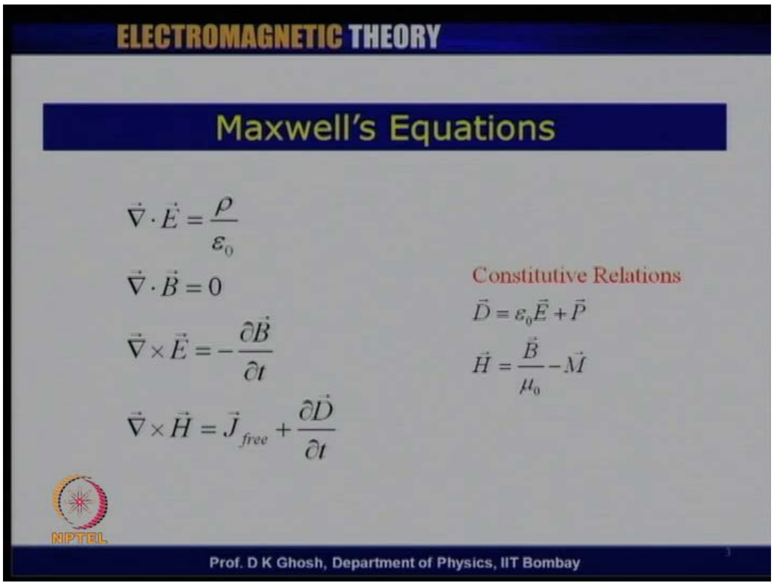
Electromagnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Module - 4
Time Varying Field
Lecture - 33
1. Angular Momentum Conservation
2. Electromagnetic Waves

In the last couple of lectures, we have been talking about the various quantities associated with electromagnetic field. For instance, we found out that electromagnetic field carries energy; it carries momentum. And today we will try to talk about another application that is electromagnetic field also carries angular momentum. And in fact we will see some very interesting consequences that might arise, because of conservation of angular momentum.

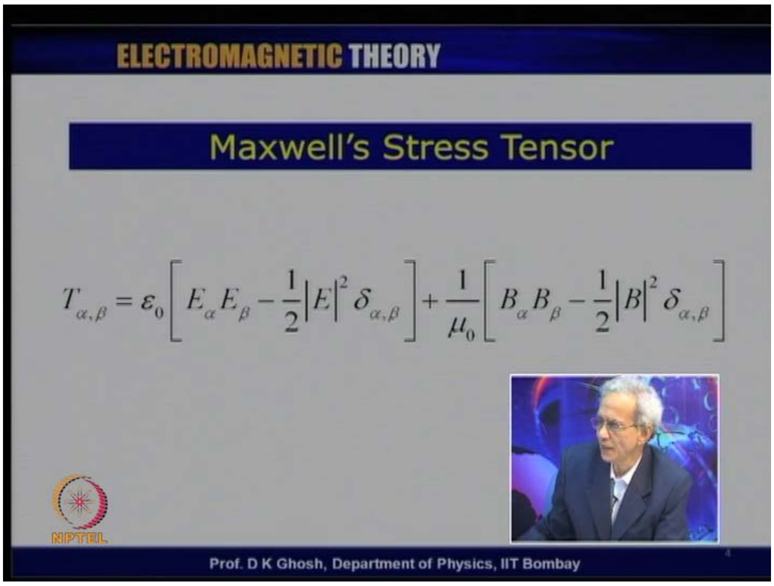
We had introduced the, what we called as a Maxwell's stress tensor and in terms of which we had tried to explain the conservation of linear momentum. And let us so in today the first part of the lecture, I will be talking about angular momentum and its conservation with respect to electromagnetic field. And later, we will go over to an application of time dependent phenomena, namely propagation of electromagnetic waves.

(Refer Slide Time: 01:27)



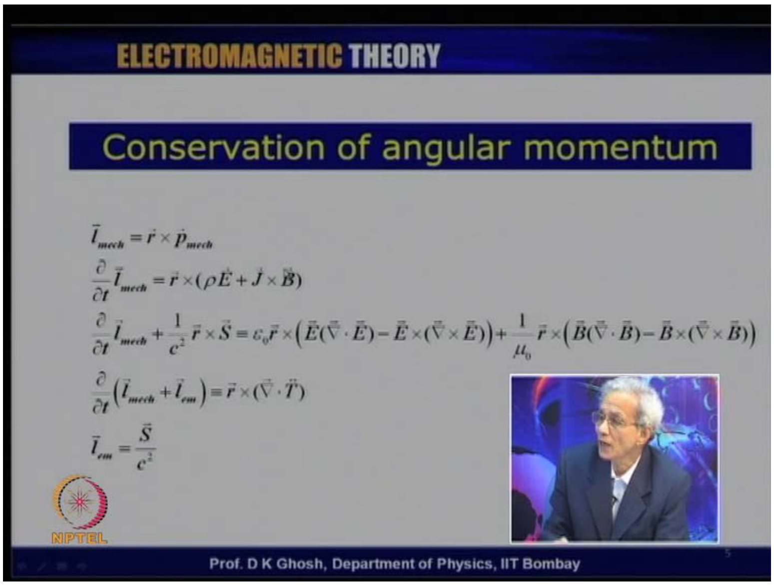
As always we have these work hours before us; the 4 equations known as the Maxwell's equation. The del dot of E is rho by epsilon 0, del dot B equal to 0, then Faraday's law, del cross E equal to minus d B by d t, del cross H, the Ampere Maxwell's law given by J free plus d D by d t. In addition, we have constitutive relations of this type: D equal to epsilon 0 E plus P; H equal to B by mu 0 minus M. In addition, we have things like, for instance, the continuity equations; or, for instance, if you are talking about a metal, may be ohms law is valid, and things like that. So, we will we will be talking about this as you go on.

(Refer Slide Time: 02:19)



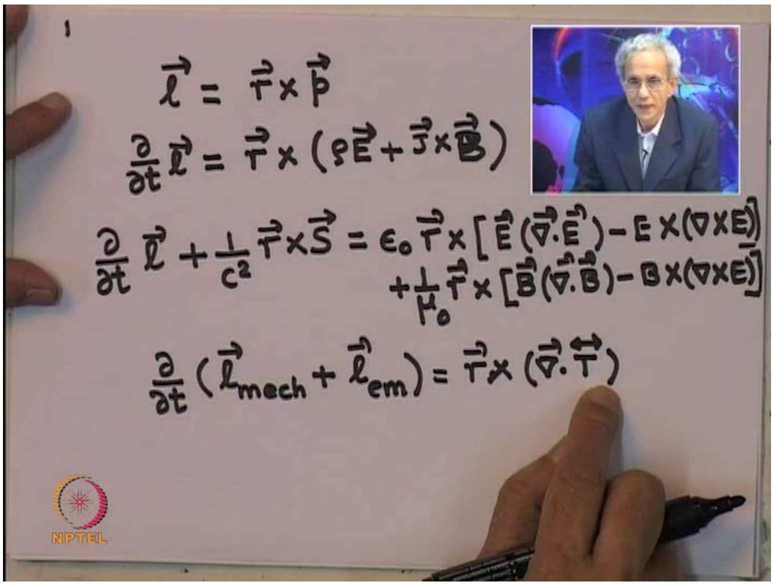
This was our expression for the Maxwell stress tensor, in terms of which we had defined the momentum density of the electromagnetic waves. Basically, it has 2 terms- one is for, from electric field, the other is from the magnetic field; and the expression is fairly simple.

