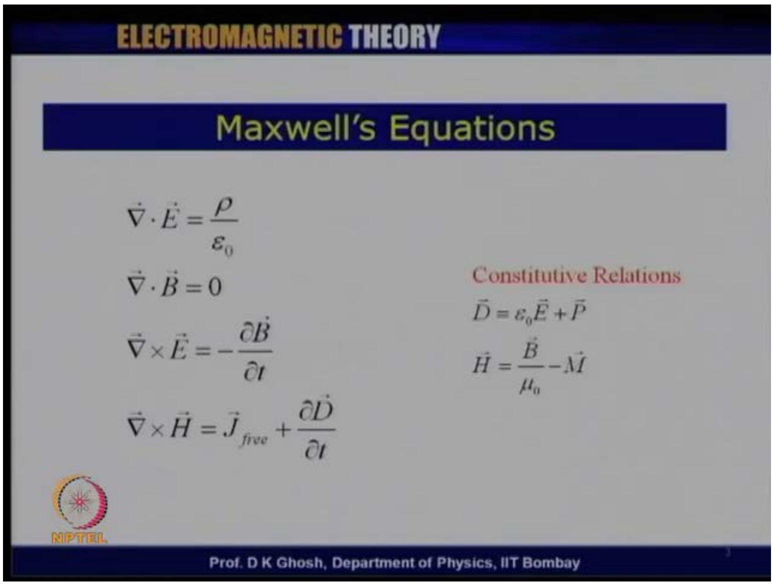
Electro Magnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Module - 4
Time Varying
Lecture - 31
Maxwell's Equations and Conversation laws

In the last lecture, we introduced the concept of gauge invariance. What we did was to realize that in an electromagnetic theory, we would like to determine the electric and the magnetic field components, which are six quantities. Now, if we put it in the language of the potential that is if you formulate the electromagnetic theory, in terms of the scalar potential which is a one component and three components of the vector potential, so there are four components we got two coupled equations. What we did is to, then realize that in defining potential, we have certain amount of choices liberty which we call as the gauge choice, using that we are able to decouple these equations. The after that, what we introduced was what was known as the pointing theorem, which talked about a conservation of energy in the electromagnetic phenomena, which brought in the concept of an energy associated with the field itself.

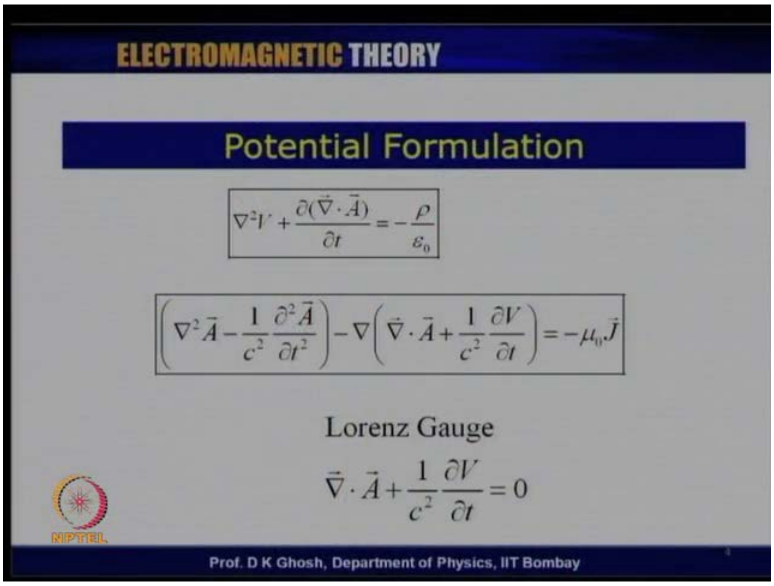
(Refer Slide Time: 01:36)



So, we will continue to make observation based on these. So, last time we had completed deriving all the four Maxwell's equation. So, del dot of E was rho by epsilon 0.

Alternatively del dot of D is equal to rho free. Del dot B equal to 0 is always true because there are no magnetic monopoles. The Faraday's law which told us that del cross of P is minus D B by D t and del cross of H than t are Maxwell's law gives me free current plus D B by D t. And there were set of two constitutive relation and those were the displacement field D is given as epsilon 0 E plus P, and the magnetic field H is given in terms of magnetic flux density B by mu 0 minus M. So, these are the complete set of Maxwell's equations we deal with.

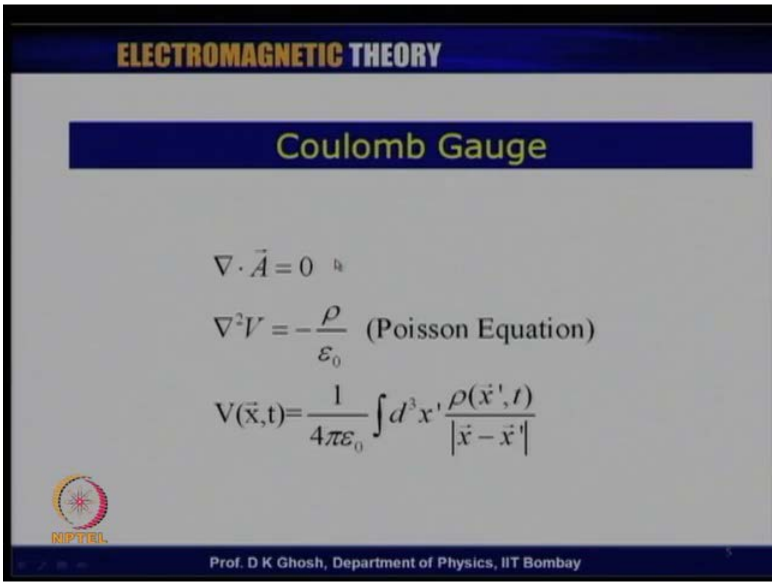
(Refer Slide Time: 01:36)



So, when we did the potential formulation, what we did is to realize that we have this equation. That they were two equations; one was for the scalar potential V which is del square V plus D by D t of del dot of a equal to minus rho by epsilon 0 and another equation for the vector potential which in addition to having del square A minus 1 over c square D square a by D t square. There was another term like this equal to minus mu 0 J. So, we talked about what is called as a Lawrence gauge, and in the Lawrence gauge we said that the relationship between A and V is that del dot of a plus 1 over c square D V by D t is equal to 0.

We also found out that supposing this condition is not initially satisfied, we could always make a choice that is A going to A plus a gradient V going to V plus a constant in such a way that this relationship becomes valid. Now, you can check that when this relationship satisfied because this quantity in the bracket is 0. So, we get del square A minus 1 over c square D square a by D t square equal to minus mu 0 J and similarly, since del dot of A satisfies this relationship this will also give me an equation, del square V minus 1 over c square D square by D t square V equal to minus rho by epsilon 0.

