Electromagnetic Theory
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Module - 4
Time Varying Field
Lecture - 30
Maxwell's Equations

In the last lecture we had introduced a concept called the displacement current. Just to recall, because it is a rather unusual idea, we had introduced, we had discussed the phenomena of charging of a capacitor.

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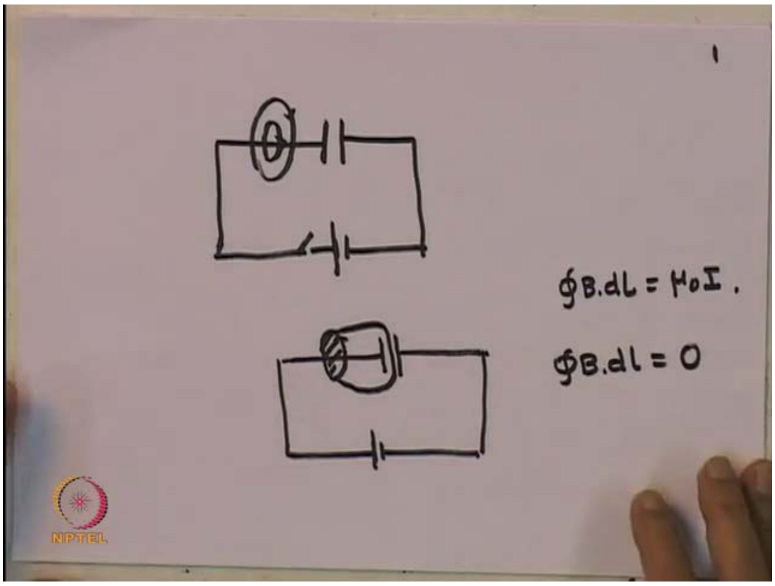

This is for example, if I have a capacitor and I have a circuit where there is a battery of course, and a key. The when I switch on the key the charges move to this plate, and it will charge this plate as positive that plate as negative, and there is a current which is flowing through the external wire during the process of charging. So, what we did is to say that since there is a current in the external circuit, the region around this current is a source of magnetic field. Therefore, this is a region of magnetic field.

So, as a result if I considered a surface which is cutting this wire, and find out what is the flux of the magnetic fields through that surface, then of course by the Ampere's law, it turns out that integral of B dot d L must be equal to mu 0 i, where because there is a current which is passing through that. So, therefore the line integral of the magnetic field in any closed loop will be given by mu 0 times i. However, there is a problem; the problem came up because we had said that this, supposing I consider just a particular loop and this is the process of charging of current. and we said that this loop in principle could be filled up by any surface.

So, for example, if it is a circular loop I could fill it up with a disk and in which case the net since the current is not equal to 0, what I find is that the your integral of B dot d L is equals to mu 0 times i. So, the point is that if you take this surface it is mu 0 times i, but on the other hand if I had the same loop, but I considered a surface like this, like a pot. Then because there is no current which is flowing through this surface, then what I get is the integral of B dot d L will be equal to 0 through the second surface. Now, this of course tell me that there is an inconsistency in the way we interpret this law.

And so what we did or rather what Maxwell did was to suggest that the during the process of charging or discharging whenever there is a change in current, the dial the charges that are there in the dielectric medium between the capacitors now, they of course, result in a motion of the charges. I am not like the motion of the charges outside, but it can be shown that it is equivalent to a current at least dimension wise. And he gave it a name displacement current. The idea was that in the outside circuit there is a real current you can call it conduction current, but if you have taken a loop which is through the outside contour, but on the other hand surface is going through the capacitor plate then of course, that is not what happens.

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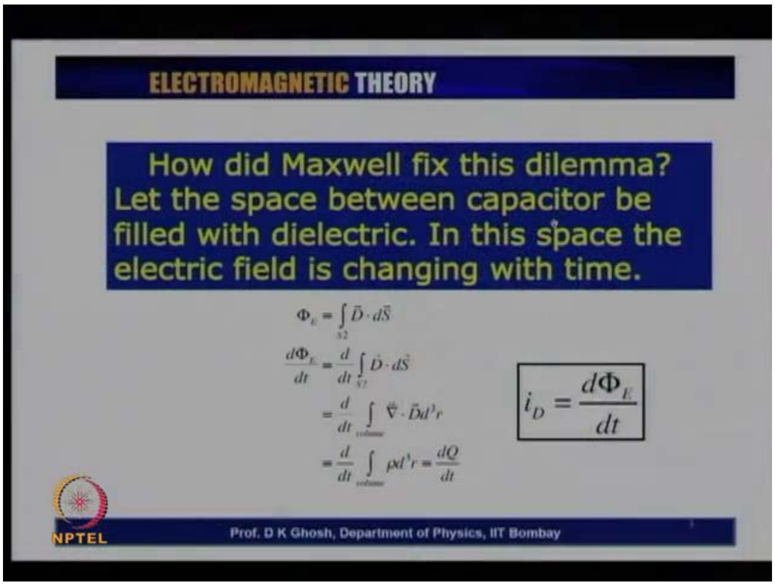

So, the question was how how does one define this?

