Electromagnetic Theory
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Module - 4
Time Varying Field
Lecture - 35
Propagation of Electromagnetic Waves in a Metal

We have been talking about the propagation of electromagnetic waves, mostly in vacuum and also in linear dielectric media. What we want to do today is to talk about propagation in a conducting medium and also what happens, when electromagnetic waves falls on the interface between dielectric like for example dielectric like vacuum or air and a conductor. So, this is basically what we will be talking about, but let us for the moment think in general terms and see what happens when an uniform plane wave is propagating let us take it along the z direction, and we will assume systems are linear, so that we have electric field, the magnetic field and the direction of propagation are still mutually perpendicular as we have talked about.

And for being specific, let us say that the electric field is along the z direction, the electric field is along the x direction, the wave is propagating along the z direction and the electric, the magnetic field is along the y direction.

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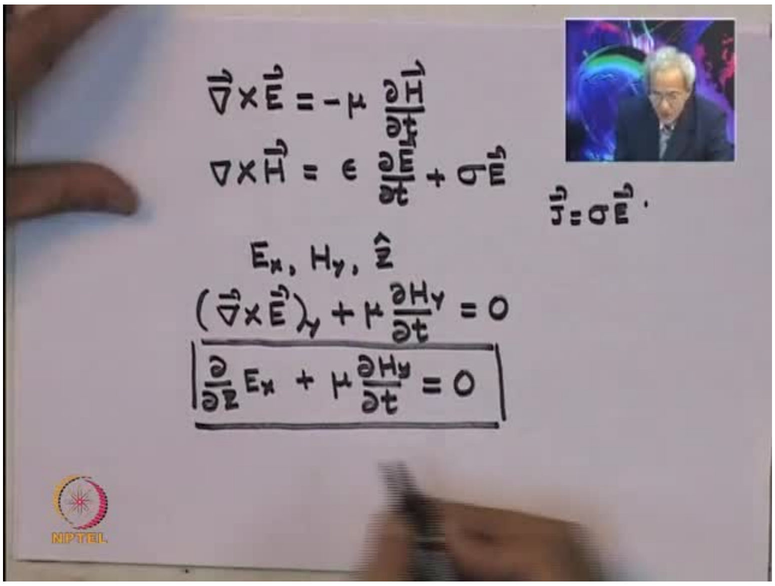

So, we we come back to the set of Maxwell's equations, we had the del cross of E which is minus d b by d t, but since we have said this is a linear medium, so we will write it as minus mu d H by d t and del cross H is well, d b by d t which is epsilon d E by d t plus J.

Now we will assume that our conductor is a ohmic conductor, so that J is equal to sigma E is what we will take. So, sigma is the conductivity, so J is equal to sigma E is what we have taken. So, since we have said that electric field is along the x direction, the magnetic field H is along the y direction and the direction of propagation is along the z direction. Let us take the appropriate components of this equation. So, the appropriate component of this equation would be, since H is along y direction, I will say that del cross E is y component, I will bring this term to the other side, mu times d H y by d t that is equal to 0 and del cross E is y component is d by d Z of E x minus d by d x of E z.

But I do not have an E z, so this plus mu times d H y by d t is equal to 0, this is one of the equation. The second equation is obtained by, rewriting the, this equation because we have said E is along the x direction, So, I will take del cross H's x component.

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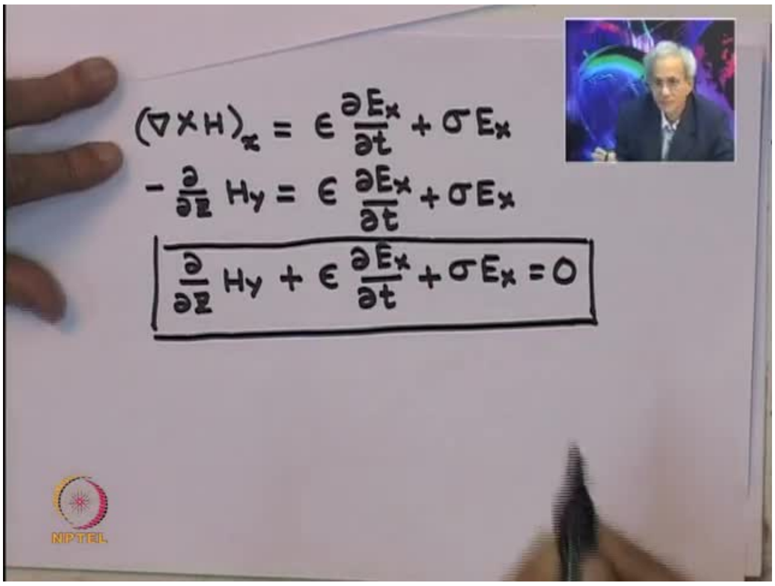

So, I will take del cross H's x component and that is equal to epsilon d E x by d t, I had sigma E, so which is sigma times E x. So, del cross H's x component is d by d y of H z and I do not have an H z minus d by d z of H y and that is equal to whatever we have written down d E x by d t plus sigma E x. So, this is the pair of equations I have got connecting E x and H y and so let us write them together. And I will write this as d by d z of H y plus epsilon d E x by d t plus sigma E x equal to 0.

So, these are the two equations, which we will be handling. This is and this equation. Now, what we, what to do now is, that since I know that I am looking for harmonic wave solutions, so I will assume that E and H go as E to the power i omega t.

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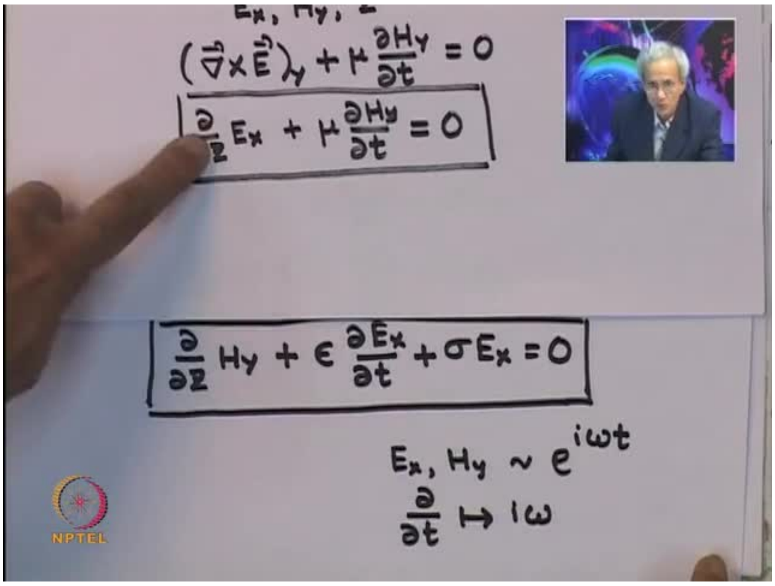

So, if you take E x and H y their time dependence goes as E to the power i omega t. d by d t essentially means multiplying by i omega, therefore these pair of equations, the time dependence is removed I will be rewriting these two equations like this.