(Refer Slide Time: 02:46)



And, so let us look at, how does one get, define the angular momentum. The angular momentum, of course, if electromagnetic field has a momentum then it stands to reason that about an origin, let us take any origin, I should be able to define the angular momentum by the standard relationship, namely l is equal to r cross P.

(Refer Slide Time: 03:15)

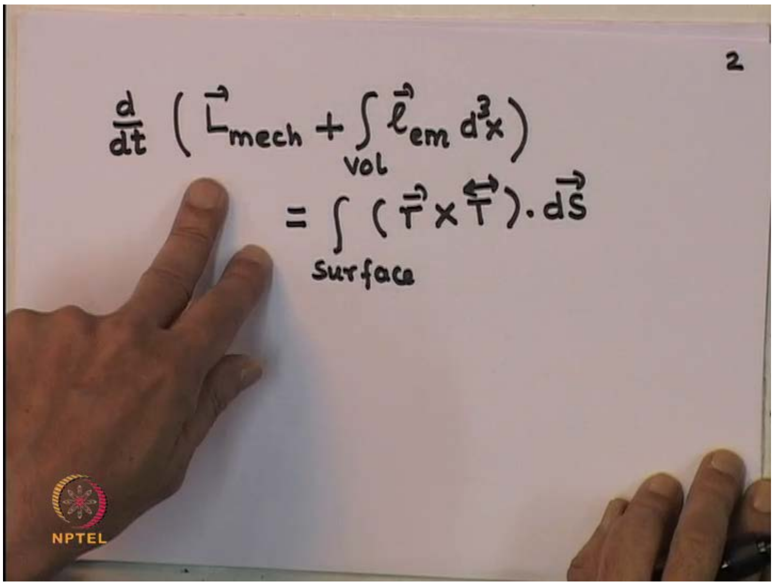


So, since, I am talking about the densities, I am using small letters. So, this is angular momentum density, which is equal to r cross the momentum density of the electromagnetic field. And, so what we will do is this, that rate of change of angular momentum, this is all in terms of the density, this is equal to r cross; remember that angular momentum's change is related to the torque that acts on it. So, therefore, r cross the force, as usual this is rho E that is the force exerted by the electric field; and of course, J cross B, which is the usual Lorentz force equation, J cross B.

So, therefore, if we, I had already shown how this expression is simplified by going over to Maxwell's stress tensor; and so as a result, since I have r cross everywhere, I will be able to borrow this expression literally. So, therefore, what I would get is, d by d t of the angular momentum density, plus 1 over c square r cross s, if you recall that I had 1 over c square S, S is the pointing vector, so I have 1 over c square r cross s; and that was equal to the part which ultimately translated as the Maxwell's stress tensor. So, I had epsilon 0 r cross E del dot E, I am not going to repeat that algebra, minus E cross del cross E, and a corresponding term from the magnetic field, which is r cross B del dot B, minus B cross del cross E.

Now, we had already simplified this term before. So, therefore, this would lead to an equation of this type, d by d t of l, this I will now put in mech, because this is mechanical angular momentum, plus the electromagnetic angular momentum; these are the total change in the angular momentum of the sources and the electromagnetic fields. So, that is equal to r cross, if you recall, this term was divergence of the second rank tensor, namely Maxwell's stress tensor. So, this of course, is S by c square, this is my electromagnetic angular momentum, and this is r cross the divergence of the stress tensor. So, this is what we had seen. So, if you now integrate it over the complete volume, what you would get, would be that statement of conservation of angular momentum.

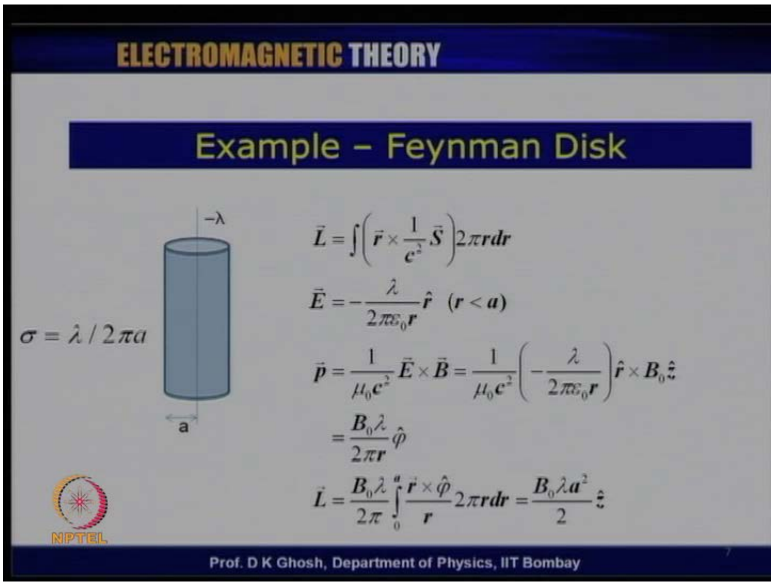
(Refer Slide Time: 06:57)



And, now that, it is being integrated; the only dependence is through time. So, therefore, this is d by d t. L mechanical, plus integral over the volume of l electromagnetic that is the momentum density d cube x. And, this change, this change results in a, if you like a flux of torque through the surface, because this is, we know that rate of change of angular momentum is related to torque, and this is of course, the total angular momentum. So, therefore, this would be a surface integral, the derivation is exactly the same as before; and this is r cross Maxwell’s stress tensor, dotted with d S.

The interpretation, as I said, the left hand side means, what is the total rate of change of the momentum on of the combined system, namely this angular momentum of the source and the angular momentum of the electromagnetic field; and that can change only if there is a flow through the surface or if you like it is a flow of torque.

