Electromagnetic Theory

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Module - 4
Time Varying Field
Lecture - 32
Conservation Laws

Last lecture, we discussed about the fact that an electromagnetic field stores energy, and we obtained an expression for the energy density of the electromagnetic field. And what we found is that if you are looking for a system which contains charges and currents, then when you talk about the change in the energy, you have to not only worry about the energy of the sources, you also have to worry about the energy that is stored in the electromagnetic field. And if you have a closed volume then the any change in the energy of the system is due to the fact that certain amount of energy could be flowing out through the closed surface of the volume.

What I want to do today, is to say is to prove that just as the electromagnetic field is a store house of energy; there is a momentum also which associated with electromagnetic field, and as well as a, an angular momentum which is also associated with the electromagnetic field. As a result, if I am looking for a closed system without any external force, then the change, I must have conservation of momentum and angular momentum. And in applying these rules of conservation of momentum and angular momentum, I need to worry about the changes in the angular momentum and momentum stored in the electromagnetic field itself.

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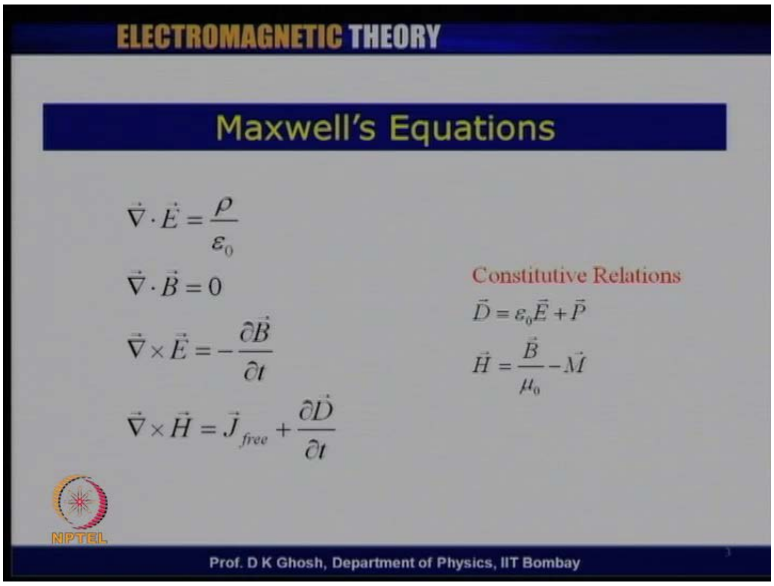

So, we will we started talking about it last time, and so we will be continuing with it today. But this is simply to tell you that this is going to be there in every lecture, that is the set of equations, which govern the electrodynamics; this is complete set of equations which are known as the Maxwell's equation. We supplement them with couple of constitutive relation between the polarization electric field and the displacement vector d; and similarly, between the magnetic field H, the flux density B, and the magnetization m. We will be using them regularly and so therefore, all most on every lecture we need to remind this.

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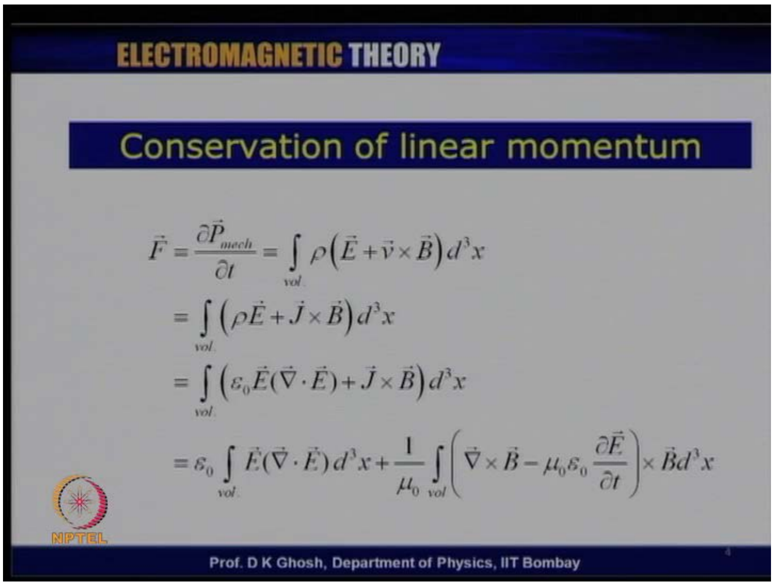

So, let us talk about linear momentum. We start with very basic definition of linear momentum, which is the force, which is the rate of change of momentum.

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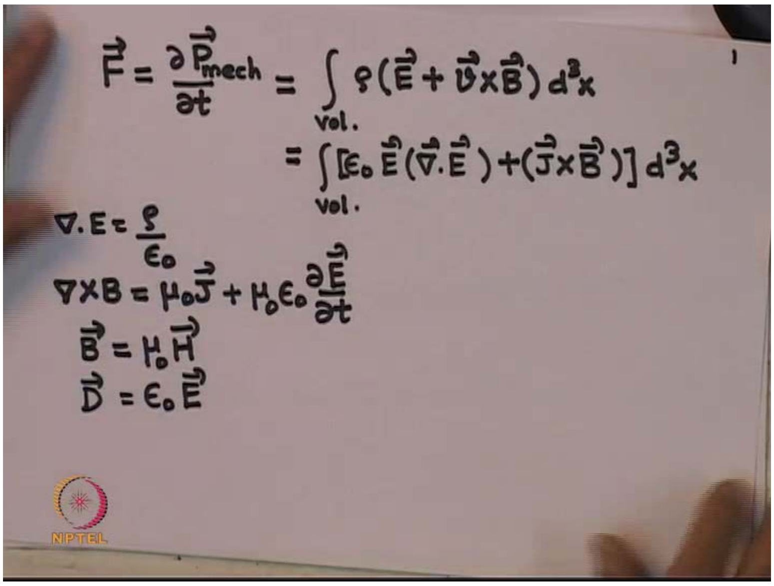

So, mechanical momentum if you like d p by d t. Now, I know that the, if I have a collection of charges and currents, then this quantity is in the continuum limit, is the force due to electric field which is rho E plus the force, Lorentz force due to magnetic field which is rho v cross B, and of course, d cube x. So, what I do now is, that we introduce instead of the E and B, the corresponding expressions from the Maxwell's equation, that we wrote down.