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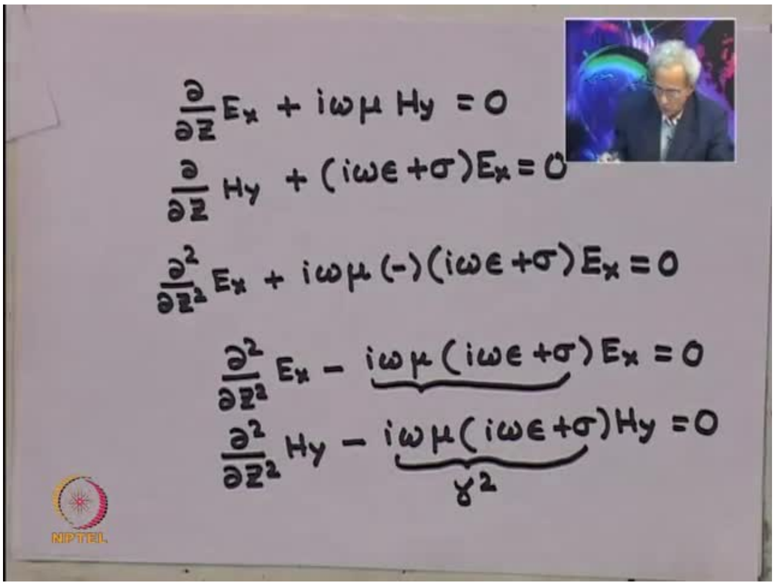

So, I have got d E x by d z plus i omega mu H y equal to 0 and the other equation is d by d z of H y plus I have got two terms in E there. So, I have got i omega epsilon and plus sigma E x is equal to 0. So, this is the pair of equations that I need to solve. So, as we have done several times, the way to solve these equations coupled equations, is to take a further differentiation of any one of these equation. For example, if I took a differentiation of the first equation, I will get d square by d z square of E x plus i omega mu d H y by d z and d H y by d z will give me minus sign i omega epsilon plus sigma times H y E x, that is equal to 0.

So, d square by d z square E x minus i omega mu into i omega epsilon plus sigma of E x is equal to 0. Parallelly we could get all most an identical equation for H y which will be d by d z square of H y minus i omega mu into i omega epsilon plus sigma H y equal to 0. I need to solve, so these are of course, decoupled now. But of course, we have to pay a price these are the second order differential equation and therefore, let us let us define this quantity here, or this quantity as equal to gamma square.

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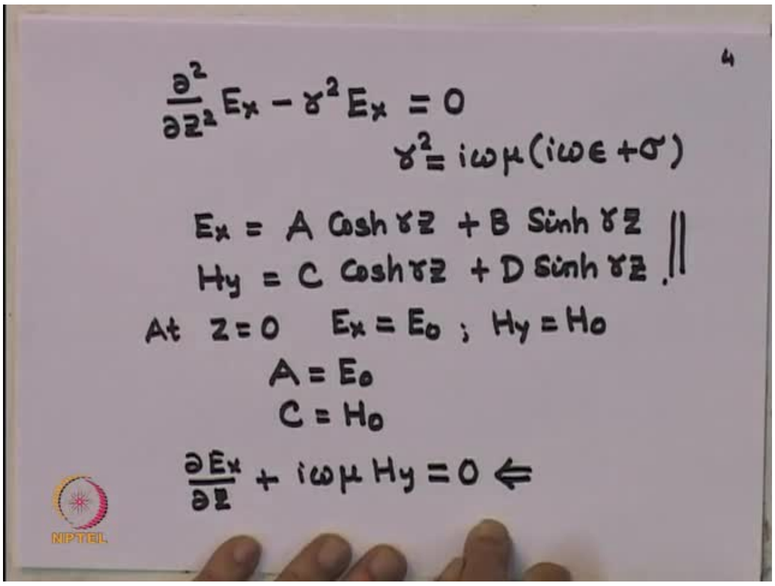

So, that gamma square is defined by this and my equations become d square by d z square of either E x or H y, I will not it twice, minus gamma square H y E x is equal to 0, where gamma square is a complex quantity, i omega mu i omega epsilon plus sigma. The the solutions of this equation is of course very well-known, we could because this is d by d z square is proportional to gamma square. So, the solutions are in terms of hyperbolic functions of cosine hyperbolic, of gamma z. So, let us write down E x is equal to some constant A cosine hyperbolic gamma z plus let us say, B sine hyperbolic gamma z and parallelly H y will become C cos gamma z plus d sine h gamma z.

Now, obviously we need to determine the constants A, B and C and to do that let us put in a condition, that supposing at z equal to 0, let us say E x is equal to E 0, so if you say at z equal to 0, E x is equal to E 0 this immediately determines the these two constants. So, A is equal to E 0 and C is equal to H 0 and I am saying H H y is equal to H 0. So, A is equal to E 0 and C is equal to H 0. I still have two more constants to determine and for that what I do is to go back to the first order differential equation that we had, where we related the curl of E with d B by d t or things like that.

So, for instance we had an equation which said d E x by d z plus i omega mu H y was equal to 0. Now, what we do is this, that we plug in these solutions into this equation so if I do that, that will give me a relationship between B and d. Let us look at what it gives me? So, d E x by d z remember that cosine hyperbolic and sine hyperbolic, they just when you differentiate them, cos hyperbolic gives you sine hyperbolic and sine hyperbolic gives you cos hyperbolic, there are no no sine changes.

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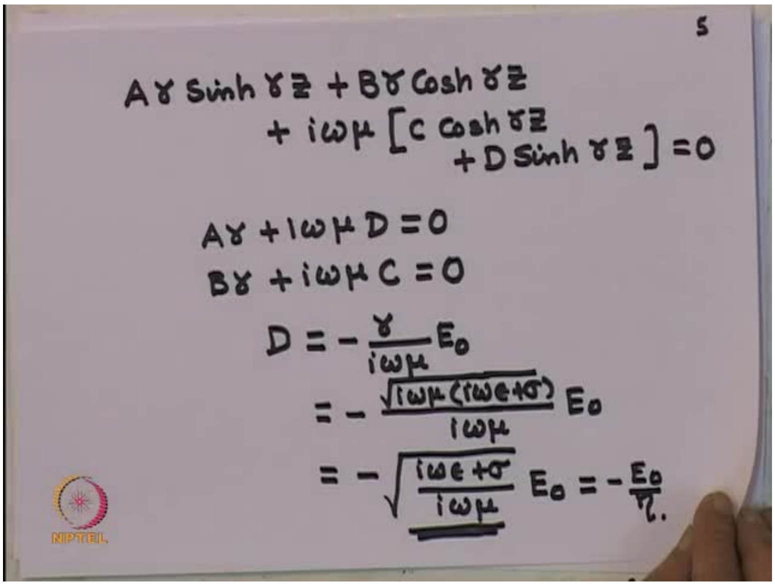

Therefore, I get d by d z of E x gives me A gamma sine hyperbolic gamma z plus B gamma cos gamma z plus i omega mu C cos gamma z plus d sine gamma z. Out of that you remember that, we have already determined that A is equal to E 0 and C is equal to H 0, so this relationship which has both sine and cosine and can be valid for all z. If I say that A gamma plus i omega mu times d is equal to 0 plus b gamma plus i omega mu C is equal to 0, so that gives me that d, which I needed to determine is minus gamma by i omega mu times A. But then A we had seen to be equal to mu 0.