(Refer Slide Time: 08:26)

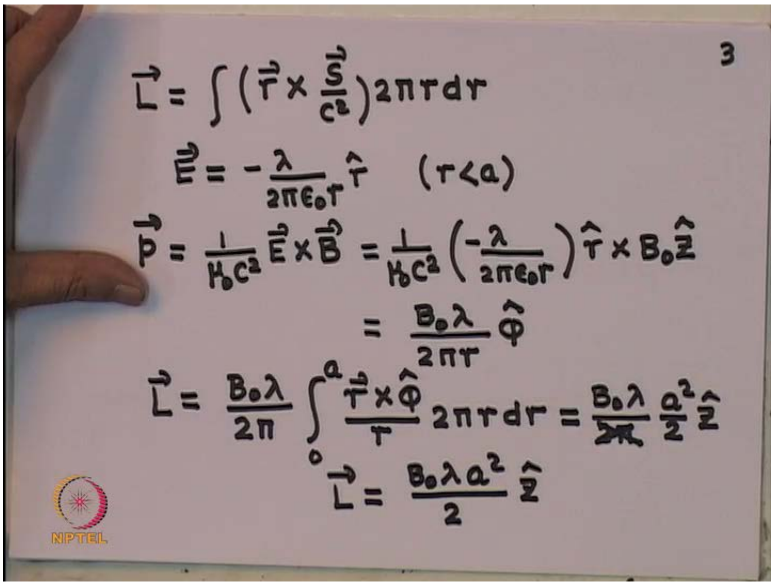


There is a, an interesting problem which I will be doing. And, this is known as the Feynman paradox; Feynman had talked about this. We are going to be using a very similar problem. So, the problem consists of the following. I have a line charge, infinite line charge, which has a charge density minus lambda; surrounding this, surrounding this is a cylindrical surface, and this cylindrical surface contains a charge density, surface charge density which is lambda, which is the same as the magnitude of this one, divided by 2 pi, namely the circumference. So, of course, this then has the dimension of charge density.

Now, the question is this, that in this situation there is electric field only within, say if this radius is a, then only between 0 and a there is electric field; that is because we have adjusted this charge density in such a way, that is equal and opposite to the charge density of this line charge. So, as a result, outside, if you use your standard Gauss's law, the electric field is 0.

So, the question is this; that when, supposing in, through this, I have a current is flowing, and I decrease the current to 0; this is, so if I decrease the current to 0, then you will find this cylinder will start rotating. The, how does it happen that the cylinder start rotating. So, this is, this as a charge density lambda, and there is a current density j. So, let us look at, why does it rotate? The Feynman paradox was because the since the system was initially at rest, it should continue to be at rest, because otherwise it will violate angular momentum conservation. The reason is that, when you talk about the conservation of angular momentum, you have to also take into account the angular momentum of the electromagnetic field, and that is what we going to be talking today.

(Refer Slide Time: 11:06)



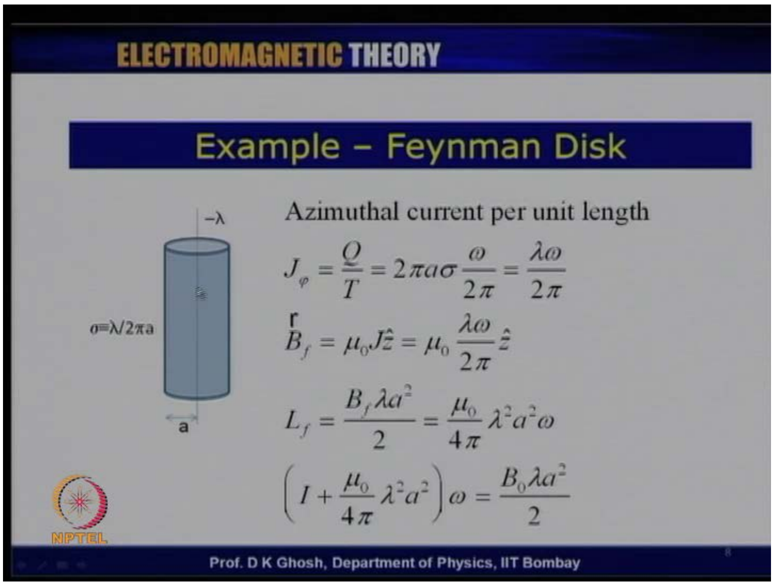
So, let us look at what we said; we said that angular momentum L was, this is I am talking about the electromagnetic field angular momentum only; in my initial angular momentum of the mechanical system was 0, because everything was at rest. So, this is r cross, the pointing vector S by c square; and, since it is total, I get 2 pi r d r and integrated overall. The, I know, what is the electric field. So, electric field is minus lambda divided by 2 pi epsilon 0 r. And, I know and of course, it is along the radial direction; and this is valid for r less than a only; there is no electric field outside, because the charge densities have been adjusted that way.

So, let us look at, what does it give me for the momentum charge density. So, momentum density is 1 over mu 0 c square. What I am doing is, to write down an expression for S. S, if you recall is E cross H. So, H is written as B by mu 0. So, therefore, this is E cross B, this is; and that is equal to 1 over mu 0 c square, the minus lambda divided by 2 pi epsilon 0 r, r, and the external magnetic field which we have put in; let us say there is an external magnetic field along the z direction, which is B 0 z.

Now, what I can do is, since I have a cylindrical geometry, so I have got r phi z. So, therefore, this quantity gives me mu 0, well c square; and recall mu 0 times epsilon 0 is equal to 1 over c square. So, that will cancel this c square there, I will left with the B 0 lambda divided by 2 pi r, and it is azimuthal, because there is a r cross z with a minus sign there.

So, if this is my momentum density, the angular momentum L becomes, let me pull out the constants first, B 0 lambda divided by 2 pi, integral from 0 to a, r cross phi divided by r, then 2 pi r d r. So, r cross phi, of course, is in the z direction. So, this is, I have got B 0 lambda divided by 2 pi; 2 pi also actually goes because of this, there is an another factor there; and this as the magnitude r, so r and r goes; I am left with r square by 2. So, when integrated gives me, a square by 2 and the direction z. So, this is the situation that I have got an angular momentum the field, which I have calculated, is given by this expression. Now, what happens, when we switch off the magnetic field?

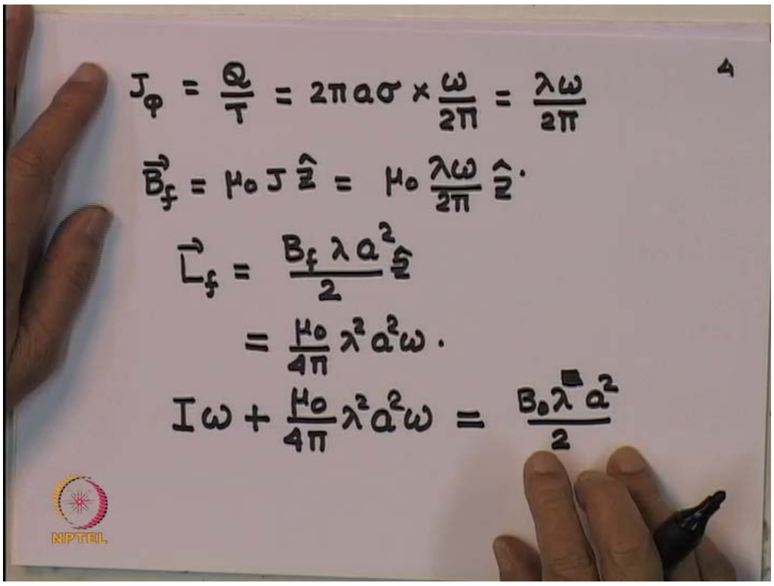
(Refer Slide Time: 15:09)



Now, when I switch off the magnetic field, in the presence of the magnetic field, there was a flux through the, this surface here; the, any of the cross sectional surface there was a flux. And, when this magnetic field is, let say slowly reduced to 0, there is a change in the flux; now this change in the flux, because you are reducing the magnetic field in the z direction from sum value is 0 to 0 results in an azimuthal current; and this azimuthal current I can calculate.