For instance, we know that del dot of E is rho by epsilon 0; and del cross B is mu 0 J. So, in this case, rho times v; rho times v is the current density. So, therefore, I first express this rho in the first equation a, as epsilon 0 times del dot E. So, epsilon 0 times, I already have a E, so E times del dot E; then I have rho v which is J, and that is 1 over mu 0, well; this is actually J cross B. So, therefore, I would write this in terms of the del cross B that I have got; and, so this quantity is J cross B. And we had seen, that in the full Maxwell's equation, we had your, so you notice that let me, I am using B and H interchangeably for the simple reason I have assumed, B is equal to mu 0 H, I am working in free space.

So, del cross B is mu 0 J plus mu 0 epsilon 0 d E by d t; and this comes because if you refer to this set of equations, you find that del cross H is J free plus d d by d t, and once again I am assuming d is equal to epsilon 0 E. So, therefore, I write everything in terms of B and electric field; of course, I could also work with H, but my system is linear. So, therefore, this mu 0 J, that is there; I write it as J is equal to 1 over mu 0 del cross B minus epsilon 0 d E by d t. So, that is my F. And so let me write it fully.

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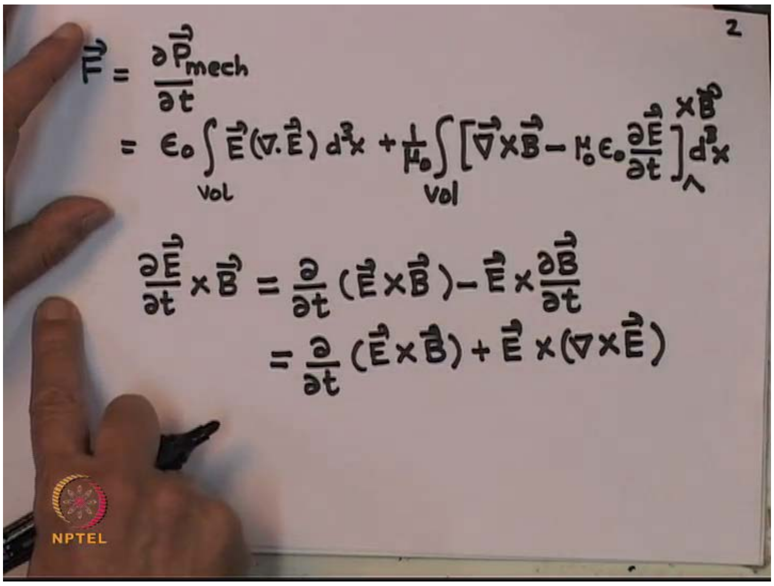

So, F is given by d P mech by d t, is equal to epsilon 0, integrals are all over volume, E times del dot E d cube x plus, J cross B which I write as a 1 over mu 0, integral over volume del cross B minus mu 0 epsilon 0, which actually happens to be 1 over c square at some stage; I will write that also, d E by d t. So, this is my expression for the rate of change of total momentum. And what I did is, to use some algebra; and so at this moment I am not disturbing the first term.

But, the, in the second term, I notice that, the, what I required is another cross E, because I had J cross B, and what I have written down is just J. So, there is a cross B there. Now, so what I do is this, there is this term we had d E by d t cross B. So, I have d E by d t cross B, which I write it as d by d t of E cross B, minus E cross d B by d t. And if you recall that from the Faraday’s law, the first term of course, I do not change, minus d B by d t is del cross of E. So, therefore, this term becomes E cross del cross E.

So, this term I will replace in, in place of this, so if I do that; but before that I will do something else. If you look at this expression, I have here 1 over mu 0, and mu 0 epsilon 0 d E by d t. So, this I have replaced by this, and this, so therefore, 1 mu 0 cancels out, I am left with epsilon 0. So, therefore, this term, when it comes to the other side will just have an epsilon 0. So, Let us write it down.

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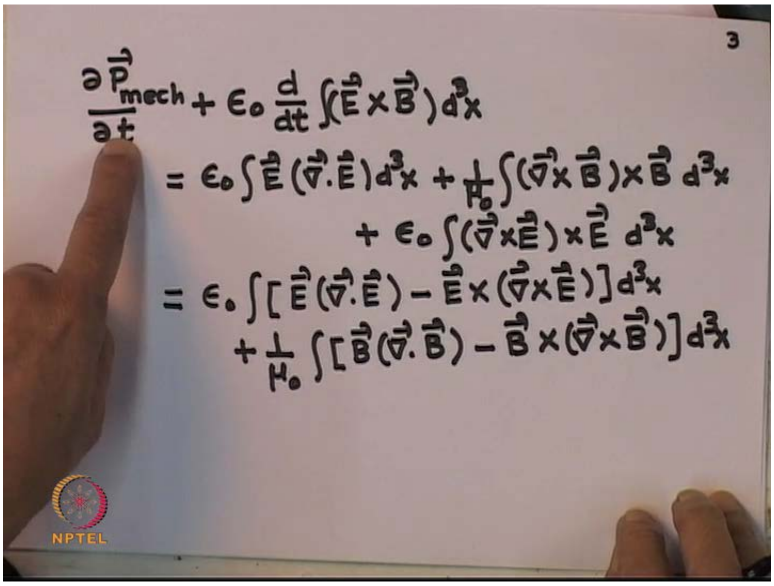

So, I get d P mech by d t; and, I will bring that d by d t of E cross term B term there. And since, this is an integral, the only variation is with respect to total time; so therefore, this is E cross B d cube x. To recall that there was a minus sign, it has come to that side. So, this quantity is equal to epsilon 0, the electric field term, which is E times del dot E d cube x, and a few terms which we just now derived namely, 1 over mu 0 integral del cross B cross B d cube x, minus epsilon 0 integral del cross E cross E, but I have change the order, so let me change the sign as well, d cube x.