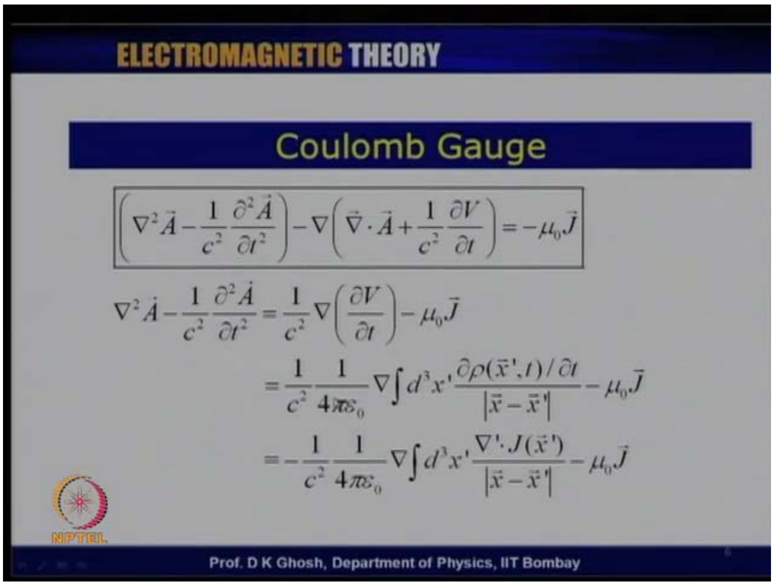
(Refer Slide Time: 04:13)



So, the two potential equations were decoupled in Lawrence gauge is a very useful gauge to work with. But if you recall we have been when we talked about magneto statics. We have been always talking about what we called as a Coulomb gauge. In Coulomb gauge the divergence of the vector potential was taken to be equal to 0 and naturally you would be asking yourself the question that is that gauge any good, the answer is yes and let us see what it does if del dot of a equal to 0, then you refer to this equation notice del square V plus D by D t of del dot of a which is now 0 is minus rho by epsilon 0. So, that tells me that del square V is equal to minus rho by epsilon 0.

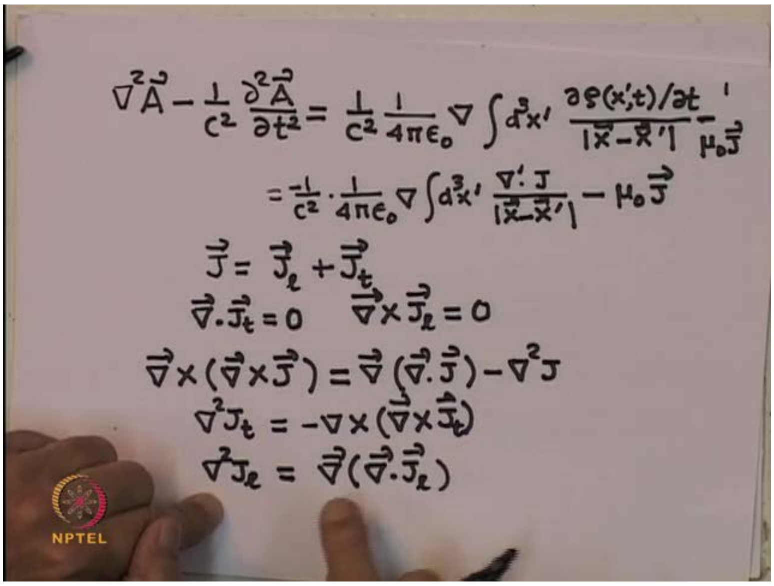
In other words potential V satisfies the Poisson's equation. So, that equation is stands by itself, which has a solution V of x t equal to 1 over 4 pi epsilon 0, integral d cube by x prime, rho x prime t x minus x prime. This of course, we know is the formal solution of this equation because if you apply del square in both sides del square of 1 over x minus x prime is minus 4 pi delta function and that will take care of this.

(Refer Slide Time: 05:41)



The problem is with the second equation. The second equation is a little complicated equation and see in Lawrence gauge this term was dropped and so that there was an equation for A itself, without involving the potential V. Now, if we simply say del dot of A is equal to 0, this equation still has a gradient of d V by d t. So, del square a minus 1 over c square equal to 1 over c square the gradient of d V by d t minus mu 0 J. Now, the question is that we have to do something to get rid of this V because what we are interested in is an equation which involves A, and let us see how it gets done. So, in this case we have just now, seen the expression for V can be written as a solution of the Poisson's equation, which is 1 over 4 pi epsilon 0 d cube x prime rho x prime by this. So, what I do here is this, since time differentiation is independent let me write it down here.

(Refer Slide Time: 07:00)



So, we will do the following. So, we have del square A minus 1 over C square d square A by d t square. This is equal to 1 over C square, I have a grad of d v by d t and the I already know the expression for the potential. So, 1 over 4 pi epsilon 0. This constant comes out grad of the potential term which is the solution of the Poisson’s equation and that is we have just now seen is d cube by x prime. We had a rho x prime, but there is a d by d t. So, it is d rho x prime t e by d t divided by x minus x prime and d cube x prime minus mu 0 J of course, is there. We are not really going to be doing much about this term till the end, because it is any way an homogeneous term which will sort of stay right till the end.

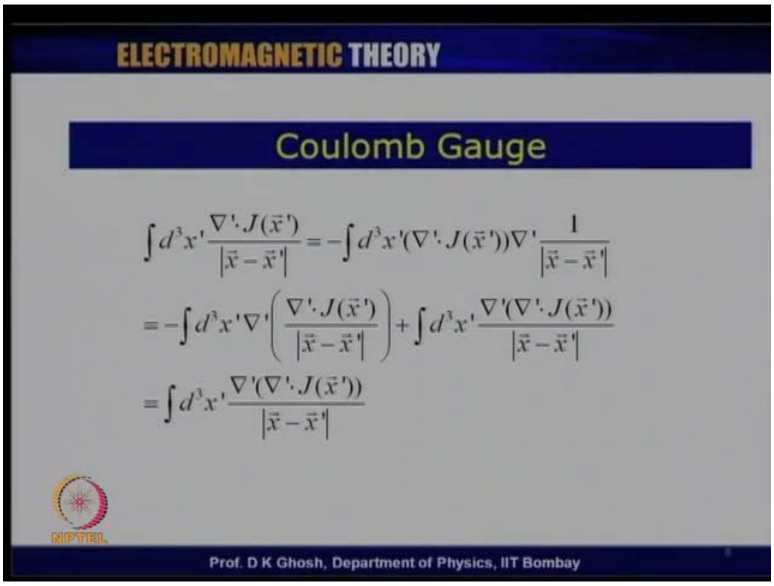
So, the question is how does one handle this equation, I have at least formally removed v and brought in a charge. But you notice that the way it comes is d rho by d t and d rho by d t is nothing but a current. Excepting that this current is J of x prime t. So, let us write it down. So, it is 1 over C square 1 over 4 pi epsilon 0, gradient of d cube by x prime and I got now, this gradient is with respect to the primed variable. So, del prime of del prime dot J, this I am writing down from the continuity equation d rho by d t plus with actually I need a minus sign there. So, d rho by d t is minus del dot J divided by x minus x prime.