So, this is, this is the expression for my angular momentum of the electromagnetic field, which actually happens to be the total angular momentum, because my system was, mechanical system was at rest. So, this is, this is the initial angular, total angular momentum. Now, let us then see what is happening at the end.

(Refer Slide Time: 16:14)



So, what I have done is, I have gradually reduced the magnetic field to 0, which has resulted in a azimuthal current; and I can calculate that azimuthal current density, J phi, which is Q by t. And, this is, there is a charge which is being, supposing the velocity that has come up, the angular velocity that has come up is omega. So, therefore, this is 2 pi a sigma, that is the amount of charge that I had, into omega by 2 pi, and, that is lambda omega by 2 pi. So, when it rotates, this is what we get.

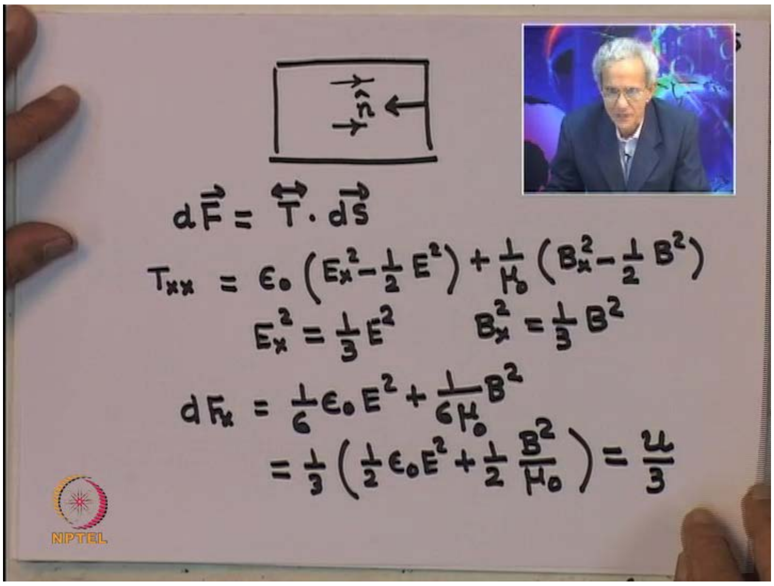
Now, these results in a magnetic, a equivalent magnetic field; let me call it B final that is mu 0 J z. So, which is mu 0 lambda omega over 2 pi, along the unit vector z. Now, since the angular momentum is in the same direction, I can borrow this expression. So, if I know that the induced final magnetic field happens to be B f, then I should be able to write down what is the final angular momentum of the field, which is B final lambda a square by 2, along the z direction. And, if you plug the value of the B final, which we just now have calculated, then this works, what I have is mu 0 by 4 pi lambda square a square omega. So, this is my final angular momentum of the electromagnetic field.

You notice that the initial angular momentum is not the same as this. So, in order to compensate for this change in the angular momentum, my system, let us assume that the moment of inertia of that system is I, then I will have I omega, plus this quantity there, mu 0 by 4 pi lambda square a square omega; this quantity must be my l I which is B 0 lambda square a square by 2. So, this is, this is what I get B, B 0 lambda a square by 2. And, this allows us to determine what should be the value of the, what should be the value of the angular velocity with which the disc will rotate.

The summary of this is, that in considering conservation of angular momentum of a system which has sources, and sources of electromagnetic field, one should take into account that total contribution, both the contribution to angular momentum from the sources as well as due to the electromagnetic field.

The next application that I talk about is that, we have seen that the electromagnetic field carries momentum. So, as a result, supposing I put electromagnetic field in an enclose, a cavity, then since this carries momentum, the walls of that cavity will experience pressure in the presence of electromagnetic field; and this actually is something which you can, sort of verify experimentally, when electromagnetic waves falls on, let us say a mirror. So, the way we will calculate this is this, that the force, supposing I have a cavity.

(Refer Slide Time: 20:38)



So, this is my cavity; inside there is electromagnetic field; and let suppose I am talking about this wall, which has this has the direction of its normal. So, the electromagnetic waves or electromagnetic field will exert a pressure on this right hand wall. And, I know that this force, supposing I look at a small area d s on this wall, then the force on that wall will be given by the Maxwell's stress tensor, dotted with d s.

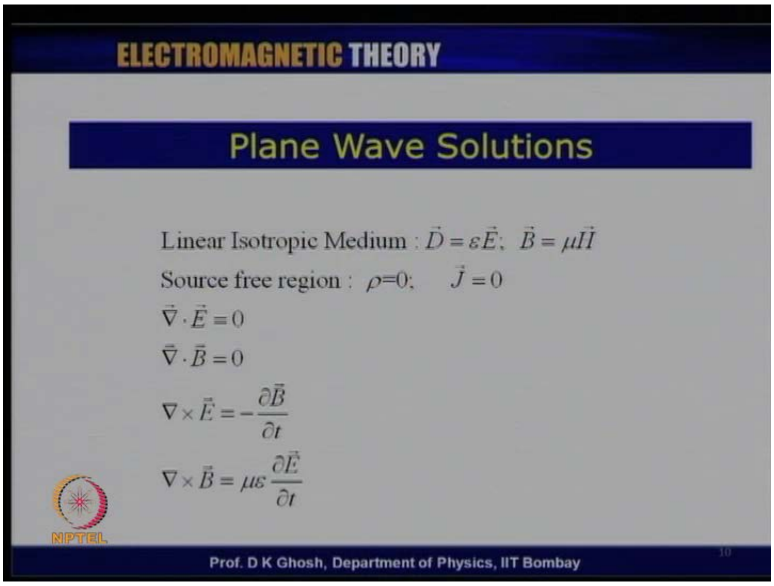
Now, since my force is in the x direction, the direction of d s is in the x direction; all that I require is just the x x component of the Maxwell stress tensor, and that we have see is epsilon 0, E x square, minus half E square, plus 1 over mu 0 B x square minus half B square.

Now, if my radiation is isotropic that it is exerting identical forces in all directions, then it makes sense to say that E x square is actually a third of E square, and likewise B x square is one third of B square. So, we plug this in, what you get is d f, of course, only the x component, is one sixth epsilon 0 E square plus one sixth B square over mu 0. And, if you look at this expression, you notice this is one third of half epsilon 0 E square, plus half B square over mu 0, which is nothing but the energy, this is nothing but the energy density of electromagnetic field. So, this is energy density divided by 3.