So, what I will do is this, I want to write this in a little symmetrically. You notice that electric field has an additional term there del dot E, E times del dot E. The other 2 terms here are very similar, del cross B cross B and del cross E cross E. However, I have a great advantage that del dot of E is equal to 0. So, I can write these symmetrically by putting in at del dot B term. So, let me write this now. So, epsilon 0 integral E times del dot E, minus E cross del cross E d cube x, and the corresponding magnetic field term which is very symmetric and that is written as B times del dot B, which is the term which we have just now added, and that is equal to 0, minus B cross del cross B.

So, rather looks horrible expression, but let us try to, sort of, see whatever we got. On the left hand side I have got d P mech by d t, plus epsilon 0 d by d t of E cross B d cube x.

And we had seen that epsilon 0 E cross B, let me recall for you the expression for the energy density.

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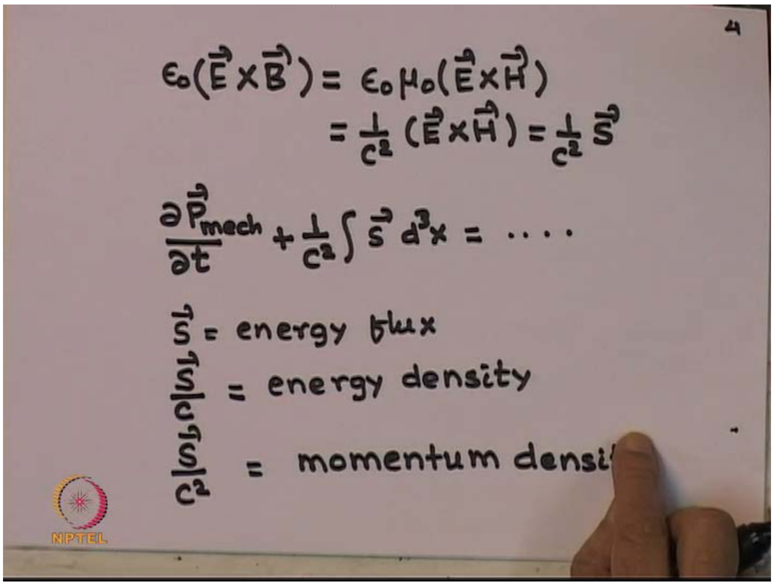

So, we had seen that epsilon 0 E cross B is, since my system is linear, B is equal to mu 0 H, so it is epsilon 0 mu 0 E cross H; and we define E cross H at the pointing vector last time, and epsilon 0 mu 0 is 1 over c square. So, this is 1 over c square E cross H, which is equal to S.

So, therefore, my expression on the left hand side I will simply write it, because the other one is rather detailed. So, this plus, 1 over c square integral S d cube x, and that is equal to that expression that we wrote down just a little while back, a symmetric expression containing between electric field and the magnetic field containing large number of terms.

Now, what else? We had said that S is the energy flux. So, let us look at a few things here. If S is the energy flux, then S by c is the energy density that will be carried by a travelling electromagnetic field, and as a result S by c square. So, this is the energy density carried by the electromagnetic field, and S by c square happens to be the momentum density, because I am dividing it by another velocity.

Now, interestingly all our derivations have so far been non relativistic, but the velocity of light with which we will later on see the electromagnetic waves propagate in free space is coming in, very naturally into the problem. So, next question is this, that; so left hand side, what I have got is rate of change of; this of course, since S by c square of the momentum density, this represents the total momentum associated with the electromagnetic field. So, here I have got the mechanical momentum, and this is my total momentum.

Now, if you remember that when we had this type of a situation, that we have a rate of change of momentum; and on the right hand side, I should have be having a force; and, so normally, we expect the right hand side to be expressed as a gradient of something like a potential, which is my force. But notice that expressing this complicated expression as a gradient of a given quantity seems to be rather difficult; and, it is this quantity which gives us a slightly different way of looking at it, and this is what I will be talking about now.

But, what I will do is this, I will express only the, because this is, these two expressions are identical with respect to electric field and the magnetic field, other than one as an epsilon 0 the other one is as 1 over mu 0; I will just do the algebra for the first, one of the terms, let us say, the electric field terms, and I will simply substitute into a similar expression for the magnetic field terms. So, let me write this term that what is that we have; left hand side we already know that we have got total, rate of change of total momentum.

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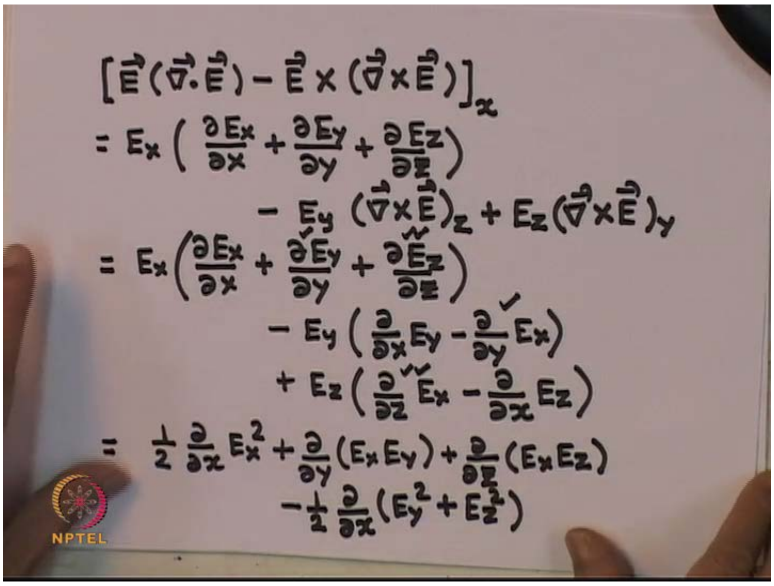

And, so therefore, the quantity that I want to now write is density, because I am not putting their integrals. So, E times del dot E, minus E cross del cross E. So, let us just look at what is the, you know this is obviously, I expect this whole thing to look like a vector, because the left hand side is a vector. So, since I cannot immediately find a simple way of doing it, let me just try to write down, what is the x component of this quantity.

So, here I have got E x; del dot E is a scalar, so since del dot E is a scalar, so I simply write d E x by d x, plus d E y by d y, plus d E z by d z, minus E cross del cross is x component, so which is E y del cross E is z component, minus minus plus E z del cross E is y component. Retain the first term the way it is, E x d E x by d x, plus d E y by d y, plus d E z by d z, and expand the 2 del cross E that I have got, which is minus E y into del cross y z, so it is d by d x of E y, minus d by d y of E x, the plus E z times, this is del cross y is y component, so it is d by d z of E x, minus d by d x of E z.