So, this is the relationship between d and A, but let us put the gamma value there. Remember gamma square was shown to be equal to i omega mu into i omega epsilon plus sigma. So, this is square root of i omega mu into i omega epsilon plus sigma divided by i omega mu times E 0 which gives me square root of i omega epsilon plus sigma divided by i omega mu times E 0 that is my d. And this quantity that we have got here, I will denote it by 1 over eta, so this will be written as minus E 0 divided by eta. I repeat the same thing, what with respect to the second equation and I will then relate the, I have already determined C. I will relate B to H 0 and what we will find, if you do that is B is given by...

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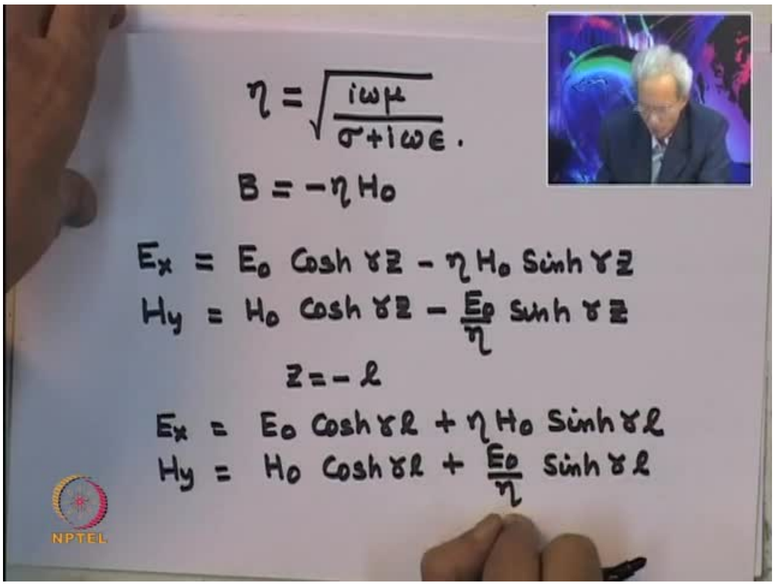

So, in this case I have defined eta to be given by square root of i omega mu divided by sigma plus i omega epsilon. And so this constant B works out to minus eta times H 0. This completes our derivation and therefore, let us combine them, I get E x equal to E 0 times cosh gamma z minus eta H 0 that is my B sine hyperbolic of gamma z and parallelly H of y is equal to H 0 cosh gamma z minus E 0 by eta sine hyperbolic of gamma z. Now, that is the pair of solutions that I have got.

Now, suppose I am talking about the surface of the conductor being at z equal to 0 and the conductor the wave is propagating in the downward direction, and therefore, what I could do is for example, I could say, suppose at z equal to minus l. Let us assume that the, I know the electric field, but I can write down, so if I, the reason for putting minus l is very trivial. I just do not want these minus sign not that it matters. So, at z equal to minus l my E x will be given by E 0 cosh gamma l plus eta H 0 sine hyperbolic of gamma l, and a very similar expression for H y namely H 0 cosh gamma l plus E 0 by eta sine hyperbolic of gamma l.

Now, supposing the electromagnetic wave has propagated a large distance into the medium, so that my l is very large. Now, if l is very large, then remember the cosh gamma l is E to the power gamma l plus E to the power minus gamma l by 2 and sine hyperbolic of gamma l is E to the power gamma l minus E to the power minus gamma l by 2.

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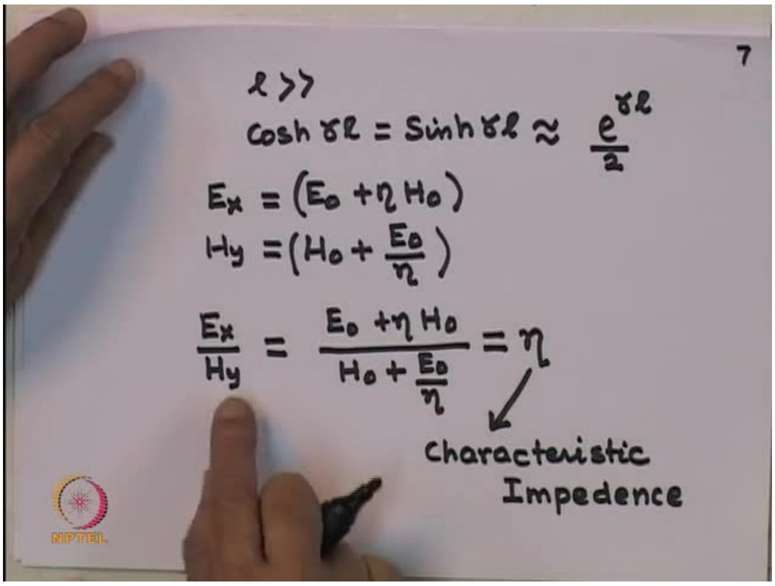

So, if l is very large, so l very large then cosh and sine both of them have the same value because E to the power minus gamma l, I can neglect them and write this as E to the power gamma l by 2, so then I will, I can write my E x is equal to E 0 plus eta times H 0 and H y is H 0 plus E 0 divided by eta. Now, take the ratio of E x to H y, so you find this is equal to E 0 plus eta H 0 divided by H 0 plus E 0 by eta. And you notice this is nothing but just eta. Therefore, this quantity eta which is called characteristic impedance, the reason for that name will become clear as we go along. We will see that eta as we have obtained as the dimension of resistance and therefore, actually it is an impendence and so the ratio of E x to H y is the value eta. So, let us let us look at what that eta actually is?

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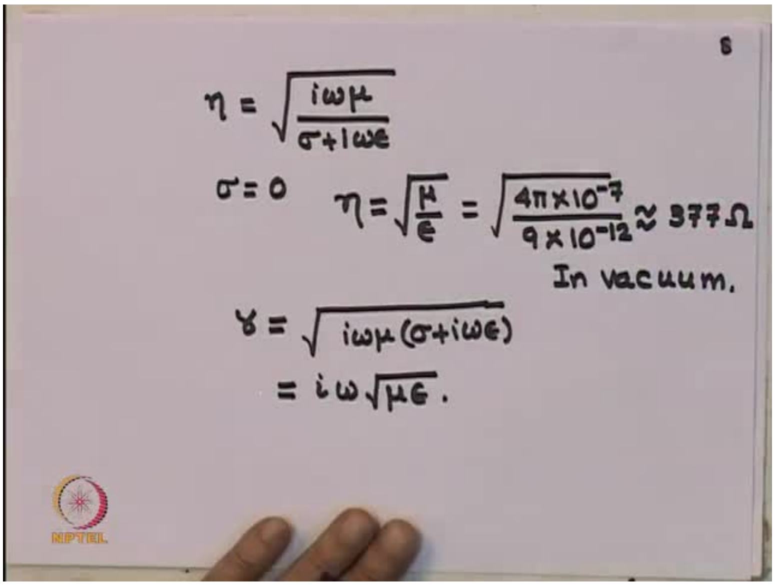

So, this eta that we have defined earlier was given by root of i omega mu divided by sigma plus i omega epsilon. Now, suppose I have a lossless medium, lossless medium means my conductivity is actually 0. So, if sigma is equal to 0, then what I have is eta is simply equal to root of mu by epsilon. And supposing this is my vacuum, that is mu is equal to mu 0 and epsilon is equal to epsilon 0. Then you can calculate this this is remember that this is 4 pi 10 to minus 7 and this is about 8.85 or let say approximately 9 into 10 to minus 12, so if you calculate this works out to 377 ohms. So, this is the value in vacuum.