We will talk about some applications of this; incidentally, this is the starting point for derivation of something like a Stephen Boltzmann law. With this I have completed the discussion of various conservation laws, which arise or which I have to be taken into account when we discuss electromagnetic fields.

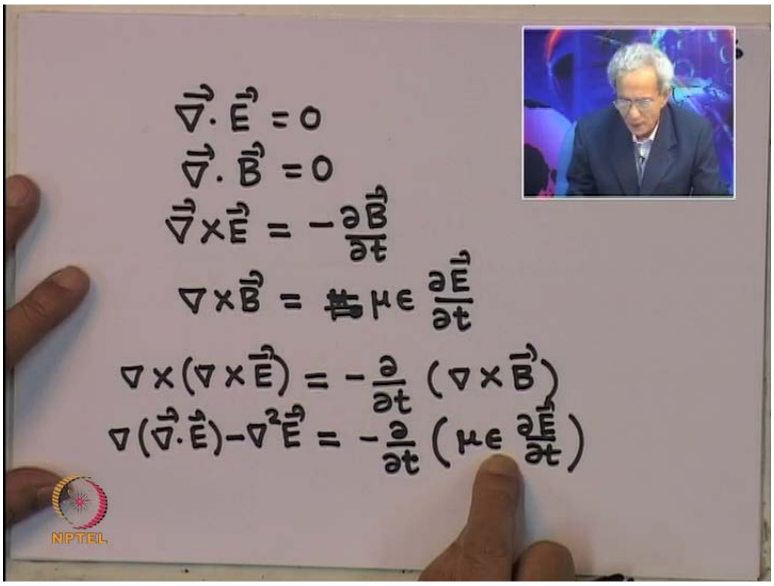
What I am going to do now, is to go over to the set of equations that we wrote down earlier, and obtain what I known as plane wave solutions to the electromagnetic field equations. The electromagnetic field equations are a set of equations which have some special solutions, and one of those special solutions is the solution of the plane wave solution, we will be talking about that.

(Refer Slide Time: 24:05)



To begin with, I will be talking about a linear isotropic medium. Linear, implying that relationship between D and E is linear; the constant of proportionality is the dielectric permittivity; and the relationship between B and the H field are also linear. So, B is equal to mu H. I will also talk about regions which are source free that is the solutions that I am going to look for, are in regions where rho is equal to 0 and J is equal to 0.

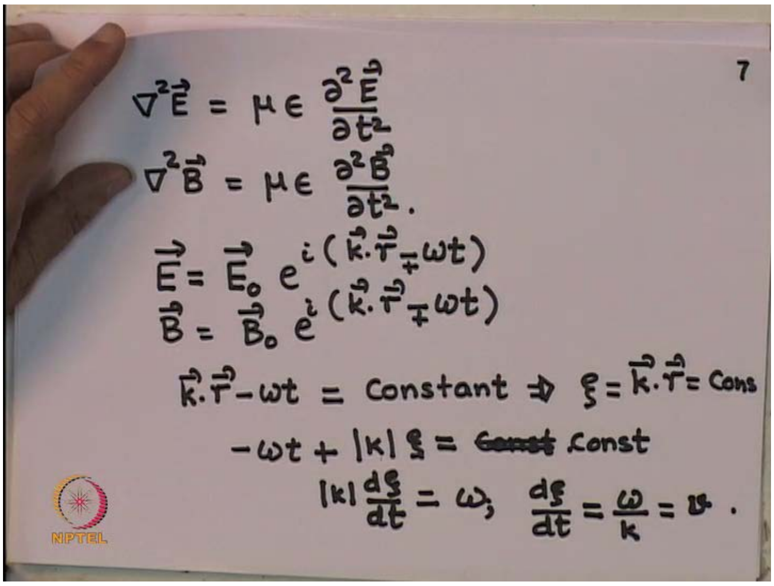
(Refer Slide Time: 24:52)



And, that tells me that in the source free region, del dot of E must be equal to 0 because there is no rho; and of course, del dot of B is always equal to 0. Now, del cross E, which is the Faraday's law is equal to minus d B by d t; and del cross B, I had a mu 0 J, which I will drop because my J is equal to zero, so this is equal to mu epsilon, let me, let me take mu epsilon, rather than mu 0 epsilon 0. If I had mu 0 epsilon 0, it would mean I am taking the solution in vacuum, but this is mu epsilon d E by d t. So, these are the basic equations with which we will work.

Now, notice, take any one of these equations. For example, take del cross E. Now, if you take del cross of this equation, del cross, del cross E, I get minus d by d t of del cross B; and of course, we have seen several times, this gives me del of del dot of E, minus del square E, that is equal to minus d by d t of, for del cross B, I will replace from Maxwell's, Ampere's law, mu epsilon d E by d t. We had seen that in the source free region, dell dot E is equal to 0. So, I get a del square E is mu epsilon, d square E by d t square.

(Refer Slide Time: 26:58)



So, let me write that down. Identically, you could have converted this into equation for the B field; and all that you need to do then is to take the del cross B equation; take del cross of this, replace for the del cross of E from this equation. So, I would get very similar equation, del square of B is mu epsilon d square B by d t square.

Now, this, these are equations, which, sort of, look very simple, but there is a problem. See, when you take del square of E, it does not mean the, there is actually 6 equations, because E and B are vectors, so I have to take component vice. But del square of E, if you take x component, is not in general equal to del square of E x; this, however, is true, if I work in Cartesian coordinate system. The, if you were to solve this in, for instance, the spherical polar, then of course, it will mix up the various components; we had seen that del square mixes up. But the Cartesian coordinate system, you of course, have del square E x, del square of E is d square of d x square E x, E x plus d square about d y square E y, plus d square about d z square E z. So, as a result solving these in Cartesian coordinate system is fairly strict simple, and that is what we are going to do for the momentum.

Now, what is it that we get now? Look, firstly, what we have seen is this; we have got del dot E is equal to 0, and del dot of B is equal to 0; this will let, I will, I will return back to this picture, in a second. So, I am going to be looking for some special solutions of this equation. And these are the, what are known as the plane wave solution. What are plane waves? The plane waves are those for which the surfaces of constant phase, which are also known as the wave fronts, there planes. So, for example, a spherical wave would be, that if you are looking at, finding the locus of all points which have the same phase, then in a spherical waves those surfaces are spheres, but in plane wave the surfaces of constant phases are planes.

