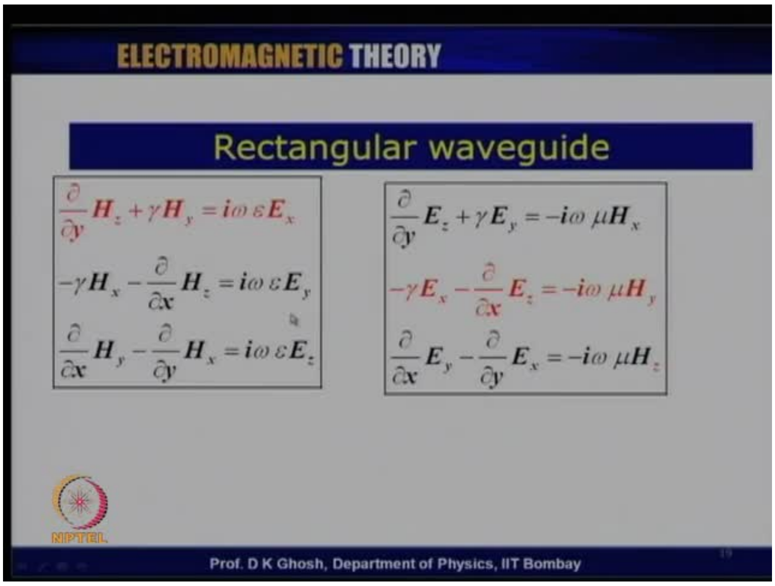

Now, we return back to the same set of equations, that we have talked about earlier, but earlier we had said d by d y is equal to 0. Now, my d by d y is not equal to 0. So, that term which we had put to be equal to 0, that is still there. d by d y of for example, this is the set of equations which we had, were d by d y of H z minus d by d z of H y and d by d z still is minus gamma. So, this is one equation.

There is a reason, why I have marked it with red, not anything special but I will be handing these pair of equations separately, then the same with the remaining components, so these are the equations that I have got. But let us look at this pair, that I have written down.

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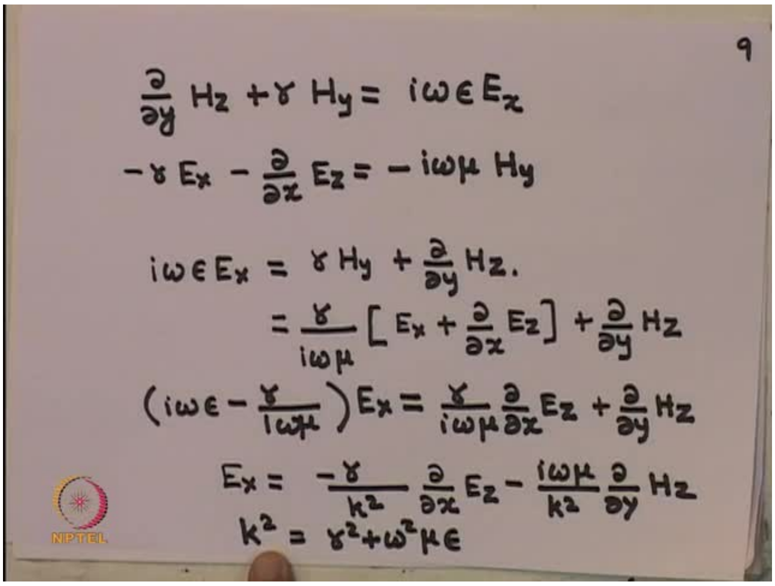

d by d y y of H z plus gamma times H y is equal to i omega epsilon E x. Here the other equation is minus gamma times E x minus d by d x of E z is equal to minus i omega mu times H y. Now, I am to use this pair of equations to express things in terms of only E z and H z and let us see how it is done? So, here I have got i omega epsilon E x, so let me write this. I got gamma times H y plus d by d y of H z and this is equal to gamma by… Let me go back to the set of equation, so what I am going to do is I am going to plug in here the expression for H y, which has E x and d by d x of E z.

So, this is gamma by i omega mu, this is this equation. I have got E x plus d by d x of E z and of course, I still have that second term d by d y of H z. So, this term E x, I bring it to that side, so I got i omega epsilon minus gamma by i omega mu of E x. This is written in terms of gamma by i omega mu d by d x of E z plus d by d y of H z. Now, notice what has happened, that E x has been written, in terms of derivatives of E z and H z. Now, you can take the remaining pairs and show that all the components, you should be, it should be possible to write it in terms of the derivatives of the E z and H z alone.

In other words E z and H z completely specify all the derivative, all the components that are there. So, let us simplify this, so this is, you can write this as E x equal to minus gamma by, just do a little bit of simplification, k square. I simply multiply this with this, you can see that this will become omega square mu epsilon plus gamma square with a common minus sign d by d x of E z minus i omega mu by k square of d by d y of h z, were I have defined k square as from here. That is and equal to gamma square plus omega square mu epsilon. So, this is my expression for E x in terms of derivatives of E z and H z. Now, I could do the same thing with other components.