And gamma which was written as equal to square root of i omega mu into sigma plus i omega epsilon, since sigma is equal to 0 and therefore, the, I have two i’s under the square roots, I will pull out an i there omega and root of mu epsilon. So, these these are the characteristic impendence and the propagation vector. In case well, I have just taken so far a pure dielectric. So, let us let us talk about the metal, but once again I will not, I will write down the full three dimensional version of that equation, which we talked about.

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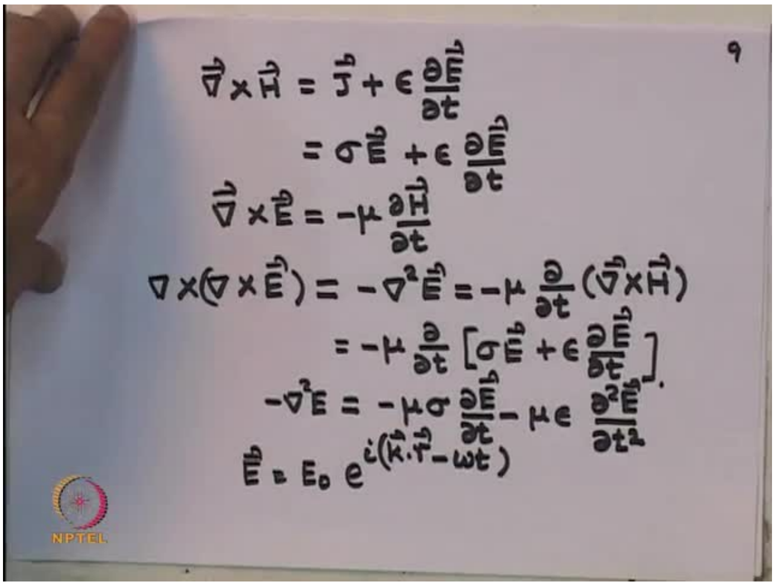

So, the process is essentially the same, I have got these two pair of curl equations. del cross H is equal to J plus epsilon d E by d t, this is essentially partly repetition because I am simply doing it in three dimension and J is equal to sigma E plus epsilon d E by d t and del cross E is minus mu d H by d t. So, a while back, we have this specialized and talked about that. Let us suppose that E is along the x direction, H is along the y direction, but let us do it in general. Supposing, I do del cross del cross E, I get, we have seen that this is del of del dot of E minus del square E and we do not have any charges.

So, it is minus del square E that is equal to minus mu d by d t of del cross H and for this del cross H, I substitute from here, and so that this is equal to minus mu d by d t of sigma E plus epsilon d E by d t. So, this is a, an equation which is minus del square E equal to minus mu sigma d E by d t, this is first ordering time minus mu epsilon d square E by d t square. So, this is what we have and we are looking for the solution of the form. Let us say E is equal to E 0, E to the power i k dot r minus omega t. So, if you do that, I have a del square, which is which will give me minus k square and I have a both a d by d t and d square by d t square d. By d t as we have seen, will give you minus i omega and d square by d t square will give you minus omega square. So, as a result the right hand side of this equation will give me E.

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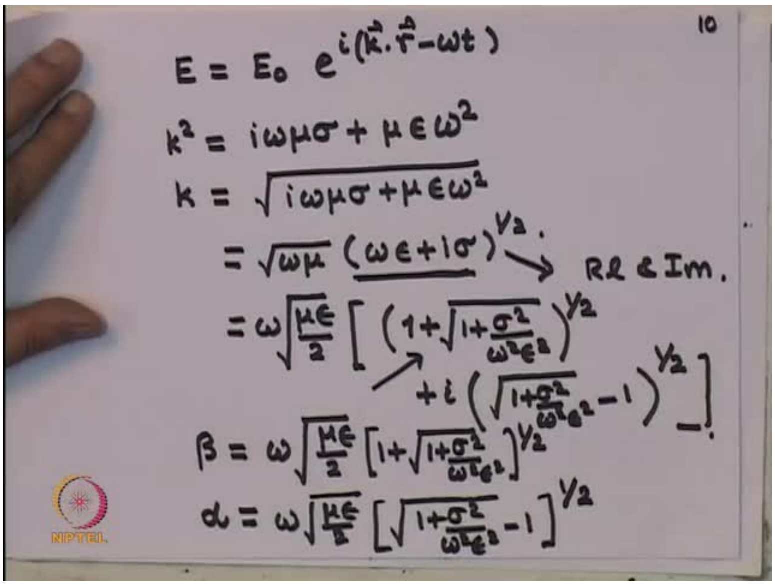

So, we have written the harmonic solution E is equal to E 0 E to the i k dot r minus omega t, this is what we want. So, the correspondingly this equation, will give then k square, that is what comes out of the del square operation, is equal to i omega mu sigma plus mu epsilon omega square. So, this is this is the relationship between the propagation well is not really propagation, but full. So, k is equal to i omega mu sigma plus mu epsilon omega square. And this, let me pull out the omega outside, that gives me if I pull out omega mu, I get omega epsilon plus i sigma raise to the power half.

Now, what I will do is this that you notice that this is the of course, real, but this is a complex quantity. So, the k has a both the real part and imaginary part. The imagine since it is appearing in the complex E to the power i k dot r. The real part of k will give you the propagation and the imaginary part of k will give you the attenuation of the wave. So, let us look at this is fairly straight forward algebra, so what you want to do is to write down this quantity. This is done standard, just write omega epsilon equal to some A cos theta and sigma is equal to some sine A theta.

So, this quantity becomes A cos theta plus i times A sine theta. So, this whole thing will become A to the power half E to the power i theta by 2 and you can determine each one of them by this trivial exercise. So, if you did all that this is a fairly straight forward, so this as to be separated into real and imaginary parts. This will give you omega this the, other complicated expression, but let us write it down. mu epsilon by 2 you get 1 plus square root of 1 plus sigma square by omega square epsilon square raise to the power half there is a square root within a square root plus i times square root 1 plus sigma square by omega square epsilon square minus 1 raise to the power half.

So, you notice that I identify this quantity here, which is the real part of k as my propagation vector beta. So, beta is the propagation, which is omega root mu by mu epsilon by 2 into square root of 1 plus square root of 1 plus sigma square by omega square epsilon square raise to the power half and alpha which is the attenuation factor, because i alpha when it goes to E to the power i k dot r, that will give you E to the power minus alpha r. And this is a very similar expression which is omega mu epsilon by 2 and this is this square root 1 plus sigma square by omega square epsilon square minus 1 raise to the power half So, what you have done is in general determine the propagation vector beta and the attenuation factor.

Now, at this stage we need to talk about what is meant by a metal? So, for we have had both the dielectric and the metal property together, so it is this ratio omega by sigma by omega epsilon, which determines whether something is a good dielectric or a good conductor. Now, if something is a good conductor, then sigma is much greater than omega epsilon.