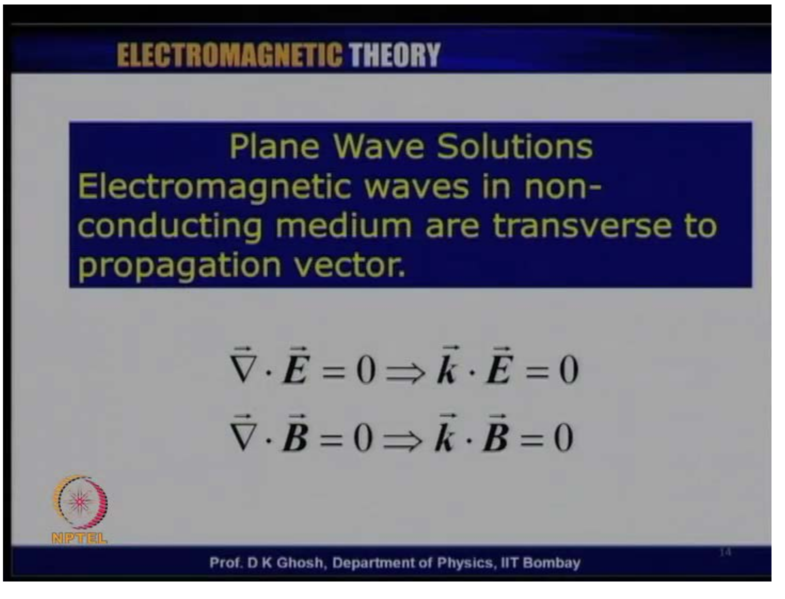
So, the solution, I am looking for is this; I am looking at E vector as E 0, e to the power i, k dot r minus omega t. Now, this is very useful, because what we can always do is to; of course, our waves are real. So, what we could do is to take an exponential form and ultimately take for example, either real part or imaginary part. If I take a real part, I can for example, if E 0 is taken as a real, then I will get cosine k dot r minus omega t. But otherwise, this is the way the algebra becomes very simple, if you work with this. Now, and similarly, B is equal to B 0 e to the power i, k dot r minus omega t.

So, let us, let us look at what is meant by then surfaces of constant phase. So, surfaces of constant phase will be those surfaces for which k dot r minus omega t is constant. So, I am looking at, incidentally I should point out that, I can also have a plus sign there, the difference is in the farmer case, when k dot r minus omega t is there, it is what is known as a forward moving wave; and if k dot r plus omega t is there it is a backward moving wave. So, the surfaces of the constant phase will be k dot r, well, that let me take a minus r omega t, that is equal to constant.

So, let us, let us assume that, at any given time, omega t is constant. So, this implies that k dot r is constant. Now, let us say, the component of r vector along the direction of k, k is known as a propagation vector, is zeta let us say. So, therefore, this zeta, which is defined as k dot r, this is constant. So, therefore, what I have is, minus omega t, plus k zeta, that is equal to constant. And, if you differentiate it now, with respect to time, what you get is, at a, sorry, this is, this is the constant, and so I have got k dot r. So, that is k zeta; and this quantity is constant. So, therefore, I get minus omega; if I differentiate it with respect to time, I get k d zeta by d t that is equal to omega, which means d zeta by d t is omega by k, plus or minus if your chosen the other sign, and which is equal to plus or the velocity.

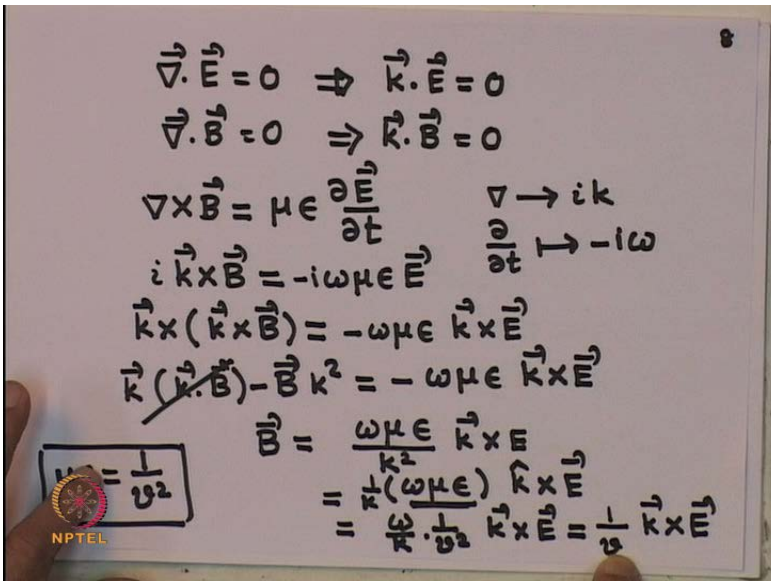
So, omega by k is the, is what is known as the phase velocity. We will see, why it is not just the velocity, but the phase velocity later. So, these are the harmonic solutions, the plane wave solutions of this equation.

(Refer Slide Time: 34:12)



So, let us look at the plane wave solution little more carefully. So, we had already seen that del dot of E is equal to 0.

(Refer Slide Time: 34:19)



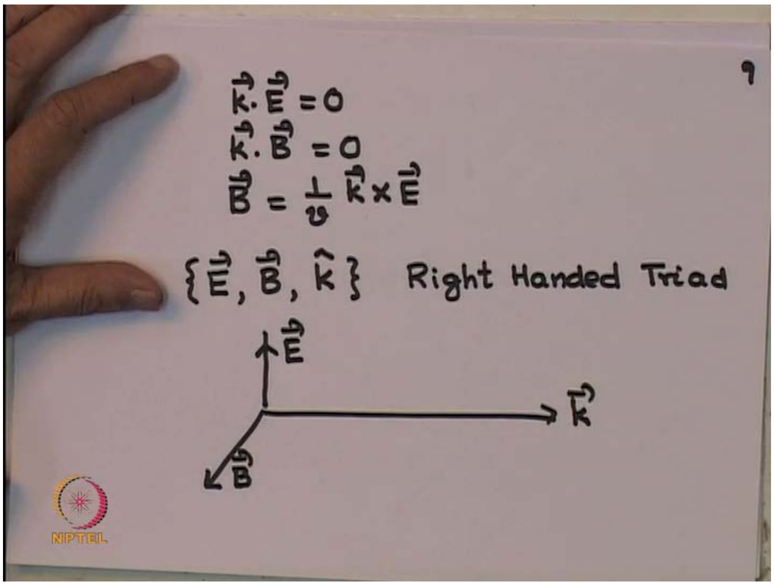
Now, since, I have got E is e to the power i k dot r minus omega t, if you took a divergence, this will lead to k dot of E is equal to 0. Parallelly, since del dot of B is equal to 0, I will get k dot B is equal 0; which tells me, remember that we are in a non conducting medium, which tells me that electric and the magnetic field vectors are transverse to the propagating vector. Thus, that is not all; that is something more that happens.

Let us take the, one of these equations. Let say del cross B equation. I know that del cross B is mu epsilon d E by d t. Now, this will result in i k cross B; operation of del is equivalent to multiplying with i k because of the exponential form that we have taken. So, that is equal to the operation. So, therefore, let me write this, the del operator is same as multiplication by i k. And, d by d t operator is same as multiplication by minus i omega, because I am looking only at forward moving wave. So, this is minus i omega mu epsilon E.