This is simply using the continuity relationship between the current and the d rho by d t. So, therefore, what I need to do is this. So, this is what I have let me further simplify it. This is the term which we are interested in looking at. But before it proceed let me make a few comment about a vector. Now, if I have an arbitrary vector J current density vector I can resolve it into two components. One I will call as the longitudinal component J l and another I will call as the transverse component J t and the relationship is that del dot of J t is equal to 0 and del cross of J l is equal to 0. Now, this is something which we have been doing all the time, saying that a vector is completely specified by specifying its curl and the divergence.

So, this also implies that if I have an arbitrary vector. I can resolve resolve it into a curl free vector and a divergence free vector. The so this is the way we would resolve a general vector. Now, notice if I now try to find out what is del cross del cross J. We have used this equation several times. We have seen that this is nothing but del of del dot of J minus del square J. So, if I now want to write down, what is del square of transverse component of J. So, if I take a transverse component, I notice that the curl does not vanish. But divergence of the transverse component is 0. So, del dot of J transverse is 0 and I will be left with them. There is a minus sign there. So, minus of del cross del cross J, I can write down del cross del cross J t, but that is the same thing because del cross of J l is equal to 0.

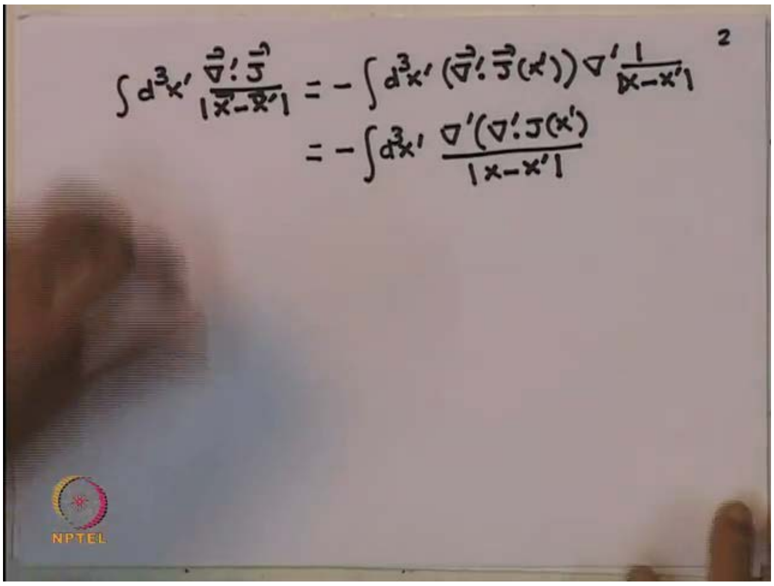
Likewise if I have to calculate del square of J longitudinal, then the curl part is equal to 0 and I am given left with gradient of del dot of J and write this also as t. So, these are the few vector relationships which I am going to be using to simplify this expression remember what I have done. I have written an arbitrary vector J in terms of a longitudinal component and a transverse component one which has the curl equal to 0, the other which has divergence equal to 0. So, that the Laplacian of the transverse component is related to del cross del cross of that quantity and Laplacian of the longitudinal component is related to the gradient of the divergence. So, let us proceed as to what does it mean and how does one use it.

(Refer Slide Time: 13:06)



So, when I turn back to what we had. So, I had minus 1 over C square. Well primarily what I had was this type of writing. So, let me let me try to see what is this integral to start with and then I will plug all these things there. So, this integral that I have got, is integral d cube x prime del prime dot J by x minus x prime and you remember I have to ultimately take the gradient of this quantity.

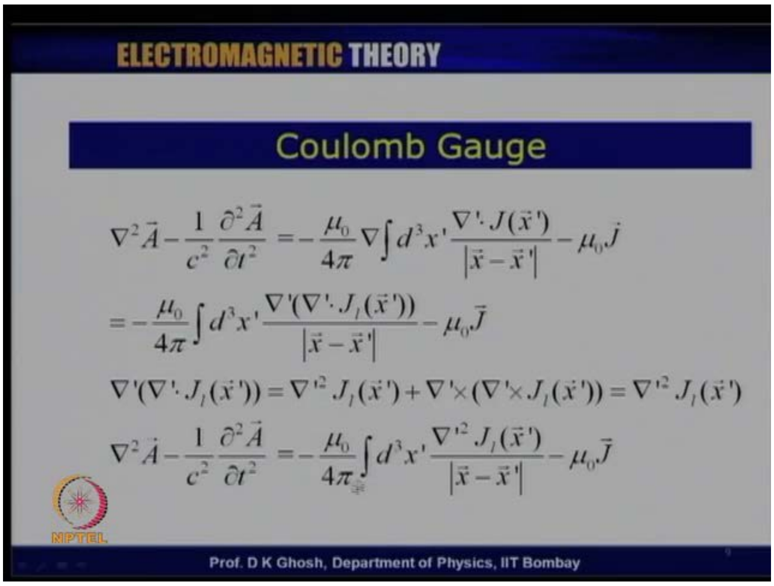
(Refer Slide Time: 13:24)



So, what do you do is this. That this quantity, I will write as by simple chain rule. What I will do is to say that this quantity is equal to. Let me write it down and then I will see why this is d cube by x prime del prime dot J of x prime. Then del prime of 1 over x minus x prime. Let me just see what I am doing. See, what we I do is this. That I say that supposing I take supposing I take a gradient of the whole thing that is gradient of gradient del prime of del prime dot J by x minus x prime. Then by general differentiation that is this factor multiplied with gradient of that and the other factor is what I had there.

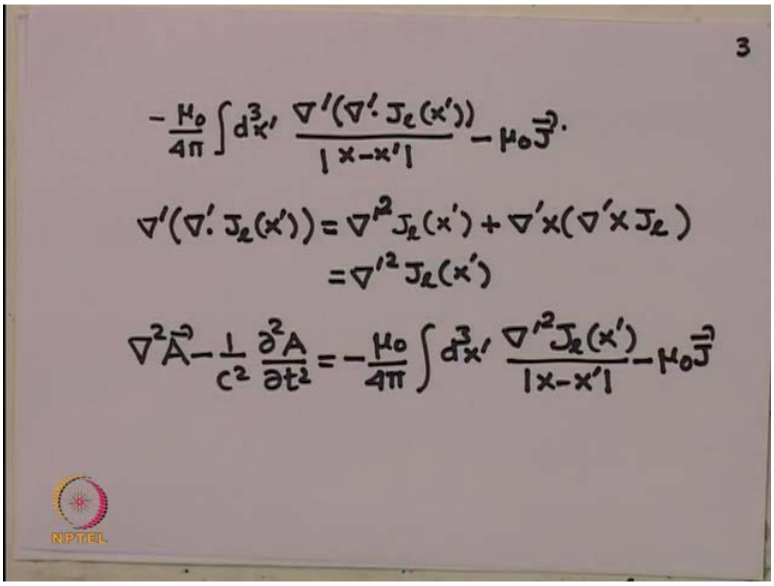
That is this quantity that I have written down and when I get gradient of this whole quantity is gradient of del prime dot J by x minus x prime minus this quantity and this when, there is a del prime of the whole thing I can use the analog of divergence theorem, to convert that volume integral to a surface integral and since, all fields current sets etc go to 0 at infinity that type of a term drops out. So, this integral that I had is this quantity. So, this is what we have written down there. Now, what I will do is this that this is equal to minus integral of d cube by x prime del prime of del prime dot J of x prime divided by x minus x prime. See what I am actually doing is this. There is a gradient to be taken outside this and I am trying to sort of see to it, that when that gradient is taken I have a quantity whose gradient is this. So, let us see what it means actually.