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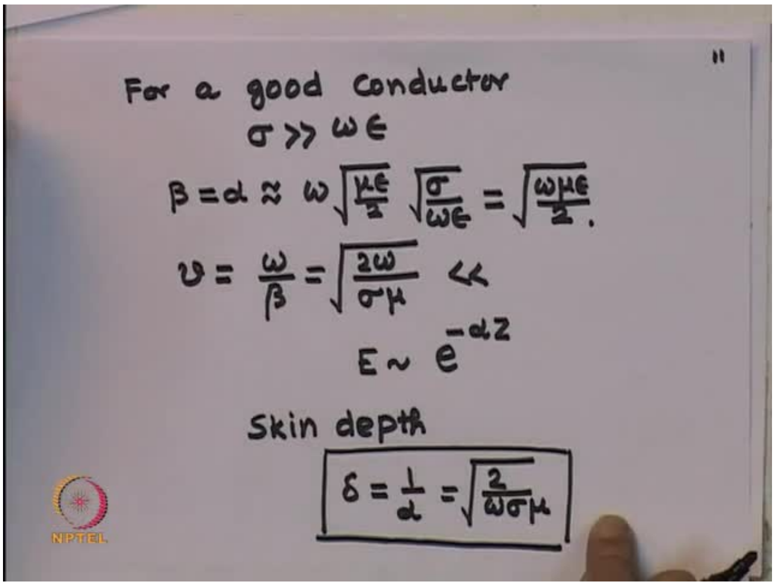

So, for a good conductor sigma is much greater than omega epsilon. As a result in this factor here, since sigma is much greater than this. You notice that I can write down all these ones can be then neglected. If the one is neglected, then for instance what I get for beta? So, beta will be equal to alpha and is approximately equal to omega into root mu epsilon by 2 and I have, first I take this square root giving me sigma by omega epsilon and then I take the second square root, which gives me square root of sigma by omega epsilon. So, if you look at that that becomes square root of omega mu epsilon by 2.

So, the velocity in in the conductor, which is given by omega by the propagation constant, so that is simply equal to square root of 2 omega divided by omega mu sigma mu and this is much less as you can see it because your sigma is a very large quantity. So, as a result the propagation speed sort of gets reduced. The other thing is that, since the alpha is this quantity and the electric field attenuated as E to the power minus alpha z, the distance for at which the strength of the electric field becomes 1 over E th of its initial value is what is called as the skin depth. Now, clearly since alpha is given by this, the value of the skin depth is 1 over alpha, which is square root of 2 over omega mu omega sigma mu.

So, firstly you notice that this tells me that the skin depth of course, becomes smaller and smaller, as the conductivity raises that is the electromagnetic field does not quite penetrate for into the medium into a conducting medium. But but let us sort of have an idea about how much is this skin depth? For example, let us take a good conductor like copper.

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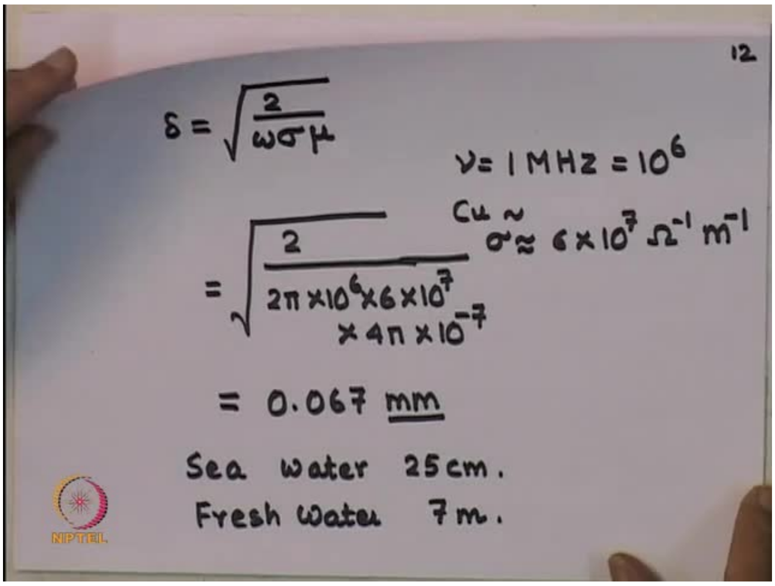

So, delta there is square root of 2 over omega sigma mu. So, let me take, let us take the omega is the frequency corresponding to, let us say 1 mega hertz. That is mu is equal to 1 mega hertz that is 10 to the power 6. Let us take copper whose conductivity sigma is approximately 6 into 10 to the power 7 ohm inverse meter inverse. It is slightly less, something like 5.58 also 6 into 10 to the power 7. So, if you calculate this sigma you get 2 divide by, now there is a omega there, so I get 2 pi mu, 2 pi into 10 to the power 6 sigma is 6 into 10 to 7. And for mu I will take the permeability of the vacuum, which is 4 pi into 10 to minus 7. You can immediately see 10 to 7, 10 to minus 7 etcetera.

Go away and this number is a rather small number it works out to something like 0.67 milli meter. You could compare this with what happens for instance in sea water, which is not as good a conductor as copper. And the, in sea water it is about 25 centimetre remember because of celerity sea water as certain amount of conductivity, but if you take fresh water it goes something like 7. So, this is this is what skin depth about. So, basically what we have said is, that when electromagnetic wave propagates in vacuum, it attenuates, it moves with a much lower speed and there is a thin area, thin surface layer at the surface where the amplitude will quickly divert.

So, in other words since I am talking about a conductor, any current that arises because of the electric field will have to be confined within this small thickness. Let us look at the reflection, we have already talked about reflection and transmittance from a dielectric medium. Now, one can very similarly, work out the reflection and the transmission from the conducting medium.

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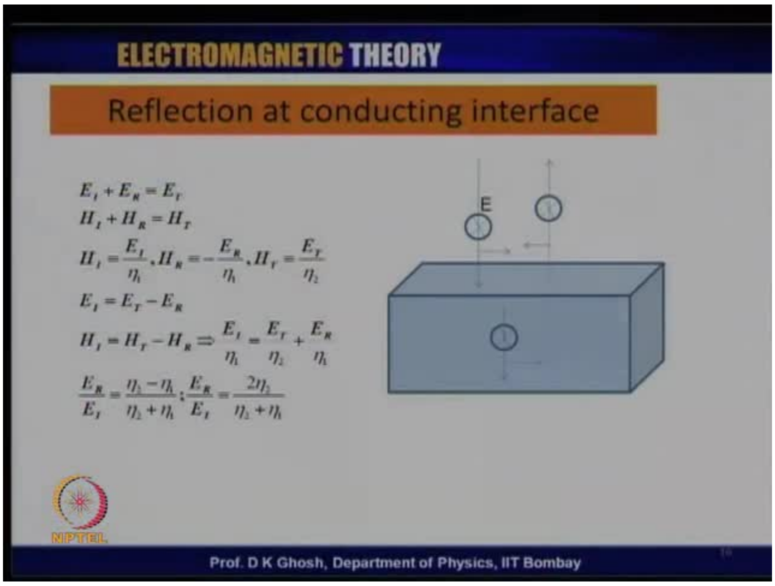

Now, the principle is exactly the same. We have seen that at the surface at the boundary between, let us assume that electromagnetic wave is falling from vacuum on to this surface here. So, this surface I will take to be z equal to 0 and as we have done earlier I will assume that the electric field is you know perpendicular to this propagation direction. And of course, the and the magnetic field will be this way. So, therefore if the electric field is to be continuous at the boundary, then I can write down E incident plus E reflected must be equal to e transmitted.