So, there are several terms there. I want to now, sort of, try to simplify this a little bit. So, first thing that you notice is E x d E x by d x can be written as a half of d by d x of E x square. I have a term here E x d E y by d y, and a termed, either minus minus plus, d by E y, d by d y of E x. So, if you combine this term and that term, what I get is, I can write that as d by d y of E x E y; and identically I have a term here E x d E z by, let me just put a double click on it, E x d E z by d z, and E z d by d z of E x. So, therefore, these 2 terms will give me plus d by d z of E x E z.

What am I left with? I am left with these 2 terms only; both of them have negative sign in front of it. And this is E y d by d x of E y, so therefore, this is half; both the terms are d by d x term, and I will write this as E y square plus E z square. If you look at this expression now, you notice that in this term if I added an E x square then I will get E x square plus E y square plus E z square, which is E square. But if I added an E x square inside the bracket, which is like subtracting half of d by d x of E x square, so I must add another half, so which will be, this half will go. So, if you now combine all these expressions, what you are getting is, because all these terms are d by d x, this is d by d x, this is d by d y, this is d by d z. So, what I am getting is, that half has gone away.

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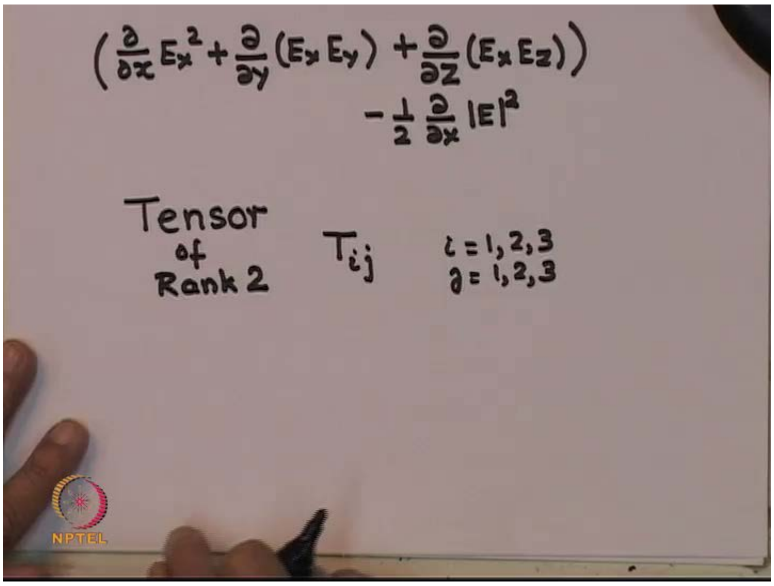

So, I am getting d by d x of E x square, plus d by d y of E x E y, plus d by d z of E x E z, and minus half d by d x of E square. And this if you recall, is just the first term, that is the x component, this is what I am trying to do. So, therefore, when I add up various components, I will get d by d y of E y square, d by d x, etcetera, etcetera.

How do I simplify this? Now, the thing is; so I have similar terms, 2 more terms, which I must add up. Now, what was found, which I will prove by first assuming the result, is that the, this quantity, after I add up the corresponding y and the z components, it can be written, not in terms of semi, and expected it to be a vector and it is a gradient of a scalar quantity. So, you notice that, when I expected this to be a gradient of a scalar that is going up, scalar high is one quantity. When you take the gradient, it becomes a vector which is characterized by 3 quantities.

Now, just as you define a scalar as a quantity having essentially a single quantity, a vector characterized by 3 quantities in cartesians x, y, z, now one can define a quantity, which is known as a tensor. Now, a tensor can be of arbitrary rank. So, for instance, a tensor of rank 2 is characterized by 9 quantities. So, for example, if I talk about a tensor T, just as a vector v is characterized by v x, v y, v z, a tensor T is characterized by a pair of indices.

For example, the components of a tensors will be T i j, where i going from 1 to 3, 1, 2, 3, and j of course, also going from 1 to 3. And one can in principle define a tensor of; this is tensor of rank 2; one can define a tensor of rank 3 with a quantity characterized by 3 indices, namely T i j k, that is of course, 27 quantities. So, this is 3 square quantities. Now, notice one thing that the reason why I cannot express this quantity as a gradient of a potential is because this seems to mix up the components; one good thing is it never mixes up old components, it mixes up maximum 2 components at a time. Now, that tells us that maybe we should not be looking for expressing it as a gradient of a scalar, but we should probably go hire up, take, talk about a tensor of rank 2; and of course, by doing an appropriate algebra reduce the tensor of rank 2 to a vector, because after all left hand side is a vector, and that is what we do here.

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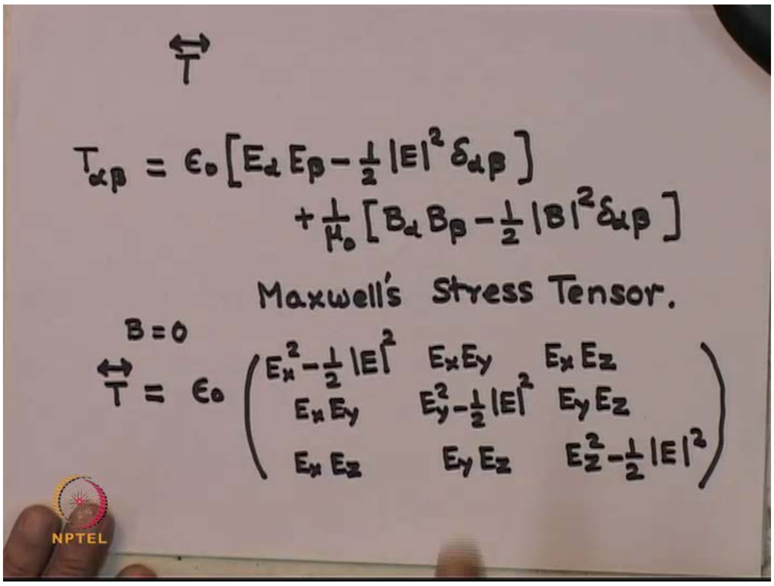

And, so what one can show, which I will show after assuming the result; that if you define a tensor of rank 2, which I will indicate by a notation like this, a double arrow. So, that the alpha beta components; alpha going from 1 to 3 is epsilon 0, E alpha E beta, minus half of E square times delta alpha beta. Well, remember that I worked only on the electric field; so what I have is, actually a corresponding term from the magnetic field as well, and this will be B alpha B beta, minus half B square delta alpha beta. This quantity has been given a name as Maxwell’s stress tensor.