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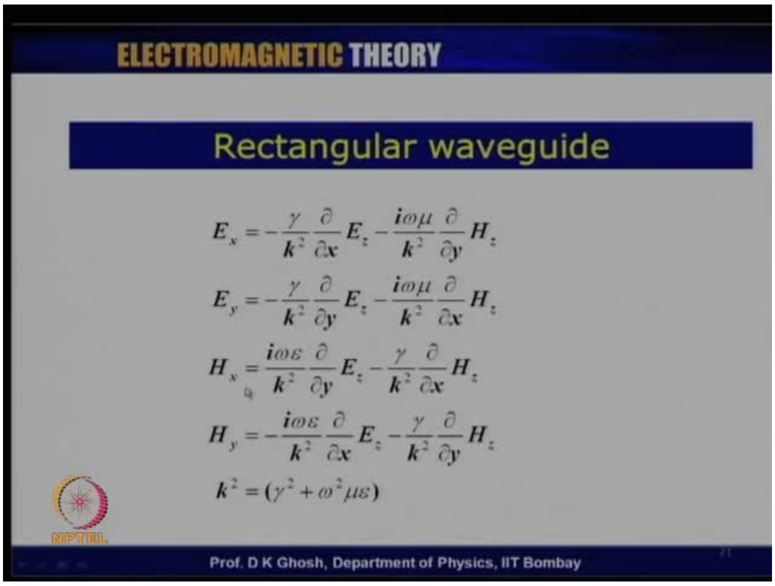

Get this whole set of E x, E y, E z, H x, H y and H z written in this fashion. So, I have got these the remaining things. Now, so what we have said is, that things are written in terms of E z H z or rather their derivatives and the remaining 4 quantities namely H x H y and E x E y, we have seen, can be written in terms of that one pair. I have shown one, I have shown the remaining. You should be in a position to work it out therefore, even in this case it seems reasonable to classify things, in terms of solutions for which, E z equal to 0, which you call has the transverse electric mode.

The solutions for which T m equal to 0, if H z equal to 0, namely the transverse magnetic waves. The first thing is the solutions, for which the transverse wave solutions, for which transverse electric mode exists, transverse magnetic mode exists. But in this case unlike the case of parallel plate waveguides, no transverse electromagnetic wave is possible, because we have seen that if both E z and H z are 0, then since all the components, remaining 4 components are return purely in terms of, the derivatives of E z and H z, all the field components will be 0.

So, in other words a rectangular wave guide of the type we have talked about, does not support T e m mode. So, the restriction is coming in now. Now, so will be talking now about, let us say T e mode. So, we will do exactly the same as, what we were doing earlier, namely we will write down the Helmholtz equation, remember this was del square of E.

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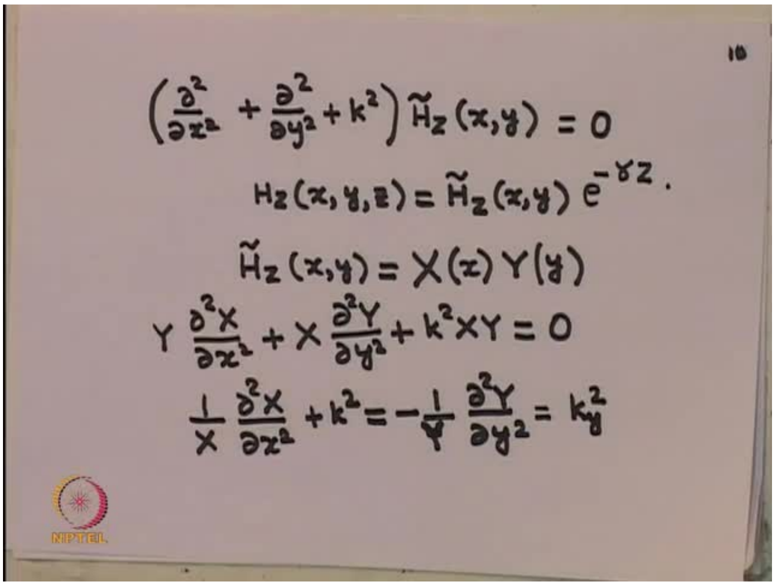

So, I get d square by d x square plus d square by d y square. Now, d square by d z square gave you gamma square, of a component let us say H z because my transverse electric mode I am doing, plus k square because on the other side I had omega square mu epsilon. So, I bring the omega square, minus omega square mu epsilon to this side and since k square is gamma square plus omega square mu epsilon, this is what I get. This times your H z, I will write it only as a function of x y because the z variation is simply e to the power minus gamma z, this quantity is equal to 0, this is our Helmholtz equation.

Now, actual solution is H z of x y z, which is equal to this H z, that I have written down. Then let us just put a tilda here, H z of x y times e to the power minus gamma z. So, when I have next job is to solve this equation, or if I am looking at that transverse magnetic mode, I will solve for the electric field. Now, the way to solve this equation is known as separation of variable method. So, what is done is, to realize that H of x y, I should be able to write it the product of something, which I have written as capital X which is the function of x alone and a capital Y which is a functional y alone.

Now, if you plug it in here, you find Y d square x by d x square plus X d square y by d y square plus k square x y equal to 0. Now, notice if I divide this equation by x y all over, I get 1 over X d square X by d x square plus k square is equal to, I take this to other side minus one over Y d square Y by d y square. Since, this depends upon X and this depends upon Y, each one of them must be constant, because otherwise such an equation cannot be valid, for all values of x and y. So, we will write this as a constant and which I will call them call it as k y square. Now, what I will do next is to take this set of separation of variable equation, obtain the solution for capital X capital Y plug it in here and then substitute the necessary boundary conditions.