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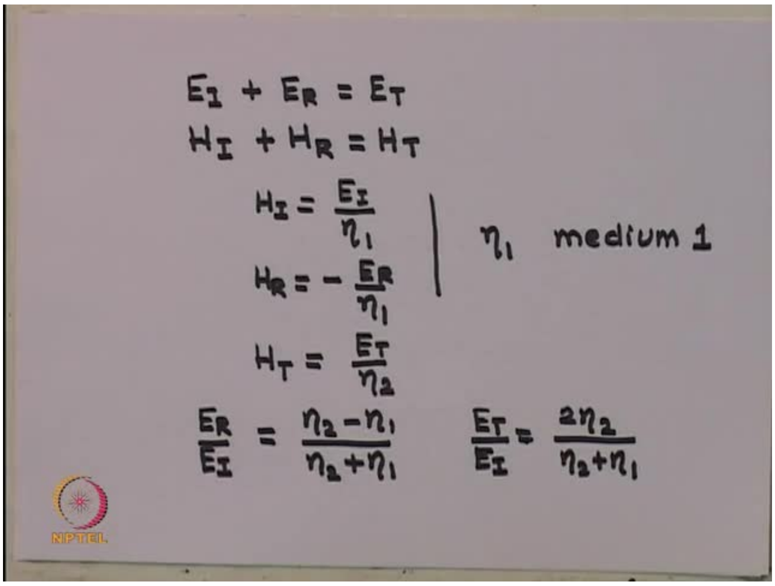

And parallelly the magnetic field is also tangential component of magnetic field is also continues. So, I will write h incident plus h reflected is equal to H transmitted. Now, little while back, we had worked out the relationship between the electric field magnetic field and we had seen that the ratio of the electric field to the magnetic field happens to be the characteristic impedance. So, for as the incident ray is concern your H I is E I by eta. So, let us call it eta 1 because that is the medium number 1. Now, so for as the reflected part is concerned, this is minus E reflected by eta 1 minus because the direction of propagation has changed and since I have assumed the electric field direction to be the same corresponding the magnetic field direction as to reverse.

I have taken eta 1 because both incidence and reflected ray they, belong to medium number 1. So, eta 1 is medium 1. The transmitted ray however H T will be E T by eta 2. So, eta 2 is my transmitted ray. Now, can easily solve this set of equations this is and you can show that E reflected by E incident happens to be equal to eta 2 minus eta 1 by eta 2 plus eta 1 and E transmitted by E incident is 2 eta 2 by eta 2 plus eta 1. And you could also solve for the corresponding magnetic field thing and the only difference that you find is that the H R by H I instead of eta 2 minus eta 1 by eta 2 plus eta 1 happens to be eta 1 minus eta 2 by that is the numerator that 2 is replaced by E 1 and this is the only difference. That is, there is a small error here, which should be H R by H I, this equation should be H R by H I.

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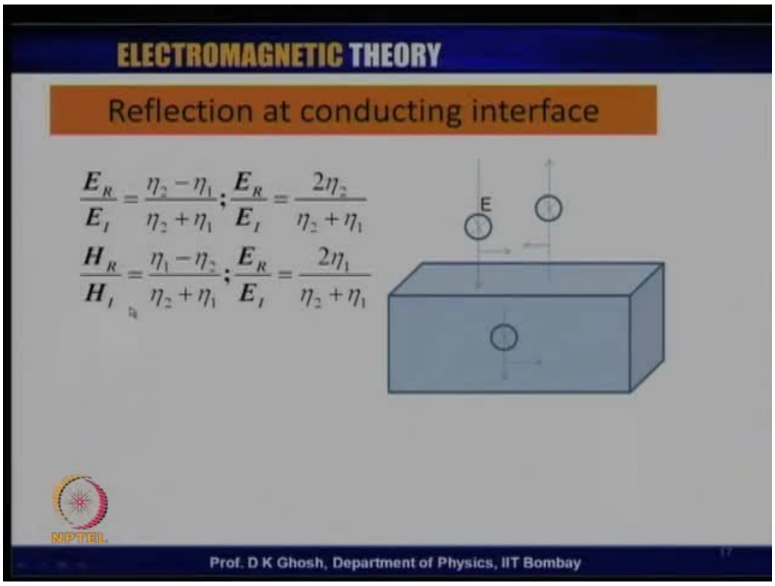

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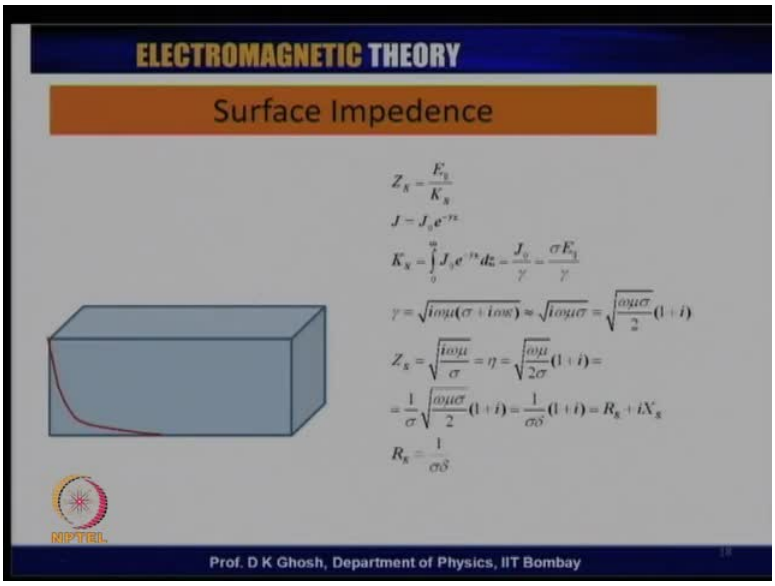

So, what we have now done to, but before we proceed, let us look at the consequence of some of these things. So, let me look at the supposing my eta on1e the medium number 1.

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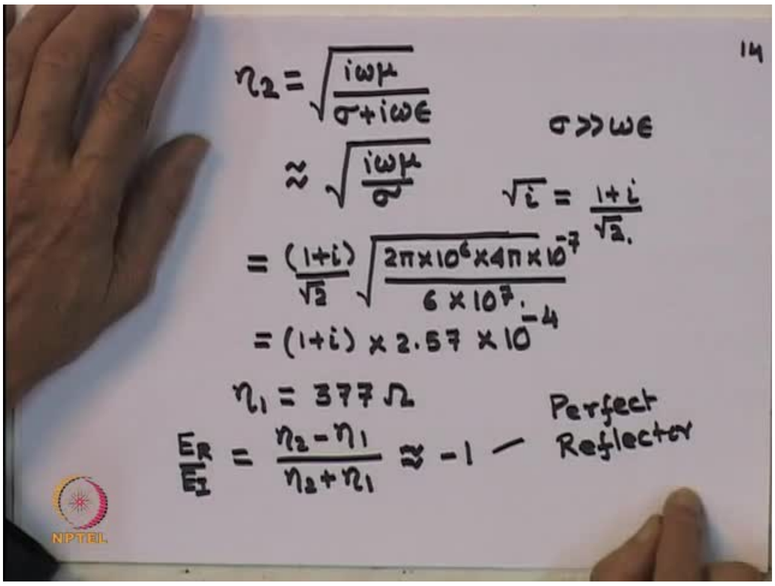

Well let me write down eta 2 first because that is a conductor. So, eta 2 is i omega mu divided by sigma plus i omega epsilon. Now, if I assume that the, it is a good conductor then sigma of course, is much larger than the omega epsilon. Though it is a complex quantity, but in a first order I can always neglect this part because this is be a rather small number. So this quantity is equal to square root of i omega mu divided by sigma. You know that square root of i can be written as 1 plus i by 2 root 2. So, you could actually calculate by putting numbers here.