What do you want to do now, is to obtain the relationship between, what I proved here, that is the, remember the integration of, this is just a x component, add up the corresponding the y and the z component, take the integration, then you should be able to show, that this is nothing but the momentum that appears on the left hand side.

So, let us, let us look at how one handles this. So, basically this is what we have written down, but let me, let me illustrate it, by sort of writing down specifically some components. For example, let me take B is equal to 0 as an example. Now, if you take B is equal to 0, my stress tensor only has electric component. So, the T can then be expressed as a matrix, which is epsilon 0. Now, remember this is x x, x y, x z; like this it goes. So, x x is E x square, because E x E x, minus half E square, because it is x x; now this is E x E y, the delta alpha beta term does not come in, this is E x E z; there is a symmetric tensor, because 1, 2; 2, 1. So, this also dou u y E x, I will write it as E x E y; this is E y square minus half E square; this is E y E z; this is E x E z; this is E y E z; and this is E z square minus half E square. So, this is, this is the way one could write down. So, this is T x x, this is T x y, this is T x z, T y x, T y y, T y z, etcetera, etcetera. And if you do not have magnetic field is equal to 0, then of course, this will become much bigger.

Now, what we want to show is this, that if you take a divergence of this quantity, then you get hold of a scalar, a vector. Now, remember that when we had a vector, if you did a divergence you got a scalar; in other words, the vector is a tensor of rank 1. So, when you took a divergence of a vector, which is divergence of a tensor of rank 1, you got a scalar which can be regarded as a tensor of rank 0. So, if I take divergence of a tensor of rank 2, I expect a vector, and that is what I am looking for. So, let us look at that.

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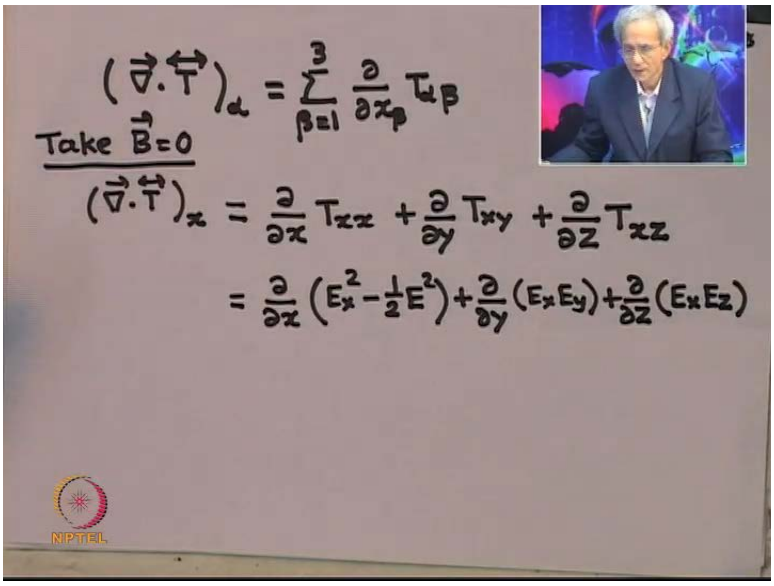

So, del dot T, T is a tensor. This quantities, for example, if I look at the alpha component of this, it is a vector, so there are 3 components there. So, this is sum over beta, beta is equal to 1 to 3, d by d x beta of T alpha beta. This beta index is summed, and so therefore, it is just that. So, let us, let us look, take again B is equal to 0, because it is a symmetric term. So, therefore, whatever I do for E, I can carry it over for B.

So, what I will show is, that if you take del dot T; and let us look at its x component, and let me take B equal to 0 for convenience. So, by definition, here is, here is the definition. So, therefore, I get d by d x of T x x, plus d by d y of T x y, plus d by d z of T x z. Now, remember, I had already derived, what is T x x, what is T x y, what is T x z. So, this is equal to d by d x of E x square minus half E square, plus d by d y of T x y is E x E y, plus d by d z of T x z is E x E z.

Now, this is precisely what we had shown to be the x component of the quantities which are there. So, notice this that I wrote down; this is the way I had written it down; this is exactly the x component. So, in other words, the Maxwell's stress tensor that we have defined, its divergence which makes it a vector gives me the right hand side of the equation that I wrote down. So, let us now repeat, rewrite the equations; they are summarized on the screen.

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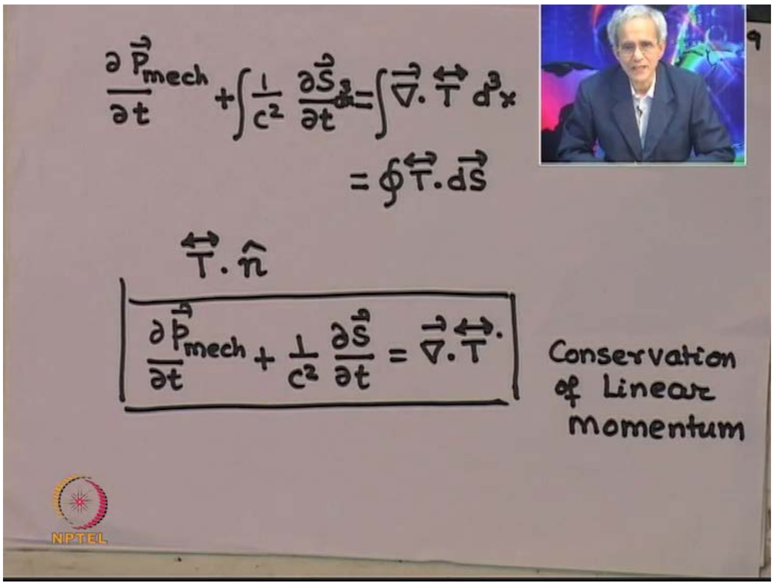

So, this is d P mech by d t plus, we had seen this is 1 over c square d S by d t, so that is equal to divergence of the Maxwell's stress tensor. So, if you now want to write down; if you remember that this is my way I wrote down, but this is total d p mech by d t, this should have been integrated out, because that was a density, so this side should also be integrated out.