(Refer Slide Time: 16:33)



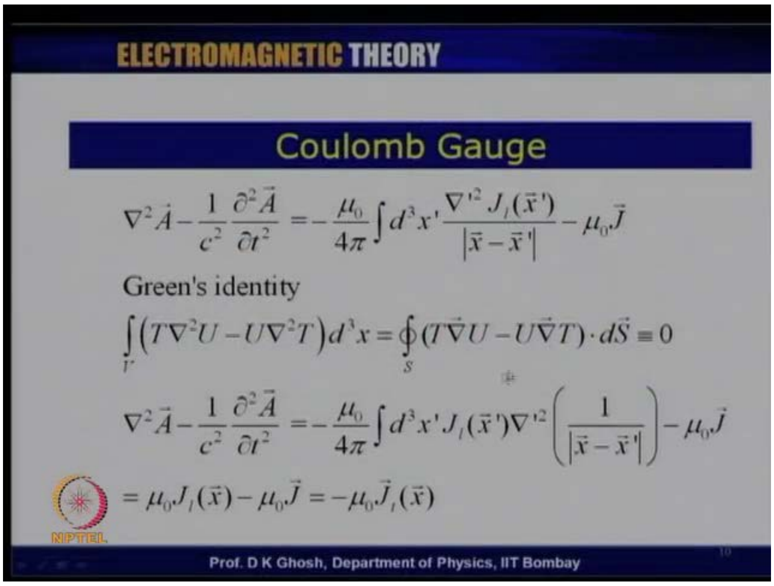
So, this tells me that del square A minus 1 over C square d square a by d t square is this and by what is just now showed this quantity is can be written as minus mu 0 by 4 pi d cube by x prime, gradient prime del prime dot J l of x prime divided by x minus x prime, minus mu 0 J.

(Refer Slide Time: 16:50)



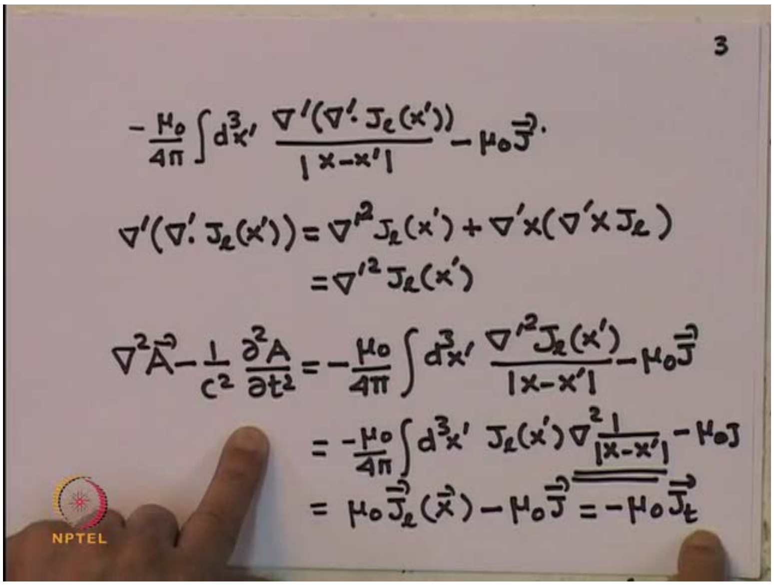
Now, look at what we are doing. Now, I am got a gradient of a divergence here. So, I will write this as gradient of del prime dot J longitudinal x prime that is equal to del prime square J longitudinal x prime plus del prime cross del prime cross J l, this is the same formula which we've been using several time del cross del cross equal to del of del dot minus del square and since the longitudinal field is called free. So, this is equal to del prime square J l of x prime. Now, if you plug in all this things. I get del square a minus 1 over C square d square a over d t square is equal to minus mu 0 by 4 pi integral d cube x prime del prime square J longitudinal of x prime, divided by x minus x prime minus mu 0 J. Now, at this stage what I do is to use what is known as a Green's identity.

(Refer Slide Time: 18:56)



Green's identity if you recall is, if I have to scalar fields T and U. T del square U minus U del square T volume integral is converted into a surface integral of this type and this surface integral because my choice of the scale functions will be the current and in this case the distance 1 over x minus x prime all these surface integrals will be 0. So, in other words what I get is integral of T del square U is equal to U del square T which means that del square J l by x minus x prime is same as J l times del square of x minus x prime.

(Refer Slide Time: 19:48)

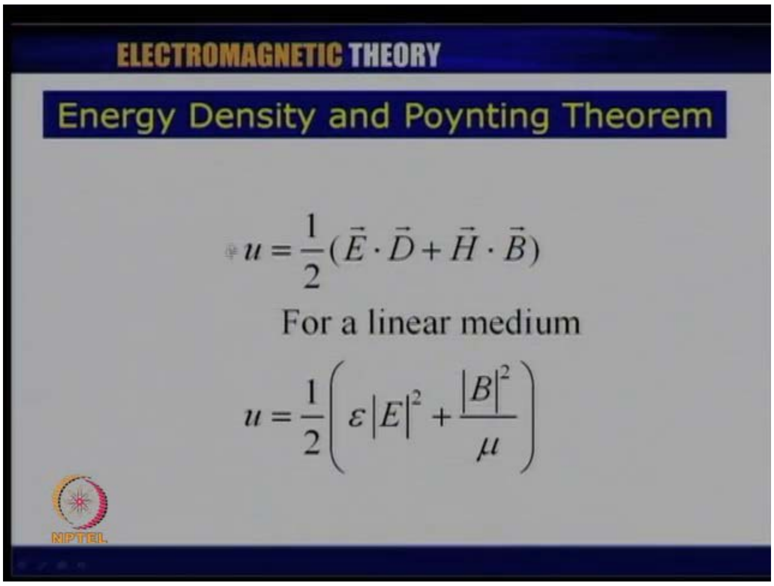


Now so once I do that so it is minus mu 0 by 4 pi integral d cube x prime J l of x prime del square of 1 over x minus x prime minus mu 0 J and we have repeatedly said that, del square of 1 over x minus x prime is minus 4 time, 4 pi times a delta function. So, therefore minus 4 pi cancels here and I will be left with simply mu 0 times J longitudinal at the point x and I had subtract to be to subtract from this minus mu 0 full J which is nothing but mu minus mu 0 times J transverse, because I had written the vector J as the difference between the full vector and the vector J as the sum of a transverse part and a longitudinal part. So, what is what has happened.