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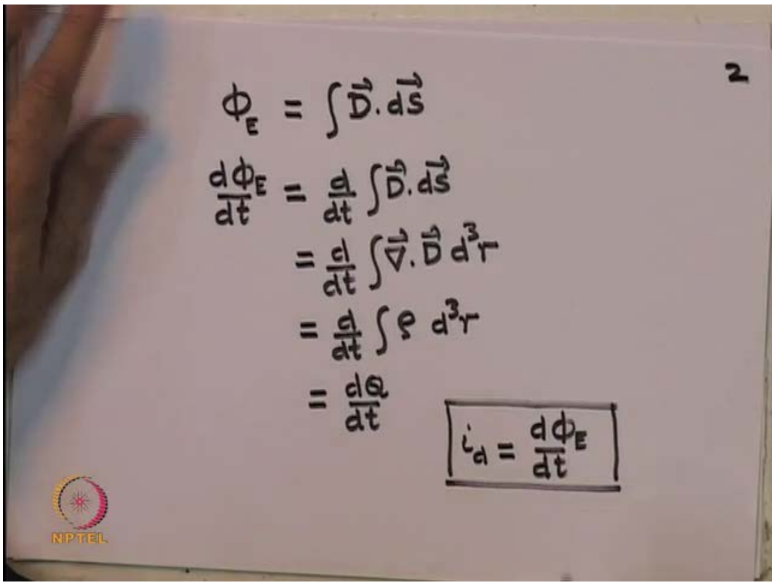

So, we said that inside the capacitor I have an electric field whatever be the say for the capacitor I have an electric field and which is let me take it as a dielectric medium. So, the surface integral of the vector D dot d s is the flux of the vector D. So, therefore the rate of change of this fix d phi E by d t which is equal to d by d t of D dot d s. And you remember that since this is D dot d s by divergence theorem, I can write this as del dot of

D d cube r that is over the volume, but del dot of D is nothing but the charge so I get d by d t of a rho and d cube r.

So, in principle this quantity has the dimension of charge because integral rho d cube r is the dimension of charge so d Q by d t is the dimension of current, only its origin is different. This origin is the the there is really no charge current which is passing through, but there is a displacement of the charges in the dielectric, and so these results in the rate of change of the electric flux being essentially having big equivalent to a current. So, when you are inside therefore, what you are done, Maxwell had done was to for various reasons, the word displacement is not particularly relevant but he had defined the rate of change of electric flux as the displacement current.

So, in other words in the outside circuit there is a conduction current which is being supplied by the battery, when the current thing is being charged. In the inside the what we have is a rate of change of electric flux and that acts very similar to an external current. So, let us look at supposing I have a parallel plate capacitor.

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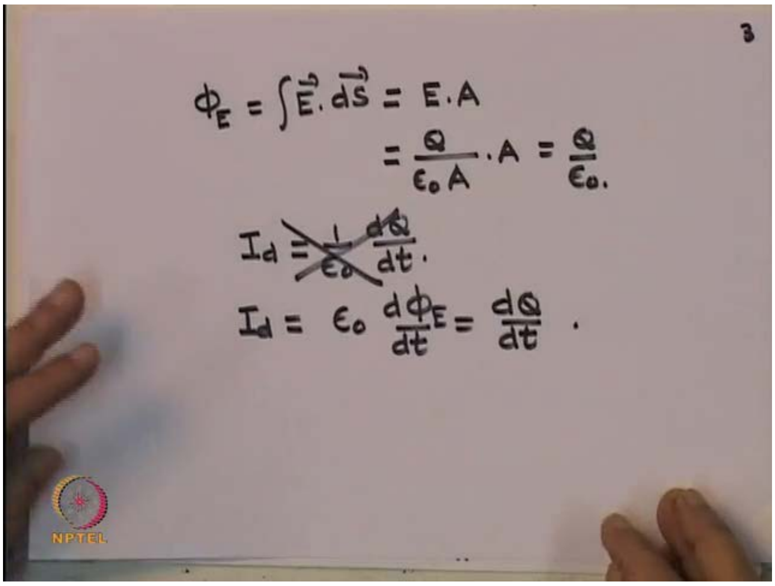

Now, if I have parallel plate capacitor then the flux of the electric field is integral E dot d A or E dot d s and I know that the electric field is uniform and therefore, it is E times the area of the surface capacitor plates. And I know this strength of the electric field in a parallel plate capacitor is Q by epsilon 0 A that divide multiplied by A that gives me Q by epsilon 0. Therefore, the displacement current is given by 1 over epsilon 0 d Q by d t.