For example, let us take the same copper which we had calculated, so I get square root of i which is 1 plus i by root 2. Then I have got square root omega which I am taking as 2 pi 10 to the power 6 1 Mega Hertz mu as again is 4 pi 10 to minus 7, copper I am taking conductivity is to be 6 into 10 to 7. You can see what is this number? See this 10, 10 to the power 7 here and there is a 10 to the power minus 1 there, so this number is a rather small number and if you calculate everything here, you get a 1 plus i into 2.57 into 10 to minus 4 this is approximate because I have done you know neglected this, but you could put in other things there.

Now, so far as eta 1 is concerned, I know it is the vacuum. Therefore, I take the standard value characteristic impedance 377 ohms. Now, if I am now calculating the ratio of the electric field reflected component to the incident component, E R by E I, which if you remember is eta 2 minus eta 1 divided by eta 2 plus eta 1. Of course, eta 2 is a, has a small complex now part, but the real part is rather small. So, there would be, this would be in general complex, but on the other hand the angle complex part will not be all that much. That is because it is eta 1, which determines this ratio primarily and this is approximately equal to minus 1, that is the ratio of E R to E I is minus 1, this is what you would expect for a perfect reflector.

If you want a better value you need to calculate these things little better and and this will be sort of may be not minus 1 may be minus 0.99 times a, an angle, so that is that is what you will get. Therefore, a good metal is also a perfect reflector, now notice something very interesting that comes out of it.

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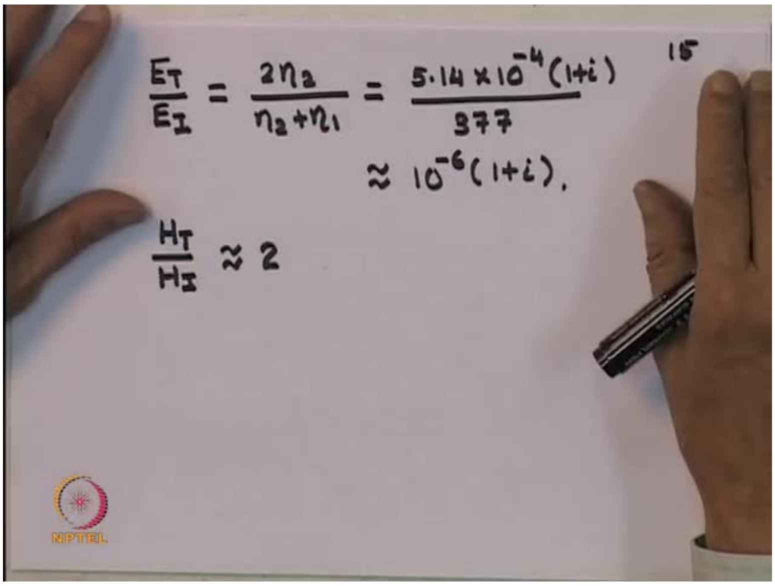

Suppose, you were to calculate E transmitted by E i you could actually immediately see what it will be. Now, E transmitted by E i is 2 eta 2 by eta 2 plus eta 1. Now, we know that the numerator denominator is approximately eta 1, which is 377 and eta 2 is a rather small number. So, you multiply this 2.5. So, I get 5.14 into 10 to minus 4 1 plus i divided by 377, I am not hiding the little complex part there. So, this is approximately because you see their numbers, this is 10 to minus 4, this is 10 to 2. Therefore it is approximately a number which is of the order of 10 to the power minus 6 times 1 plus i.

So, in other words the transmitted electric intensity or the electric field amplitude is substantially reduced. It is 10 to the power minus 6 times 1 plus i. Now, if you did the same thing which your H T by H I, you would realize this will become approximately equal to 2 and the reason is not very far to see, difficult to see while the electric field is reflected with a phase change the magnetic field is not reflected with the phase. But that its direction of propagation as changed, so as a result in order to maintain continuity, since the magnetic field is reflected without a reversal i expect H T by H I should be twice the, should be a approximately equal to, so this all stands to the last thing.

I would talk about is, we have just now seen that the electric field does not penetrate much into the medium. So, if since the electric field does not penetrate much into the medium, gets attenuated very quickly, I expect the electric field whatever is it penetrates, depending upon the skin depth to be contained within a small thickness of from the surface.

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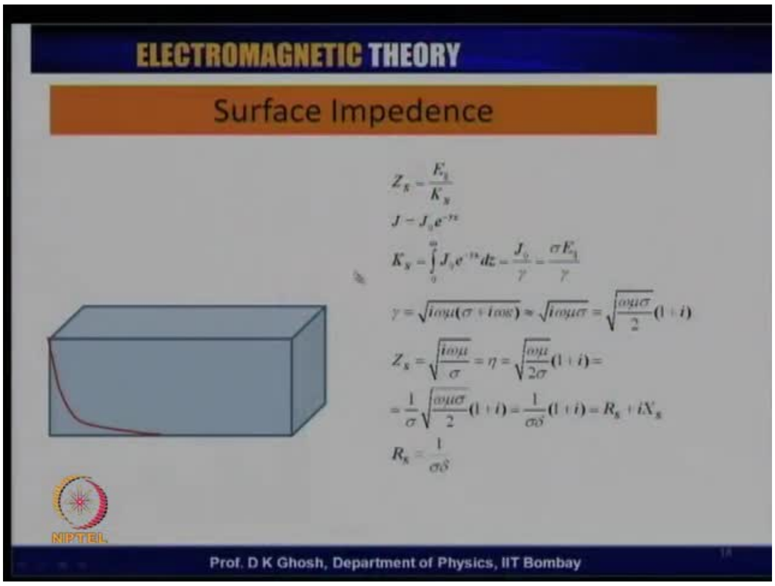

So, what we do is, we define what is known as a surface impendence. So, the surface impendence, remember the impedance is defined basically in terms of the ratio of the electric field to the current.