My equation for the scalar potential was already decoupled and that simply give, gave me del square phi is equal to minus rho by epsilon 0 there's a Poisson's equation in this case I have an inhomogeneous wave equation. We will be talking a lot about wave equations in the next few lectures. So, we have del square A minus 1 over C square d square a by d t square is minus mu 0 times that transfers current in the problem. So, in other words we have succeeded in Coulomb gauge also to decouple these equations. So, that is a comment on the gauge invariance and discussion of two important gauges there.

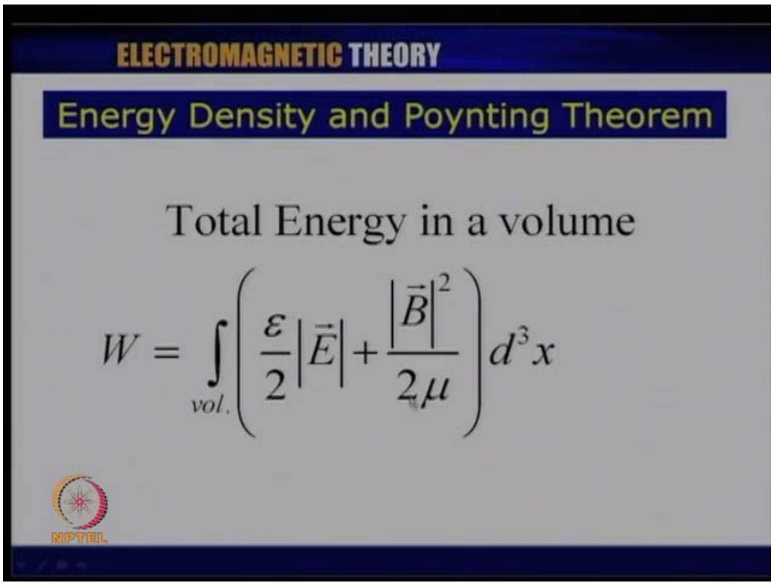
Now, let us look at the other thing that we did last time, is to say that if I have an electromagnetic field, the field is a storehouse of energy and I defined or I obtained a an expression for the energy densities.

(Refer Slide Time: 22:13)



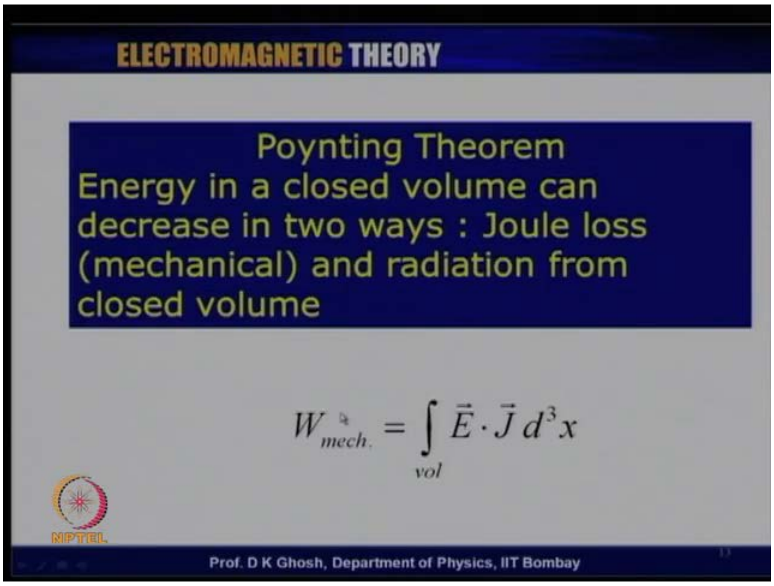
There is one part which is the electric part and one part which is the magnetic part and we had shown last time that the energy density u is given by half E dot D plus H dot B. Now, I will be confining myself to linear medium and so I will be using only E and B vector so that the energy density expression becomes half of absolute E square because D is epsilon E and half of B square by mu. So, this the form in which I will be using it.

(Refer Slide Time: 22:44)



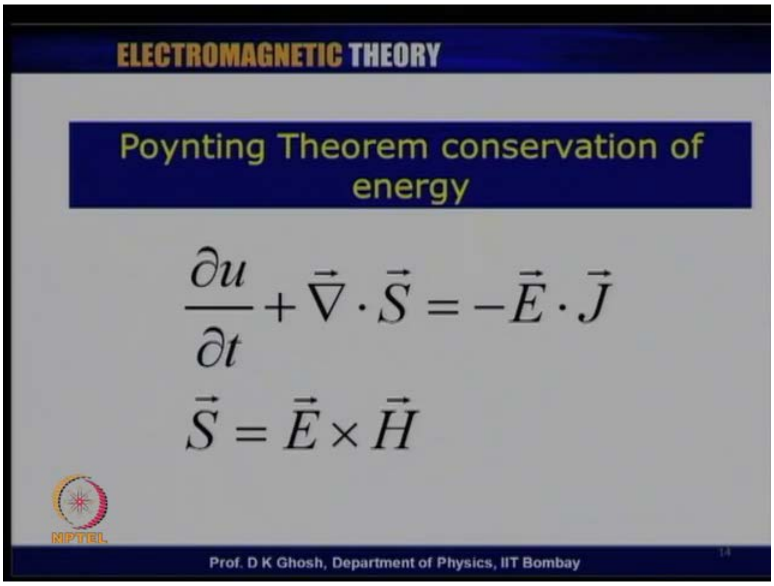
The total energy in a given volume del E is given by the volume integral of this quantity so that is the total energy in a given value.

(Refer Slide Time: 01:36)



So, what we said is, but if I have an, a field or a collection of charges on currents in a closed volume, then there is this is of course, a place where there is energy stored. Now, this volume can lose energy first of course, is by deception, which you called as the mechanical loss and secondly physically radiation leaving the surface, which defines this volume and we had seen that the mechanical work done is essentially the joule loss which is integral over the volume or E dot J D cube x. And the, we defined we also found there is a flux of energy from the field to outside the closed volume, and the defined a quantity which is called the pointing vector, which is basically an energy flux term.