So, this is what actually epsilon 0 I beg your pardon the displacement current is actually d epsilon 0 d phi by d t therefore, that takes care of this 1 over epsilon 0 there and I am left with d Q by d t. So, this is what we have been calling as the displacement current.

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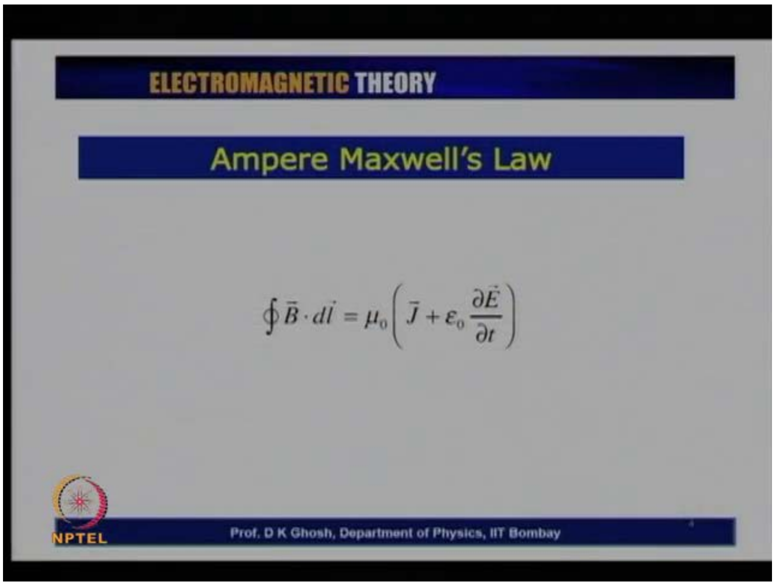

So, in the region between the capacitor plate, the electric field or electric flux changes with time and this has the same effect. Rewrite the integral of B dot d L now, which was equal to mu 0 times i and plus I have now adding epsilon 0 d phi by d t which is equivalent to A.

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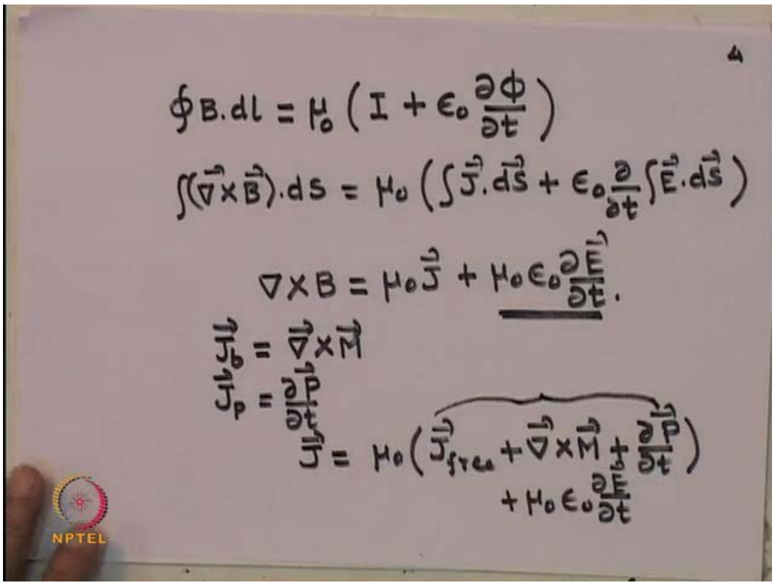

I could of course, convert this to a through the usual way of converting this to a differential form, that is right B dot d L as equal to del cross B dot d s so I get del cross B dot d s. And on that side I have current which is of course, J dot d s plus epsilon 0 and I have d by d t of flux is electric field dotted with d s. So, notice what I get is since this is arbitrary, I can convert this into del cross B equal to mu 0 J plus mu 0 epsilon 0 d E by d t. So, this is the time which we have been calling as the displacement current, but let us look at what are the various contributors to this del cross of B? The any current whatever be the source of the current can go into this J, we have already identified a few of such things.