Now, what you do is this; i and i will go away; multiply the, take the cross product of both sides, with k for instance. Let us find out what is k cross k cross B, and that is equal to, well, there is a minus sign; I have minus omega mu epsilon, and then k cross E; this is k cross k cross B, and I know, which can be written as k times k dot B; this left hand side is k times k dot B, minus B times k dot k which is k square, that is equal to minus omega mu epsilon k cross E; k dot B is equal to 0, because the magnetic field is transfers to k. So, this term will go away; minus will take care of that.

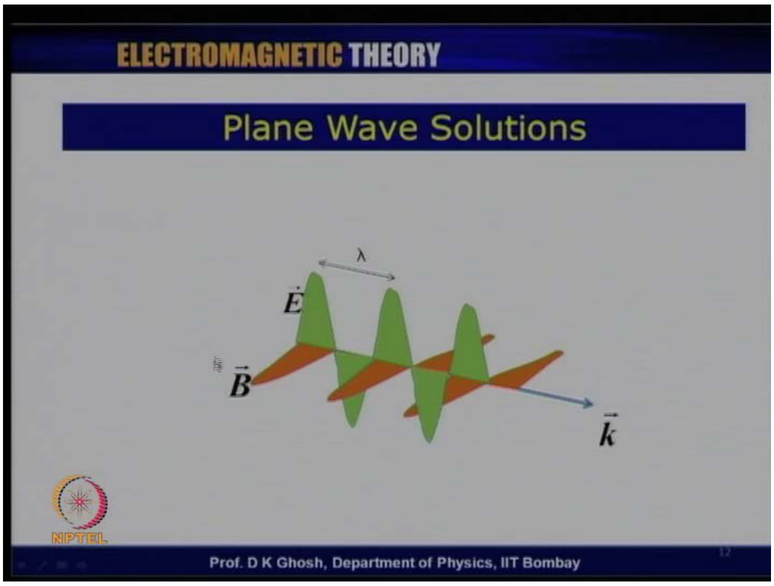
So, therefore, the magnetic field B is omega mu epsilon by k, k cross E. So, you could rewrite this as omega mu epsilon, unit vector k cross E. Let us examine this, mu times epsilon; mu times epsilon, we have seen is 1 over velocity square. This is seen from this equation directly, because this is then a wave equation whose velocity is 1 over mu epsilon. So, therefore, this quantity is written as; this is omega; I should have add a k square there, so therefore, there is a 1 over k still outstanding there. So, I have a omega by k, and mu epsilon is 1 over v square k cross E, and omega by k gives me 1 velocity. So, therefore, this is 1 over v k cross E. And, if I am doing this thing in vacuum, then this velocity is of course, just the c, namely the velocity of light. Let us see what I have got so far.

(Refer Slide Time: 39:12)



What I have got is, k dot E is equal to 0, k dot B is equal to 0, B is equal to 1 over velocity times k cross E. What is the import of these equations? These tells me that E and B are perpendicular to k; and this tells me that E and B themselves are perpendicular to each other. So, it means that the electric field, the magnetic field and the propagation vector k, they form a triad, a right angle triad; they are mutually orthogonal and they form a right handed triad.

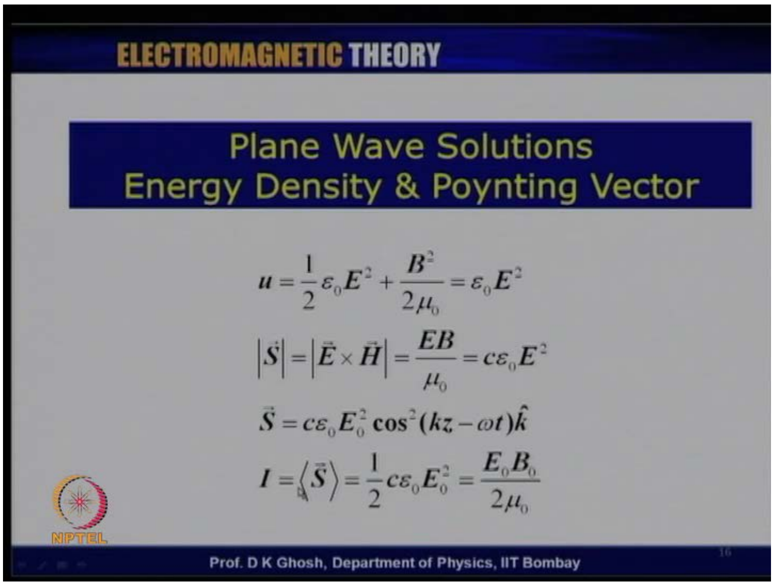
(Refer Slide Time: 40:17)



I will take you back to this picture that I had shown you earlier. So, what it means is this; the, since the electric field magnetic field and the direction of propagation r perpendicular to each other, the, suppose I take the direction of propagation as the z direction, then electric and magnetic field will lie in x y plane; and in that plane the electric field and the magnetic field will be perpendicular to each other. So, what it means is this, that if I have a propagation like this; suppose I have a electric field this way, this is magnetic field and this is my propagation vector.

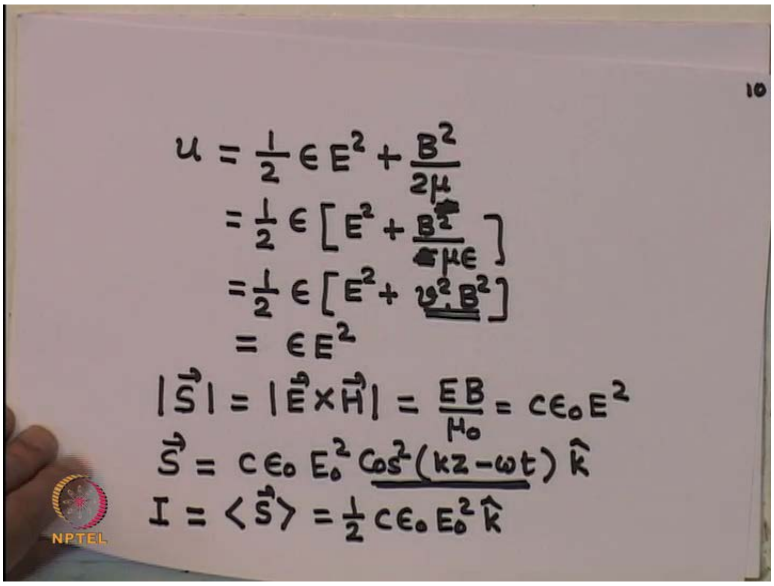
And, as the wave progress, this one moves along k, so that, E and B always remains in one plane. And, what you see there, it is just a pictorial representation; since, E is varying with time sinusoidally, this is the green picture, perpendicular to that the B is also varying with time; one section of this is mixing, but that is what the whole thing is about.