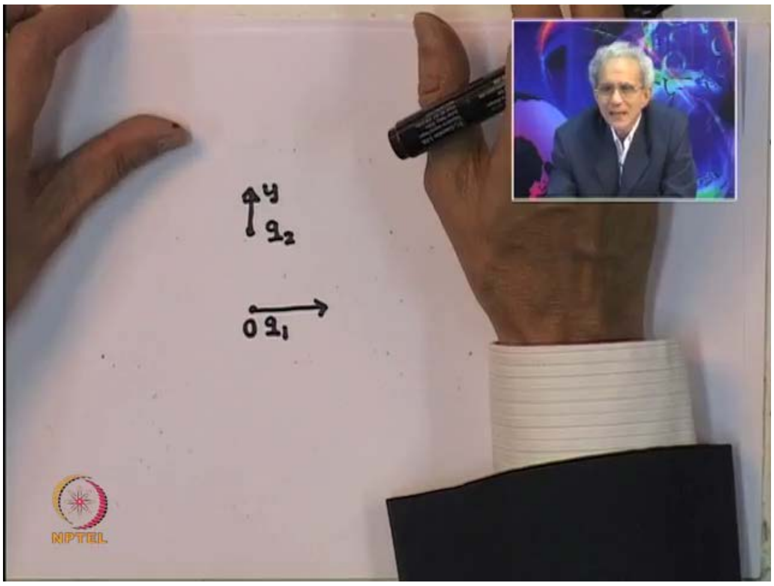
(Refer Slide Time: 23:39)



So, that the rate of change of the energy density is given by the a change due to the physical movement of energy flowing out from the surface binding surface and of course, there is joule (( )) term and this quantity E cross H is what we had defined as the pointing vector and this is in some sense, a statement of conservation of energy for the case of electromagnetic fields. The next question that naturally comes to us is that, if the electromagnetic field has a energy, is it possible it also has movement. The answer is yes.

Actually a more rigorous way of looking at it can come only after one has done relativistic formulation of electrodynamics, but I will give you some simple way of looking it. So, let us suppose I have a chariest particle which is supposing the charged particles are moving in the plane of the paper and there is a charged particle which is moving along let us say the x direction let us call it q 1 and there is a charge q 2 which is moving along the y direction and I am looking at and this charge I am looking at the force on this charge at the instant when it is passing through the origin this is of course, moving like this.

(Refer Slide Time: 25:17)



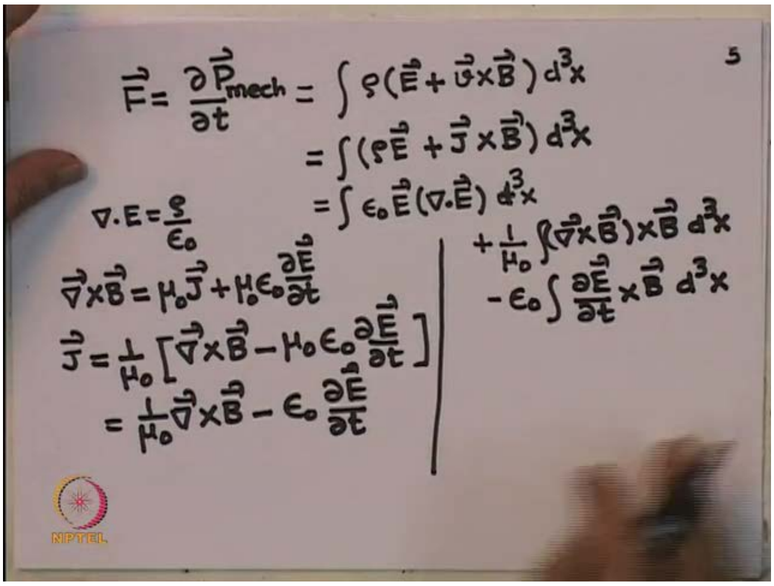
Now, notice. So, far as the electric fields are concerned there are no issues because the this distance at the instant when this is going towards right. This distance is known which is let us say d the force on q 2 due to q 1 is 1 over 4 pi epsilon 0 q 1 q 2 by d square and equivalent opposite would be the electric force due to q 2 on q 1, but let us look at what happens through the magnetic case. Now, now the these are electric charges which are moving. So, moving charges their effect is the same as that of a current. So, here I have a current which is flowing along the x direction. This is the y direction. So, therefore the, if you look at this current this gives rise strives to magnetic field in the z direction.

So, this field because there is a charge which is moving in the y direction the field is in the z direction. So, V cross B which is y cross z which is along the x direction therefore, this charge will be subject to a force because of the magnetic field generated by this charge which is flying past this point of. However and since there is a magnetic field this charge will experience a force which we have just now talked about. But let us look at what happens, to force due to q 2 on q 1. Now, feel due to q 2 on q 1 at the origin is 0. It is moving along this line. You can sort of intuitively see it, but on the other hand more rigorous justification can be given, if you knew that if there is a charged particle moving with a velocity V then what is the direction in which the magnetic field acts.

Then this here it is 0. So, in other words this exerts a force on this, but that one does not exert a force on them. In other words there is a an apparent violation of Newton's third law of action being equal and opposite to reaction. The way to circumvent this is that, see if there is a force on this there is certain amount of momentum. There is certain amount of change in the momentum and if these are not the only bodies that are being talked about, there is a third thing here and that third thing is the electromagnetic field. Now, if it is possible for us to transfer certain amount of momentum to the electromagnetic field, then there is no violation of Newton's third law.

The main reason is that Newton's third law action reaction becoming equal and opposite led to a conservation of momentum and these two forces not being equal and opposite was an apparent violation of the conservation of momentum. So, therefore we intuitively at least can understand that electromagnetic field carries momentum and what we are going to do now, is to look at what, how does one conserve momentum in case of an electromagnetic field. So, let us go back to a an expression for the force.

(Refer Slide Time: 30:00)

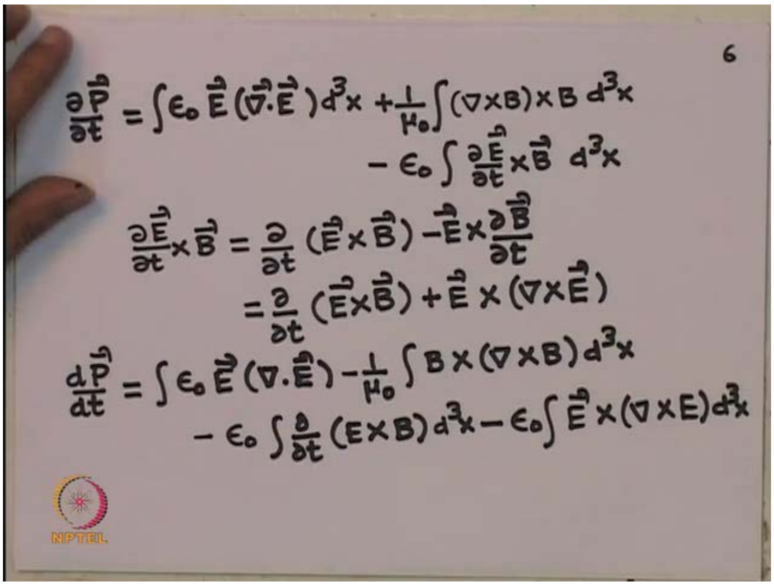


So, the expression for the force is rate of change of mechanical momentum d p by d t and let us write it in terms of the force. So, in the continuing limit my charge distribution is subjected to an electric which is rho electric field E plus rho V cross B which is due to the Lawrence force of magnetism and this times d cube x. Normal to make some modification to this equation, firstly I will use rho times V is nothing but the current J. So, this equation will be rho times E plus J cross B d cube x. now, what I will do is this. I know del dot of E is rho by epsilon 0 and what I have is this so what is right this instead of rho times E I will write it as epsilon 0 E times del dot E p.