For example, we know that I we have of course, the free current that is the charge current. So, then we have also seen that there could be bound currents if I am looking at magnetized material, and we had obtained that bound current is given by del cross of M, the other possibility is that if there is a time variation in the polarization d P by d t which essentially gives me some sort of polarization current if you like. You can check the right dimension. Therefore, my total current which contributes to the first term in this equation is mu 0 J free, that is the usual charge current plus magnetizing current del cross M plus d P by d t that is which you have called as, and of course mu 0 epsilon 0 d E by d t which is my displacement current so I have all sorts of currents there. So, these three things taken together is what I have been writing as J. So, let us rewrite this in a slightly different fashion.

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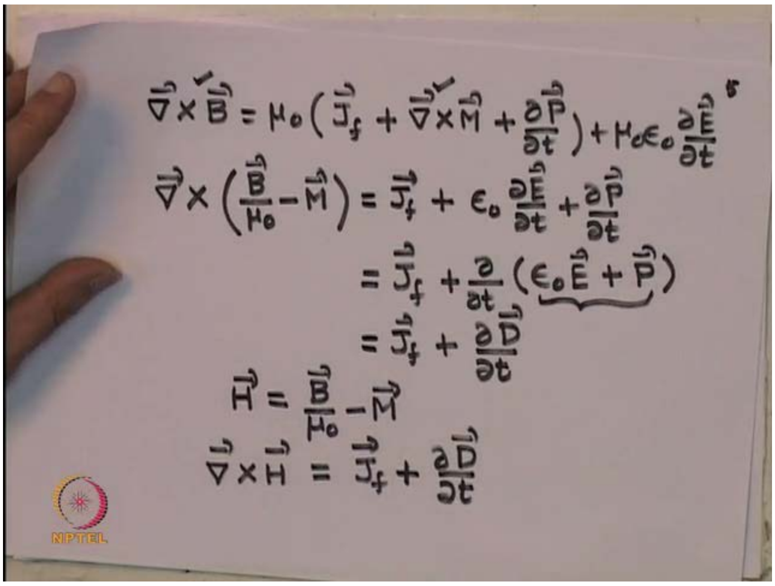

So, we have said that now del cross of B is mu 0 J free plus del cross M this is the bound current plus d P by d t plus mu 0 epsilon 0 d E by d t, we will write in a slightly compact fashion. Now, I could bring that magnetization term outside and divide all over by mu 0, so I get b by mu 0. Mu 0 is a constant minus M, so that takes care of this term as well as that term is equals to J free because I have already divided by mu 0. So there is nothing else and I am left here with epsilon 0 d E by d t plus d P by d t, which gives me J free plus d by d t of epsilon 0 E plus P.

Of course, recognize that this is nothing but our definition of the vector d, so this was, then this will become J free plus d d by d t and this quantity here B by mu 0 minus M were defined as the magnetic field H. As we told, said earlier that traditionally it is the H field which has been called as the magnetic field, and the B field has been having different type of names like; field of magnetic flux density, magnetic field of reduction, but of course, we have been using them interchangeably assuming that there is no confusion.

So, the this equation then has given me del cross of H is equal to J free which is the form we had before we introduce the displacement field plus d D by d t. With this we complete our last of the equations which is the Ampere's, Maxwell's modification to the Ampere's law.

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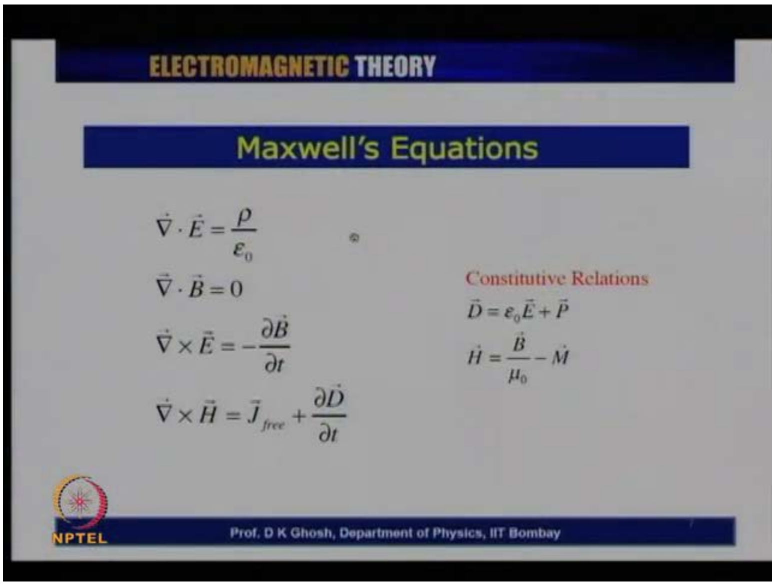

This is a good time to collect all the Maxwell's equations together.