And, as a result one can prove a theorem very similar to the way we prove the divergence theorem. So, this quantity will give you the surface integral of T dot d S, where, exactly the way we did it, T dot n can be interpreted as the momentum flux which is normal to the boundary surface. Now, if instead of this total momentum, total electromagnetic field momentum, you want to write it in terms of the momentum density, then what you get is, d p mech by d t, small p I am using for density, plus 1 over c square d S by d t, is equal to del dot T.

The interpretation of this is very similar to the way we interpreted the energy density. So, I have got on the left hand side, the net electromagnetic momentum and the mechanical momentum, and any change in this can happen due to the momentum flux that is going out of the surface. So, in some sense, this equation is an equation for conservation of linear momentum. I will give some very simple illustrations there, of this.

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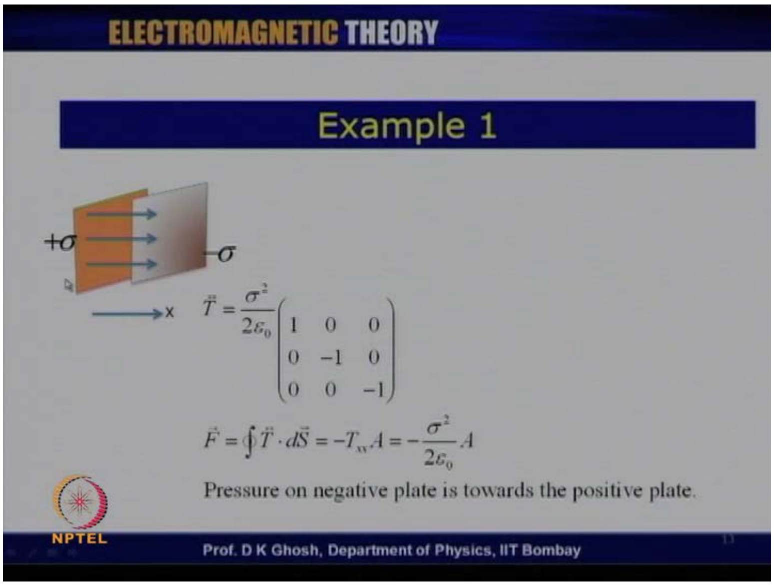

One is a rather simple example; I will give a little more complicated example later. Just consider a pair of capacitor plates, we know that. So, let me take as the positive x direction; this is the positively charged plate, this is the negatively charged plate. So, the electric field goes from the positive charge plate to the negative charge plate inside the, between the capacitor plates.

So, if you look back, and look at what is the field, the Maxwell's stress tensor is like; the, you notice that the stress tensor is E x square minus E square, now remember electric field is in the same direction as the x direction. So, therefore, E x square and E square are the same, which means this terms should be E square by 2; and since, there are only x components, any cross terms will be 0. So, this tensor will be a dynometrics, and here what I have written down is the strength of the electric field which is sigma square by 2 epsilon 0, 1, minus 1, minus 1. So, that is this should have been squared. And the, if you look at F equal to T dos, T dot d S, then you can simply find out, how much is the pressure on the negative plate.

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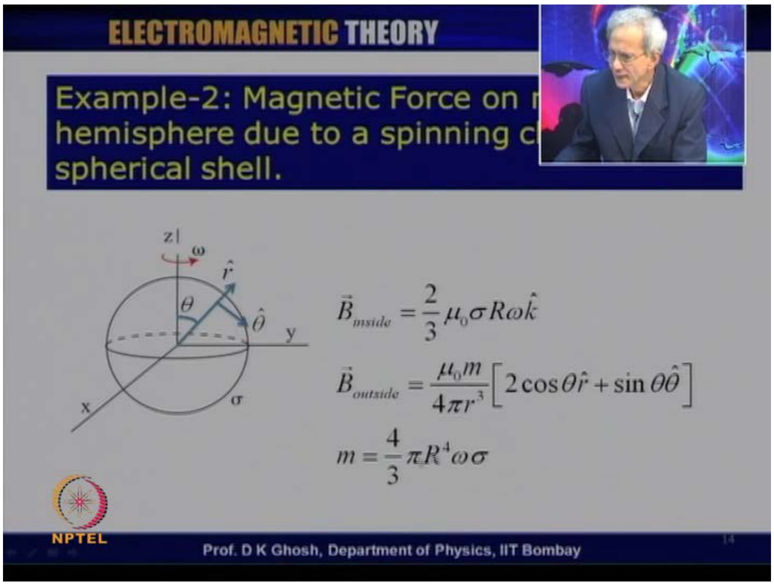

Let me take a rather difficult example. And this is a problem which we had talked about earlier, when we talked about the way to discuss the magneto statistics; but this time I will, I am looking at it as an illustration of, how to use Maxwell's stress tensor. So, this is, I have a shell, which is a charged shell, and the charge is, charge density is sigma; and, this is rotating with an angular velocity omega. What do you want is, to find out, how much is the force exerted by the southern hemisphere of the charge sphere on northern hemisphere. So, it is a rotating charge disk.

Now, you, in the during, when we did magnetic statics, we, I have taken up this type of a problem and we had seen that, here is the coordinate system; this is your radial vector, perpendicular to that along the direction of increasing theta is the direction of theta vector, and this will be z, this is a standard cylindrical coordinate system if you like. Now, the magnetic field inside was constant. Magnetic field inside was given by two third mu 0 sigma r omega, along the z direction. Magnetic field outside had this expression, mu 0 m by 4 pi r cube, where m is a magnetic moment and which is related to this by 4 pi by 3, actually it is r cube into r omega sigma, but r 4 omega sigma; and this is the expression for the magnetic field outside.