The second equation that I will be using will be del cross B as we know that, this is the Maxwell’s equation ampere’s law mu 0 J and modified according to Maxwell’s is mu 0 epsilon 0 d E by d t. So, what is I am going to do is this for this term, I have got a J there. This J I am going to replace by difference between these two things and that gives me. So, let me write it here. So J is 1 over mu 0 dell cross B minus mu 0 epsilon 0 d E by d t, which is 1 over new 0 dell cross B minus epsilon 0 d E by dt. So, what I am going to do is to for this J I am going to write this. So, let me write it as two different integrals. So, this is d cube x plus 1 over mu 0 integral del cross B cross B d cube x and I have a minus epsilon 0 there, so minus epsilon 0 integral d E by d t and cross B is already there d cube x.

So, the expression is becoming a little more complicated, but on the other hand we will be able to substantially simplify it. So, at this movement it looks bad. So, let us look at what am I got so far. Let me re write this equation, so that we can look at every term separately.

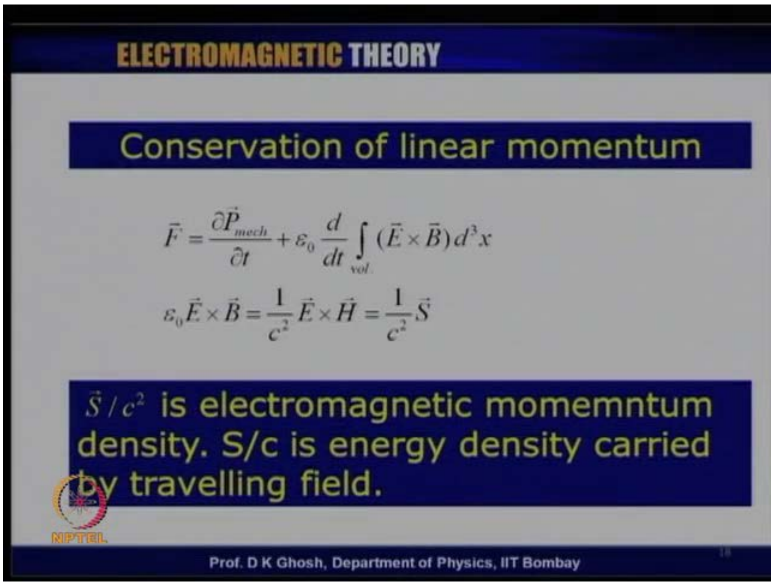
(Refer Slide Time: 34:10)



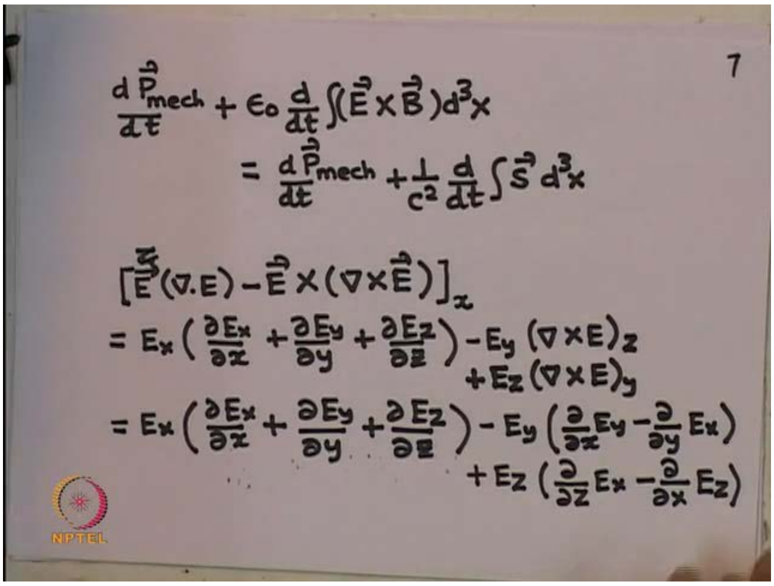
So, d p by d t is equals to integral epsilon 0 E del dot E d cube x plus 1 over mu 0 del cross B cross B d cube x minus epsilon 0 integral. I have got d E by d t cross B so d E by d t cross B d cube x. Now, I am going to change this equation to a little more symmetric form. So, first let us observe d E by d t cross B can be written as d by d t of E cross B by chain rule minus E cross d B by d t and this second term of that , this is d by d t of E cross B I do not touch it. Minus d B by d t is nothing but del cross of E. So, therefore this is plus E cross del cross of E. Now, you see the reason why I am doing this. In this term I had a del cross B cross B or B cross del cross B. I wanted to treat the electric electric and the magnetic field very symmetrically. So, I have this term there and so I have got this term there.

So, with these what I get is d P by d t is equal to epsilon 0 E dot E times del dot E minus 1 over mu 0 integral B cross del cross E del cross B d cube x minus epsilon 0 integral d by d t of E cross B minus epsilon 0 integral E cross del cross B. This term that we have got, epsilon 0 d by d t of E cross B. You recognize what is this term. I defined E cross h as S so that was my pointing vector so this is nothing but the rate of change of the pointing vector. There is a term I have a E cross B here what I need is E cross h so there is a mu 0 that will come out so mu 0 into epsilon 0 will give me 1 over C square. So, this term will go to the other side which I will write down, but what we are looking for a terms of the remaining. So, let us look at these remaining terms and what are they. So, what I have got is the following.