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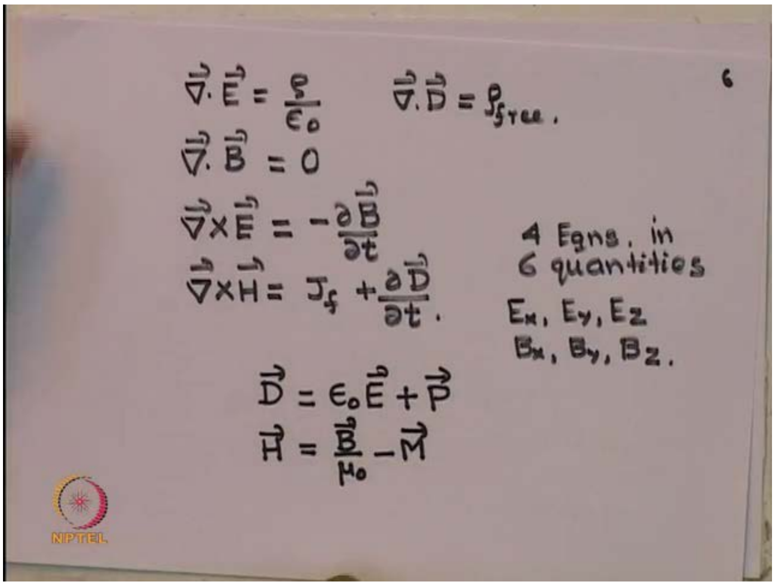

So, firstly of course, we had del dot of E is given by rho by epsilon 0, del dot of B is equal to 0; this are the two static relationship which you have not have not really changed. Notice that this pair are electrostatic and the magneto statics and the sources are the same. No magnetic monopoles so as a result del dot of B zero, electric charges exits so del dot of E is rho by epsilon 0. Then we had del cross E which in static case was 0 is now given by Faraday's law as minus d B by d t del cross of H is equal to J free plus d D by d t.

Occasionally you would also write this equation, as the del dot of D is equal to free charges. So, these are my four Maxwell's equations which will form the basses of discussion in the remaining part of the course. But these, so these are four equations in six quantities. The six quantities are the three components of electric field namely, E x, E y, E z and B x, B y, B z the magnetic field. You must supplement these set of equations with what is known as a constitutive relation and this constitutive relations is the relationship between D electric field E and the polarization vector P, d is equals to epsilon 0 E plus P and H which is given by B by mu 0 minus M.

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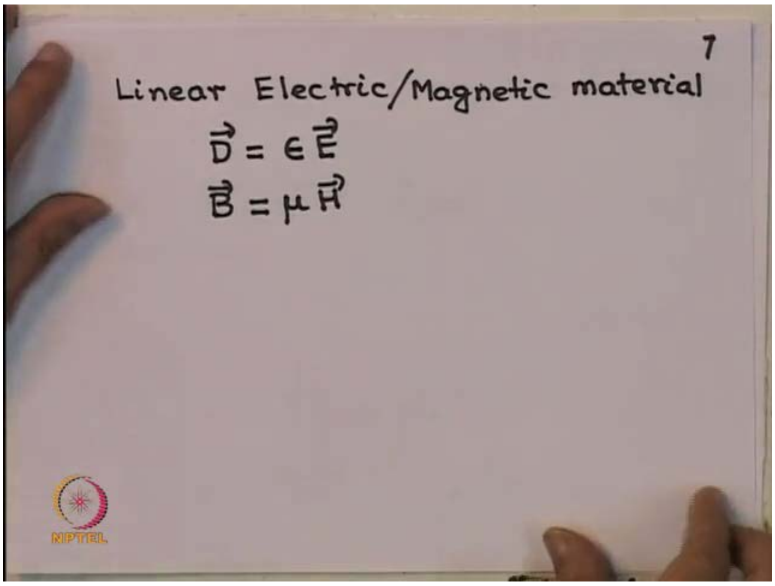

Frequently for convenience we will be dealing with what are known as linear materials and for linear material, linear electric or magnetic material for which the relationship between B and H or that between D and E will be linear. Therefore, the D is written as a quantity called epsilon the permittivity times the electric field, epsilon 0 was the permittivity of the free space, this is just the permittivity of the medium and the magnetic field B or the magnetic flux density B is given by the permeability of the medium times H.