(Refer Slide Time: 41:33)



The, let me also then talk about a few other quantities associated with the electromagnetic field, electromagnetic waves. For example, we had shown that the energy density is given by an expression like this. Half epsilon 0 E square, plus B square by 2 mu 0. And, we have we have already seen that B is equal to 1 over v k cross E.

(Refer Slide Time: 42:09)



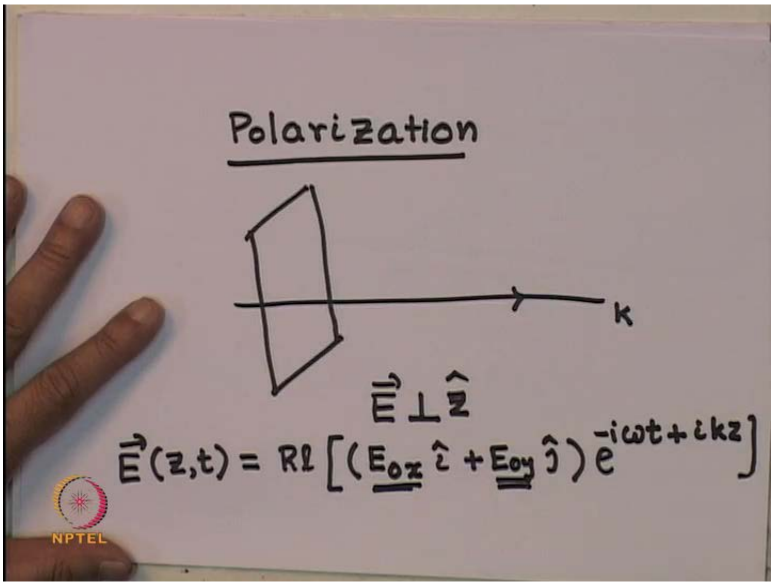
So, I can simply rewrite this; that supposing I want to write down, what is energy density. So, I have got half, well, there is epsilon 0 E square, but I am in a medium. So, let me just generalize. Half epsilon E square plus B square by 2 mu 0, 2 mu. And, if I take half epsilon out, for instance; I will get E square plus B square by 2 mu, B square by mu epsilon; mu epsilon is, 1 over mu epsilon is v square. So, 1 over 2 epsilon E square plus v square B square. So, this is nothing but E square itself. So, therefore, this is epsilon times E square. So, this for example, is one of the ways in which you can write down the total energy density.

The pointing vector S, or let us take it magnitude is E cross H, I would digress a little bit and, sort of, you must have noticed that, whenever we write pointing vector, we prefer to write it as a E cross H, and not as a E cross B, though occasionally that would be true if B and H are linear, but a general expression is E cross H; the reason is that if you recall, the H field is the field due to real sources; and the difference between the B field and H filed for the instance, could arise because of the magnetization; and the magnetisation currents being bound currents, they cannot transport energy. So, therefore, E cross H is the pointing vector.

And, this is nothing but well, I am talking about linear medium. So, it is E B by mu 0. You could also write it as, for example, c epsilon 0 E square. If you look at the pointing vector itself, this will be then c epsilon 0, E 0 square; supposing I am looking at the real part of that exponential, then cos square k z minus omega t along the direction k.

The intensity, which is defined as the time average of the pointing vector; at then take an average of this over a time period, which works out to half. So, this is equal to half c epsilon 0, E 0 square k. So, this is the intensity.

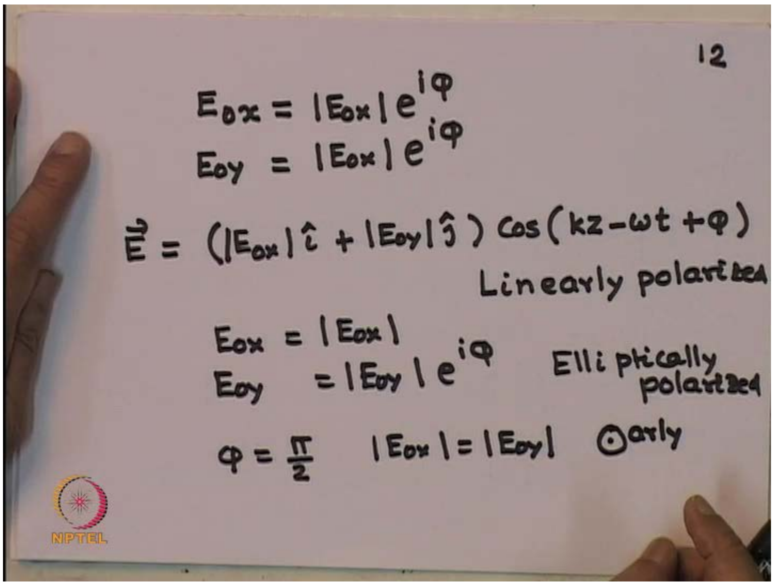
(Refer Slide Time: 45:29)



Now, before I proceed further, I would like to make a comment on what is known as the state of polarization; this is something which is already familiar to you, but let me make some comments in any case. We have said, let me for the moment talk only about electric vector, because the electric vector B being transverse to the magnetic vector. So, if I talk about electric vector, you can make similar conclusion about the magnetic vector as well. So, what we have said is that, the electric vector lies in a plane, perpendicular to the direction of propagation. Now, if that is true. So, this is my direction of propagation, and let us say that this is a plane which is perpendicular to it, the electric vector can be in any direction there.

Now, so we had written that since electric vector is perpendicular to the z direction, I can express a general electric vector E which depends upon z and t, by let us say the real part. Now, since this is in the x y plane, I can write it as, for example, E 0 x times i, plus E 0 y times J, and of course, the exponential factor e to the power i omega t minus, well, minus i omega t, plus i k z. Now, in general, these quantities E 0 x and E 0 y, they are complex.

(Refer Slide Time: 47:32)



Now, let us take some special relationship. So, suppose my E 0 x, supposing is equal to E 0 x magnitude, times e to the power i phi. Now, let me also say that E 0 y is also equal to its magnitude, and suppose I have the same phase, suppose let it have the same phase, then my electric field will be written as, well, real part of; so E 0 x magnitude i, plus E 0 y magnitude J, times cosine of k z, minus omega t, plus phi. Notice, this that, in this case, the magnitude of the vector changes from 0, because cosine can take 0 value, to, of course, E 0 x square plus E 0 y square. But its direction remains constant.

So, the electric field vector, as it propagates, points in the same direction; such a thing is what we are calling as linearly polarized wave. Now, what we could do is, to take different values; for example, supposing there is a phase difference between the x component and the y component, supposing this is E 0 y e to the power i phi; now this will then be, in general, an elliptically polarized wave. If these 2 amplitudes are equal and the phase difference happens to be pi by 2 phi is equal to pi by 2; and E 0 x is equal to E 0 y, I get what is known as a circularly polarized wave.

Next lecture, we will be talking about, how to obtain from electromagnetic theory standard relations, like for example the reflection and refraction of electromagnetic waves.