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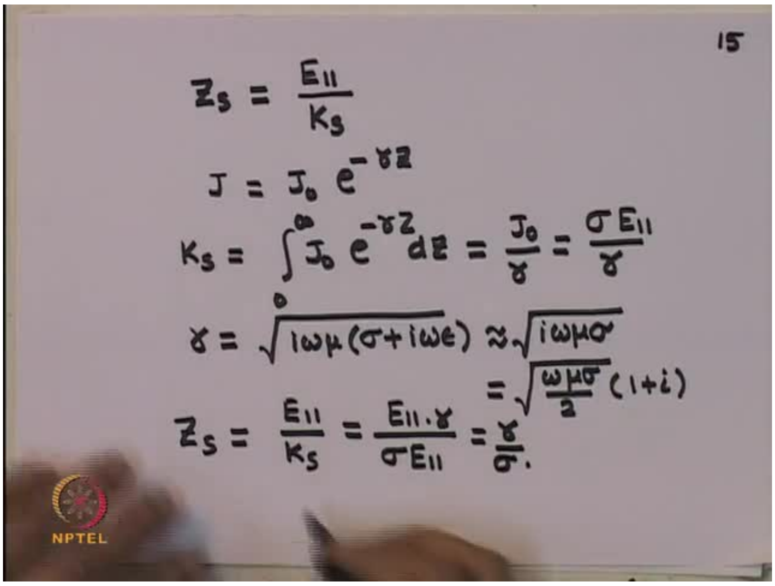

Now, in this case, thus surface impendence will be defined as the parallel component of the electric field, which is giving raise to that current divided by the surface current density. That is the current density, which is running or the current density, which is on the surface. Now, I do, I calculate this. Now, I know the current is confined within a small thickness, now let us assume that there is no reflection of the electric field from that thickness. That is since the electric field gets attenuated, I can assume that the essentially it is the infinite medium and therefore, the current profile the current density profile can be J 0 into the E to the power minus gamma z.

And surface current density will then be, I simply calculate, I simply integrate over the all space. Ideally it is to be integrated over that thickness, but then the I assume that the skin depth is rather small. So, as a result the electric field sort of does not penetrate much, so if I integrate this from 0 to infinity that is perfectly legitimate. And that is equal to simply J 0 divided by gamma and if you recall J 0 is nothing but sigma E parallel this divided by gamma. So, this is my surface current density, now what is my gamma now?

Gamma if you recall is i omega mu into sigma plus i omega epsilon and since it is a good conductor I replace it with i omega mu sigma and as before since square root of i is 1 plus i by root 2, this is omega mu sigma by 2 into 1 plus i. So, therefore my z s which is e parallel by k s and that is E parallel by sigma E parallel into gamma. So, this is equal to, well gamma by sigma and I have already calculated what is gamma and I divide it by sigma.

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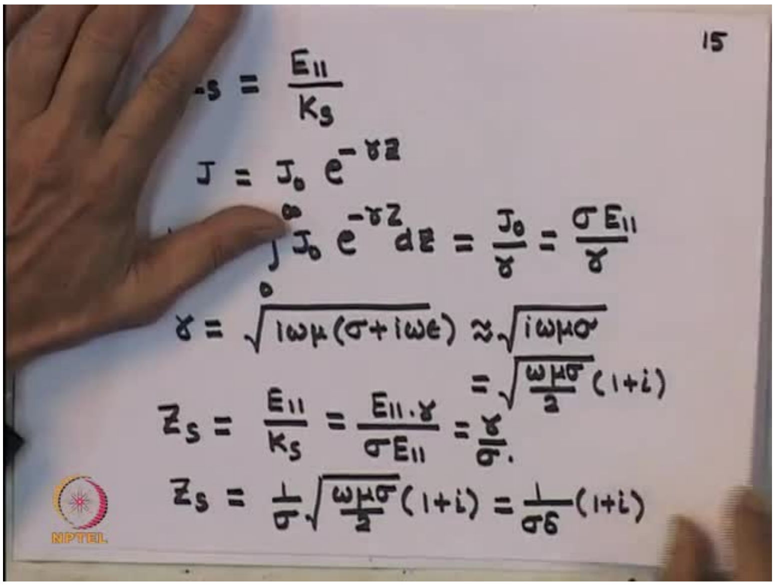

So therefore, my surface impendence is one by sigma into root of omega mu sigma by 2 into 1 plus i and since you recall that the delta the skin depth is square root of 2 by omega mu sigma. So, this is nothing but 1 over sigma delta into 1 plus i, so as a result what I get is that, the surface impedance which has a resistance part and a reactance part.

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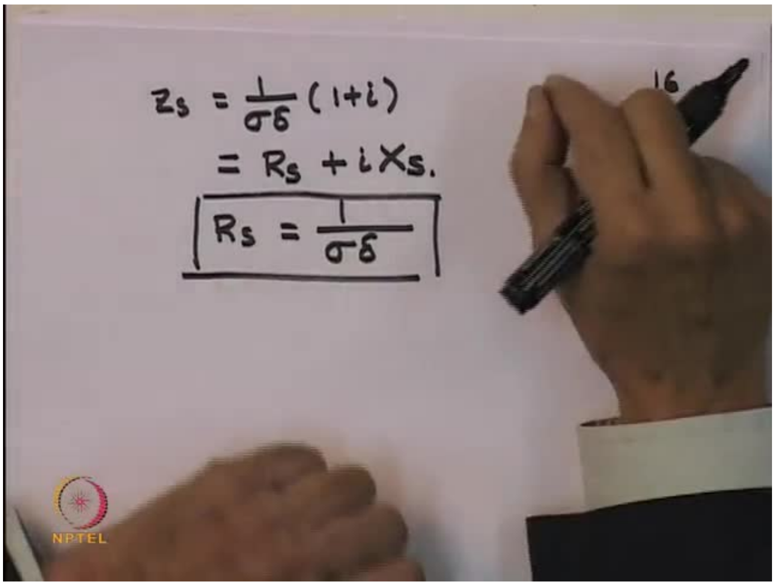

So, this is z s is equal to 1 over sigma delta into 1 plus i and this is surface resistance plus i times surface reactance and this surface resistance. Then is, simply equal to 1 over sigma delta.

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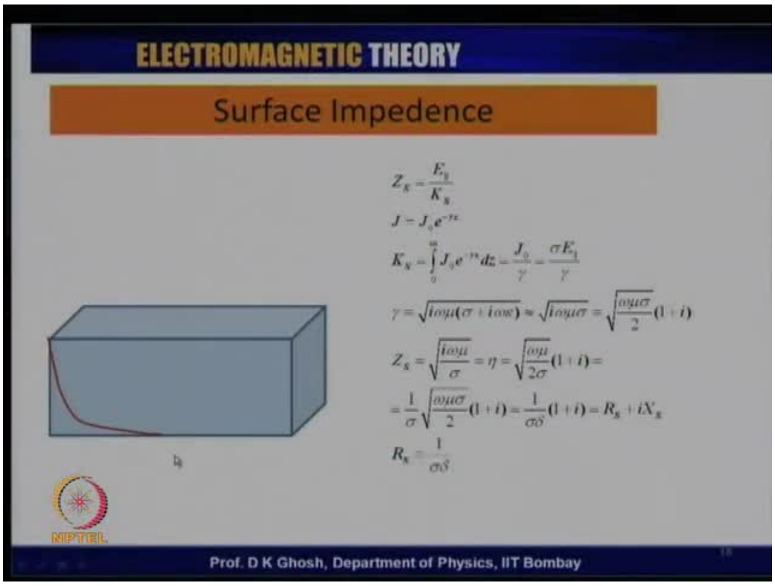

So, what it means is that the current profile is like this. The current is confined to a small thickness and the resistance that the surface provides depends of course on the conductivity of the medium. So, 1 over sigma as expected and also it is inversely proportional to the skin depth delta.