Once again mu 0 was permeability of vacuum, so these if you are dealing with linear magnetic material these would be the supplementary relations. At this stage what we are doing to do is to bring the potentials back, and see whether we get some advantage by writing these equations in terms of the potentials.

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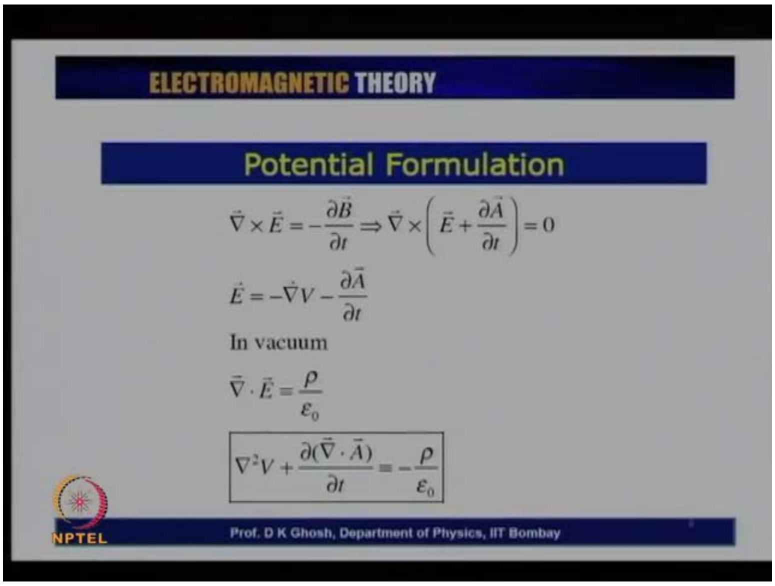

If we recall we had defined two potentials, one corresponding to the electric field and one corresponding to the magnetic field; of course, later under certain situations we had seen that even for a magnetic field, we could define a magnetic scalar potential.

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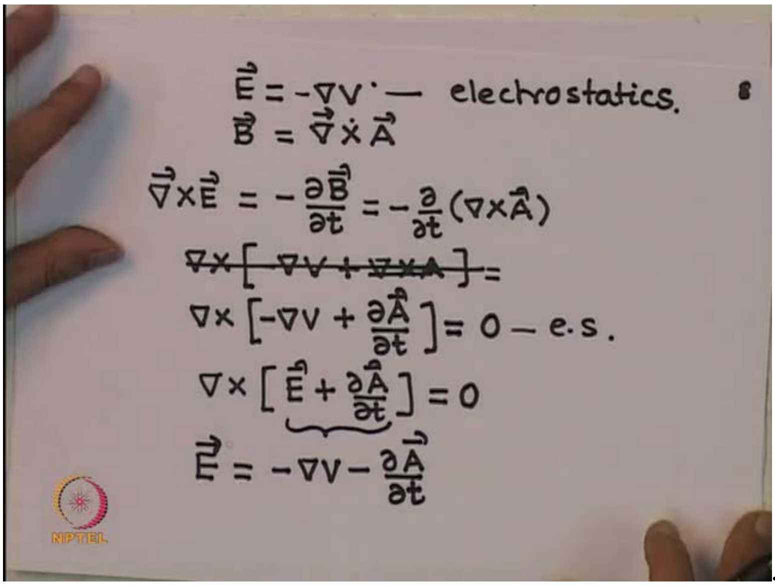

But we will be dealing here with electric field being written as minus gradient of the potential V and the magnetic field B is the curl of the vector potential A. So, let us look at the two curl equations and see how what happens to that. So, we had an equation which is del cross of E the Faraday's law is equal to minus d B by d t. So, what notice that since B is equals to curl a I can write this as d by d t of del cross A by taking the terms to the left side and writing so this can be del cross E is minus grad V and plus del cross A.

So, this is what will, this is equal to 0 I am sorry this is d by d t of d A by d t, let me rewrite it. I get del cross minus grad V plus d A by d t. Now, look at this that if I have del cross, if I have del cross E plus d A by d t equal to 0, I should be able to define this was actually electrostatics. Now, if I look at this equation I will find that this is a more proper equation when I am dealing with, the time varying phenomena because this is questionable now. Therefore, if del cross of E plus d A by d t equal to 0. Remember in electrostatics I had in electrostatics I had del cross of E equal to 0, that is what gave me the definition that E could be written as minus gradient of E.