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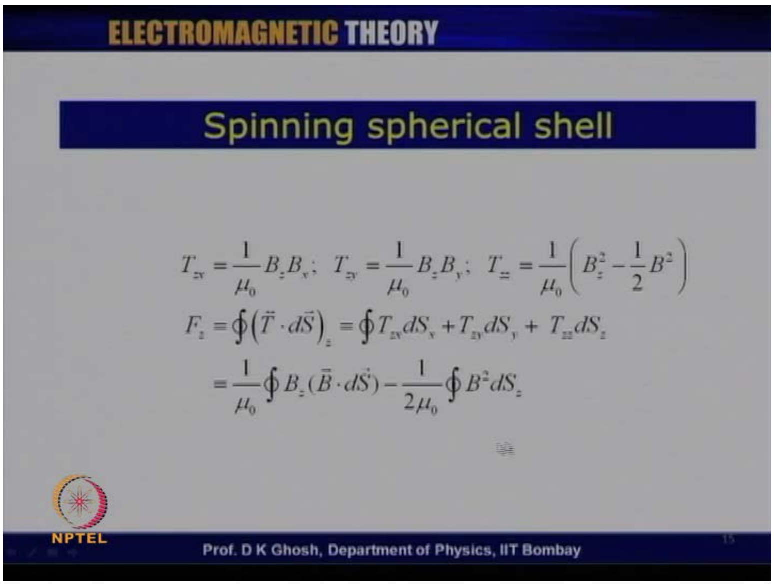

So, what we are going to do is, to try to find, write it in terms of the Maxwell's stress tensor. Let me, let me emphasise that this is not this simplest way of doing this problem, but nevertheless we are looking at it, as in application of the Maxwell's stress tensor. So, let us look at what we have.

So, firstly, you have to realize that by symmetry the force on the northern hemisphere must be along the z direction. Now, since it is along the z direction, I should be looking for only F z. So, if I look at F z, we have seen F z can be written as the z component of del dot T, which has T x y, T x z, and things like that. So, let us, we are not interested in all the T x's. So, we are interested in T; one of the components is fixed as z.

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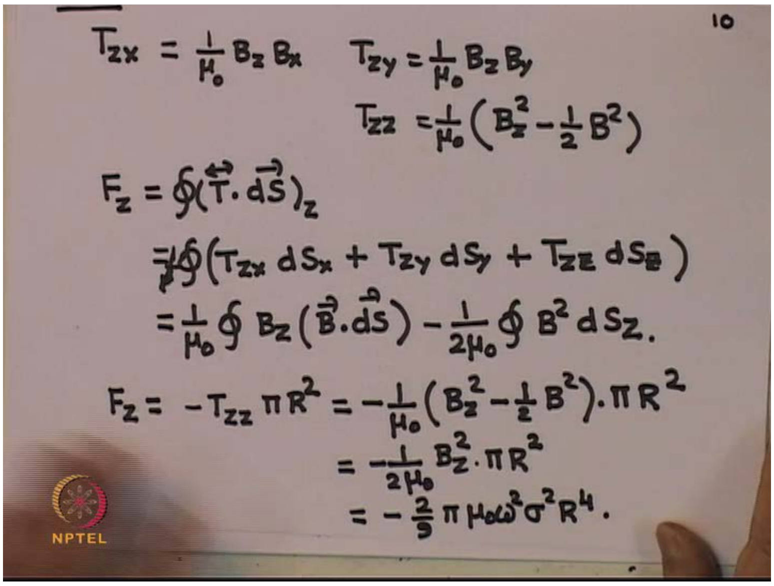

So, T z x; remember; there is another point, that so far I was singling out the electric field. Now, in this case, I am only looking at the magnetic force. So, as a result, I will assume that the electric field is 0. Now, this is not really true, because the electric field will be always there, the electric force will be always there because there is a charge density there. So, therefore, but that is a different part, whether it is spinning or not there will be electric force, I am not looking at that.

So, T z x is 1 over mu 0 B z B x. T z y is 1 over mu 0 B z B y. And the diagonal component which was T z z is 1 over mu 0, remember this was B z square minus half B square. And how much is F z? This we had seen. F z was the surface integral of T dot d S. And T dot d S is a vector, though there is a dot, but you must realize, this is a dot product of a tensor, so therefore, it is z component; which is equal to T z x d S x, plus T z y d S y, plus T z z d S z.

Now, let us look at, how we can write this. T z x is B z B x, so this is B z B x d S x. So, I get a, other than B z, I get a B x d S x. From this term I get a B z B y, so I get B z into B y d S y. So, therefore, I can get a 1 over mu 0, I have forgotten. So, let me, now, 1 over mu 0 will come here, 1 over mu zero. So, what we have seen from these 3 terms, I will get a B z and a B x d S x plus B y d S y plus B z d S z, which is nothing but B dot d S. T z z, you remember had an additional terms, which is, so I must write it as minus 1 over 2 mu 0, integral B square, this time B S z only. So, this is my F z.

Now, I am going to calculate both of them separately now. So, in other words, I will have to calculate a contribution due to the internal field, which you have seen is constant, and an external field. I will illustrate just one of those things. So, look at, what is F z, for the fields inside. So, when you are looking at the inside, the cap that is there, is the equatorial plane. So, the only thing that I have there is the pi r square, and the magnetic field is constant there, along the z direction. So, as a result, this simply gives me minus T z z and pi r square, minus because it is in negative direction.

And, I know what is T z z? So, which is 1 over mu 0 B z square minus B square by 2 into pi r square. And we have seen that inside B square and d z squares are the same. So, therefore, this takes care of minus 1 over 2 mu 0, B z square or B square whatever, into pi r square. Substitute the constant field which we have here, which is 2 third's mu 0 omega sigma r whole square; and, you can simply write this as minus 2 by 9 of pi mu 0 omega square sigma square r fourth. So, this is, this is my force on the hemisphere, northern hemisphere due to the field which is inside. Now, let us look at what is happening outside. The outside expression is a little more complicated, but the principle is more or less the same.