(Refer Slide Time: 38:14)



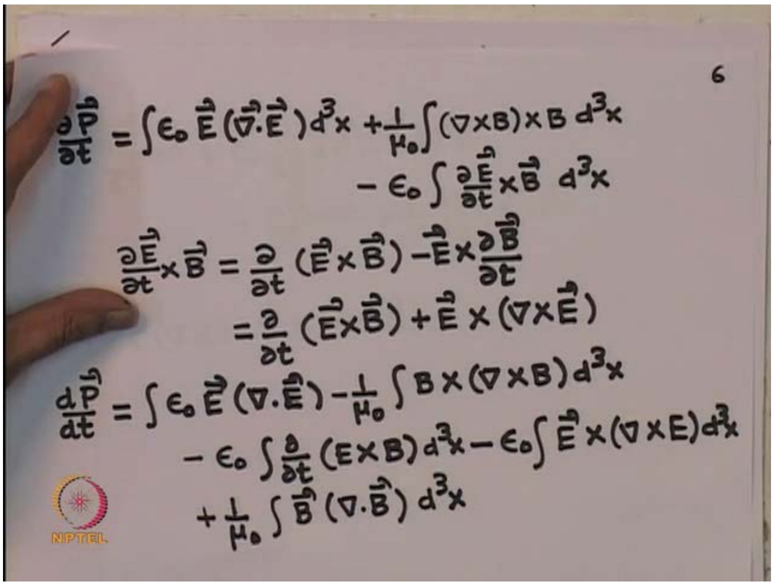
(Refer Slide Time: 38:19)



I have got d P by d t d P make by d t plus epsilon 0 d by d t of integral E cross B d cube x equal to asset of expressions which I am coming back to, but first I want you to realize that this term is nothing but epsilon 0 mu 0 E cross h and epsilon 0 mu 0 becoming equal to 1 over C square all that d P mech by d t plus 1 over C square d by d t of S d cube x and that is equal to these terms which we have written down, which had B cross del cross B, E cross del cross E, E del dot E etc. I will come back to those terms, but let us first look at what this is telling us. See notice that S by C square has to be electromagnetic momentum density because after all, this is momentum P mech and therefore, the integral of S d cube x must also have the dimension of momentum.

So, therefore this is momentum density, S by C square is momentum density and you can check that S by C is nothing but the energy that is density energy density that is carried by the field. So, so this is the momentum density associated electromagnetic momentum density associated with the field, but let us now look at the remaining term. The remaining term were let me just show it for the electric field. So, you notice there is a E del del dot of E minus E cross del cross E. So, let us look at E del dot of E minus E cross del cross of E. Magnetic field term looks a little asymmetric it says B cross del cross B. There does not seems to be a term similar to this, but on the other hand I can simply physically add it that is I will do that because I have this term.

(Refer Slide Time: 41:22)

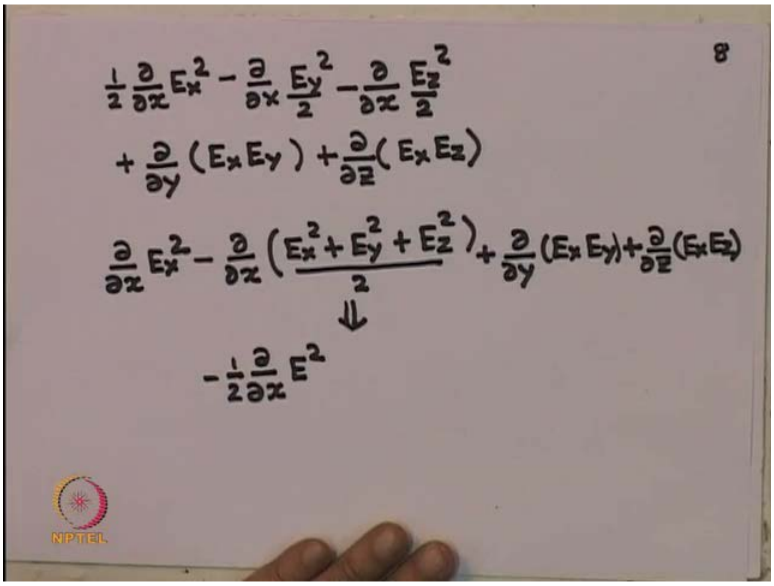


So, there is a proper minus sign. I will write down plus 1 mu by mu 0 1 over mu 0 integral B del dot of B. I could do that because del dot of B is identically equal to 0. So, therefore this is like adding 0 to that equation. So, that has the an effect of making these expressions a symmetric with respect to electric and the magnetic field. So, I will carry out the algebra from one of those fields and I will simply substitute that at a later stage for the magnetic field as well. So, let us look at what is it that I have so I have this expression E del dot of E minus E cross del cross of E. There is of course, a vector. But then I expect back. Now, to find out what is it is form.

What I am going to do is to find out what are the components of this take for example, the x component of this I will calculate x y z component and then we will be able to write in a particular fashion. So, this quantity is E x del dot of E is already a scalar. So, that is d E x by d x plus d E y by d y plus d E z by d z. Minus x component of that and that is minus E y del cross E z component, minus minus, plus. So, E z del cross E is y component. So, this is E x by d E x by d x plus d E y by d y the divergence term fully, minus E y del cross y. Dell cross E is z component. So, d by d x of E y minus d by d y of E x plus E z y component so d by d z of E x minus d by d x of E z. There is a lot of term, but I will be able to write it in a remember, this is an x component I will have to add to its the y component and the z component also.

But let us look at how to simply this x component. So, what I have got here this. I have got E x d E x by d x which is nothing but half of d by d x by E x square. So, so this term is half of d by d x of E x square.

(Refer Slide Time: 44:53)



Let us let us look at other derivatives with respect to x. Here is one term and here is another term. Look at what is this? This is minus E y d by d x of E y **so** which is minus d by d x of E y square, but 1 by 2. Let me write it 2 here and likewise the other term is also d by d x of E z square by 2. There are other terms there, which are like this. d by d y of E x E y, you can see this that this is E x times d E y by d y and d y E x times d y. So, both the terms are there. So, we write this as plus d by d y of E x E y and exactly the same way I have another term here which is d by d z of E z E x. Looks a little asymmetric, but notice I can add a term my d by d x of minus E x square by 2 here to take care of this half I will be left with d by d x by E x square minus d by d x of E x square plus E y square plus E z square by 2 and then this terms which are d by d y of E x E y plus E x E z d by d z of E x E z. Notice the symmetry now, that has come in. This is nothing but d by d x of E square.

When I add up the there is of course, a factor of 2 there so let me write it as minus over 2. When I add up the terms, which come from the y and the z component I will get d by d y of E square and d by d z of E square and what is that? That is simply the gradient of E square by 2, there was an epsilon 0 in front of it and since, the magnetic field is identical you are going to get a minus a gradient of the total energy density there, which is because you have an energy density and you are looking for a gradient of that therefore, this is the momentum density. So, this term will give you the momentum density term, I am going to be just doing the algebra today and point out and next time, We will do a complete discussion because it takes a bit of a time this term is d by d y of E x E y d by d z of E x E z and d by d x of E x E x, because this is E x square.

So, this term along with the components that will come up with y and z it is somewhat of a different type so what I am going to get here is a set of terms which I haven't quite come across because this term which is d by d x of e square and d by d y d by d z which is nothing but the gradient of the energy density. So, that is the momentum density term. So, I have a term which is associated with the pointing factor term. I now, have a term which is with the amount of energy that is stored in the field. The term that is remaining which is still in a big mess, that is the term which will be associated with the momentum of the electromagnetic field and we will continue from here in the next lecture.

So, to summarize we have been able to write the momentum density of the electromagnetic field in terms of three separate components. This is so far it has been algebra, next time we will combine these to find out how, what what are the interpretation of each one of these terms.