But now, del cross of E is not equal to 0 so as a result it is this quantity which can be expressed as a gradient of a scalar potential. Therefore, what I do is instead of this I define that E is equal to minus that E plus d A by d t is minus grad V. So, as a result E should be written as minus grad V minus d A by d t instead of just minus grad V. So, this is this is the way we get express electric field in terms of a potential V and a vector potential A.

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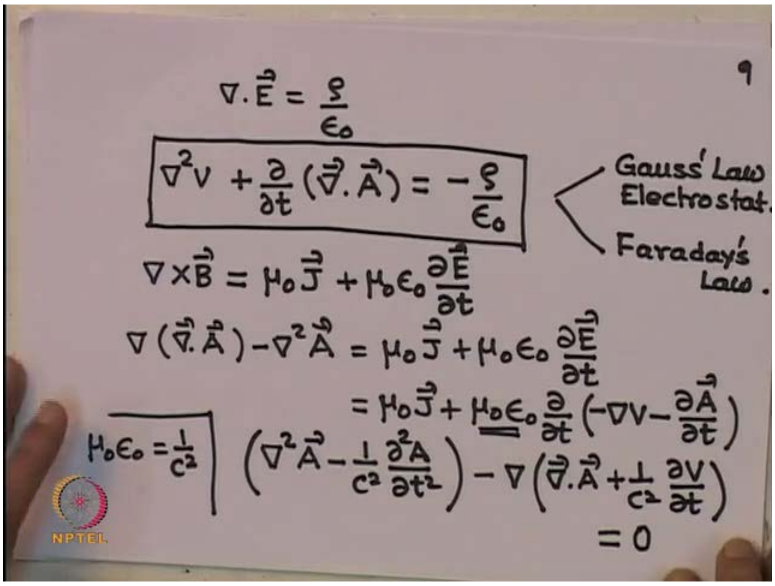

Now, if if I am in vacuum I know that del dot of E is equal to rho by epsilon 0 so del of E I will rewrite in terms of the potential. I am trying to express everything in terms of potential law so this was Faraday's law and I am now combining it with the Gauss's law of electrostatics. So, del dot of E is the same as minus del square V minus d by d t of del dot of A. Therefore, I can write this as del square V plus d by d t of del dot of A is equal to minus rho by epsilon 0.

So, this is essentially contains the pair of equations namely the electrostatic Gauss's law and the Faraday's law. Now, let us repeat that job for the other pair of law law that we have so this is one equation which we keep aside for future. I have del cross of B which we discussed just now is mu 0 J plus the displacement current term mu 0 epsilon 0 d E by d t and B will be written now as del cross of A and E by whatever we had just now talked about, namely minus gradient of V minus d E by d t. So, B is del cross del cross A which I know is del dot del dot of A minus del square A is equal to mu 0 J, I do not do anything to that term plus mu 0 epsilon 0 d E by d t.

And the idea is to remove the direct reference to the fields and replace them in terms of the potential so which is equal to mu 0 J plus mu 0 epsilon 0 d by d t of minus gradient of V minus d A by d t. So, let us combine these two by writing you recognize that mu 0 epsilon 0 is 1 over C square, C is the velocity of light in vacuum. So, what we will do is I will write this equation by first writing a term which we will later on will see that looks like a wave equation form so del square A i take to that side. I get del square A minus 1 over C square that is the mu 0 epsilon 0 d A by d t d square A by d t square plus rather minus a gradient of so I am brought this gradient term to that side.

So, that is del dot of A and the other gradient term is here so I get, I have 1 over C square, this minus because of this common minus is going away and I am left with d V by d t, this is equal to 0. So, a rather clumsy equation, but let us look at this equation. So, let me write down this pair of equations in one place so that we can discuss it reasonably.

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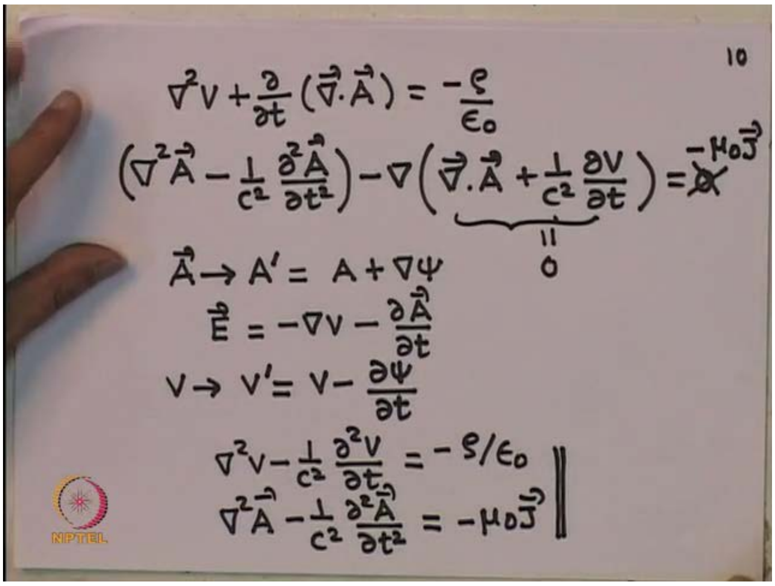