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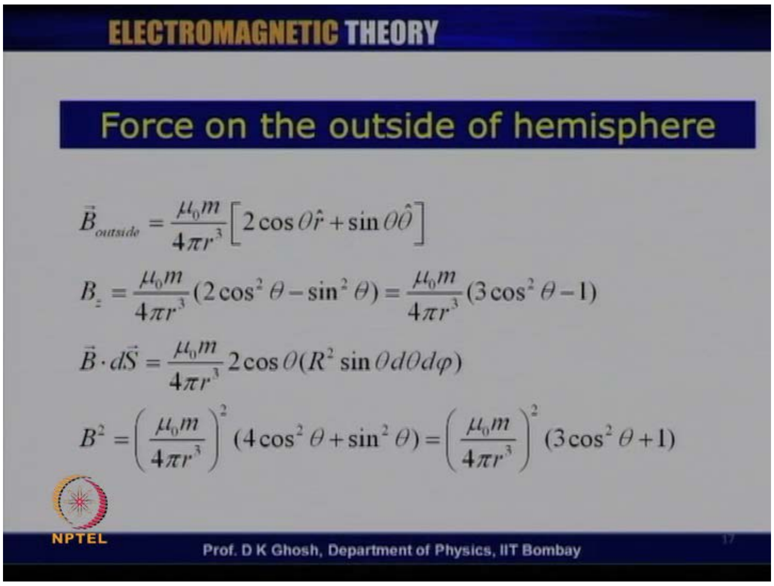

So, look if you look at it, B outside is mu 0 m by 4 pi r cube, I have told you m, I given you an expression, into 2 cos theta r plus sin theta theta. And I mean, remember my first and last B z times B dot d s; so I need that. And of course, there is another term which is the second term, which is d S z.

So, what is B z? B z is r dot z. Now, r dot z is cos theta. I already have a cos theta here. So, that gives me 2 cos square theta. And what is theta dot z? Theta dot z is minus sin theta, minus because the angle theta is the direction the angle that the radial vector makes with the z axis, so that, increasing theta direction is, gives you a minus sign. So, it is 2 cos square theta minus sin square theta. And so you can rewrite this as mu 0 m by 4 pi r cube 3 cos square theta minus 1.

What is B dot d S? Well, the magnetic field is given by this. Now, d S on the surface of the hemisphere is along the radial direction. So, therefore, I do not have to worry about this term, I simply worry about. So, r dot r is 2 cos theta, and the surface element is r square sin theta d theta d phi. So, if you plug this in, and the secondly, I need a B square term, which is simply taking the square of that, modulus square of this, which is 4 cos square theta plus sin square theta; add them up, you find B square is given by this expression, times 3 cos square theta plus 1. With this you fix both the B z and B square, which are required for the calculation of the stress tensor.

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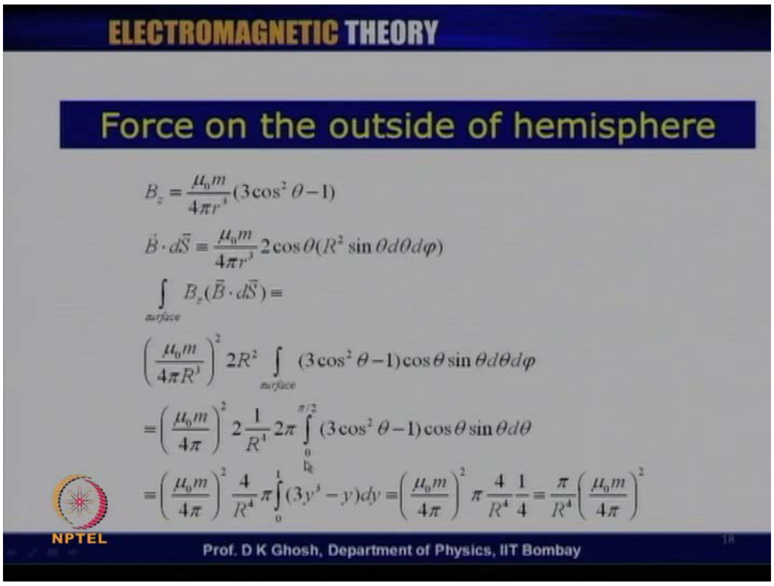

And so this is what we have bring down. So, let us look at any one of those integrations. So, B z was this, B dot d S was this, what is the surface integral of B z B dot d S. Now, this is absolutely trivial, because all that is you have to realize is, these are cos theta integration, and there is a sin theta d theta there. The phi integration does not come into the picture. Since, it is a hemisphere, the integral limits are from 0 to pi by 2. You work that out, you get substitute for m, you get an expression like this.

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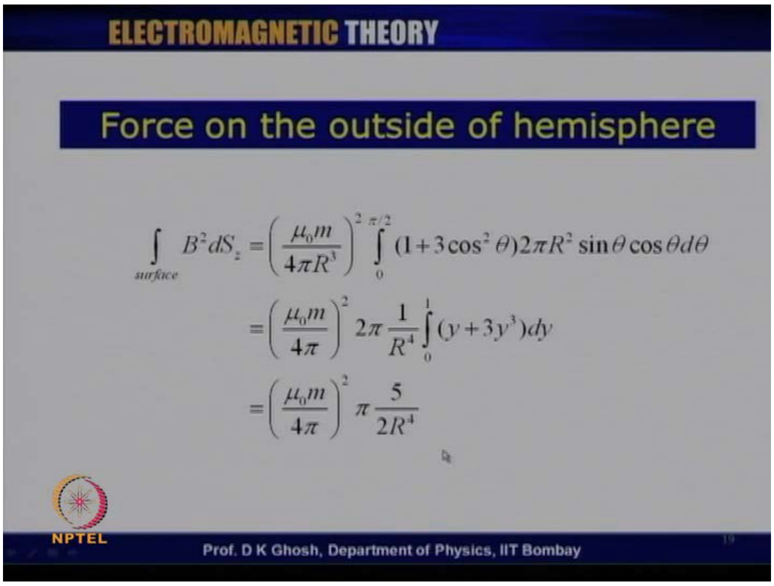

I can repeat that for the second term, which is B square d S z. Once again the integrals are absolutely straight forward, because the integrals are on cos square theta, and the surface element gives me a sin theta. You add this off. So, what I need to do now is to add up all the 3 things that we have got. 2 terms of the field outside and the term that we obtain for the field inside, and you get that the force that is acting on the northern hemisphere, is given by an expression of this type.