So, I have an equation which is del square V plus d by d t of del dot of A equal to minus rho by epsilon 0 and a second equation which is del square A minus 1 over C square d square A over d t square minus gradient term which is gradient of del dot of A plus 1 over C square d V by d t is equal to 0. So, what have we achieved this equation is equivalent to two curl equations that we have written down. Now, the two equations of magnetism that we had written down, this equation came from Faraday's law and then Gauss's electrostatics.

So, this is now what do you want to do is this, notice one thing that this equations are not decoupled. This is this contains both V and A, and this also contains both V and A, so what is the advantage that we have got? The advantage we have received so far is, instead of equations four equations in six quantities I have two equations in four quantities, I have vector potential which is a vector so I have three quantities there. I have a potential V which is a scalar so there is one quantity there, so I have got four, but of course, the equations are coupled. Now, so what we will do is this that we will try to do what is known as a Gauge transformation.

Now notice one thing that we know that there is an indetermination with respect to definition of the vector potential that is I can always let A go to A prime which is equal to A plus A gradient of a scalar function and similarly, with respect to this scalar potential I can always add a constant. So, you notice that my electric field E was minus grad V minus d A by d t. Suppose, I let V going to V prime equal to V minus d psi by d t, then if I put in a condition that del dot of A plus 1 over C square d V by d t equal to 0 suppose this quantity is put to be equal to 0, then these two equations will be decoupled.

The reason why it will be decoupled you can see it, if del of A is 1 over C square d V by d t is 0, then in this equation I can substitute for del dot of A minus 1 over C square d V by d t, so that will give me del square V minus 1 over C square d V d square V by d t square. There is one d V by d t here, there a d V by d t there so that quantity will be equal to minus rho by epsilon 0. And if this quantity equal to 0 this equation already gets decoupled namely del square A minus 1 over C square d square A by d t square equal to 0 equal to minus mu 0 J, well I should not have written it 0 it is equal to minus mu 0 J. So, if this condition is satisfied I get a pair of decoupled equation for my potential and this condition is what is known as Lorentz gauge.

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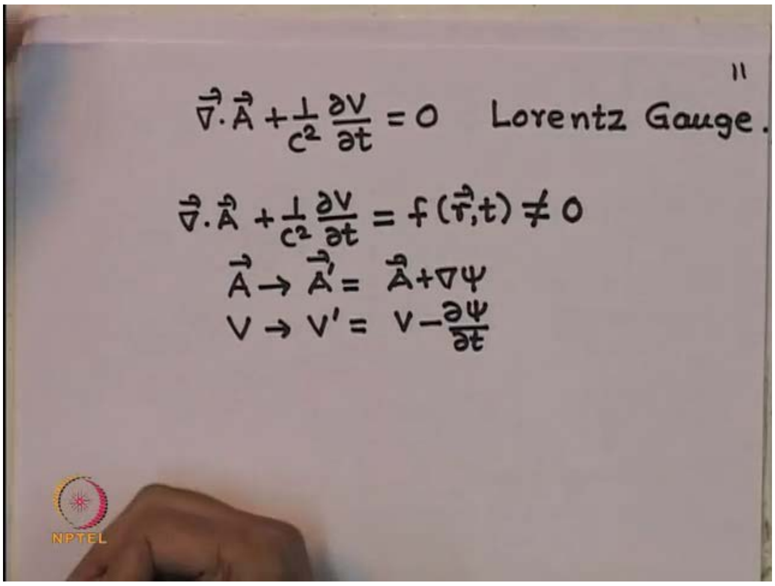

So, del square del dot of A plus 1 over C square d V by d t become equal to 0 is what is called as the Lorentz gauge. So, in Lorentz gauge gauge my equations for the potentials are decoupled. The question is can I always ensure that such a condition is satisfied? The answer yes, the suppose for some reason I have got an A and a V which for which this equation is not satisfied. So, let me say del dot of A plus one over C square d V by d t supposing this is equal to sum function of position and time and this is not equal to 0.

Now, in this case what I can do is to do a gauge transformation that is let A go to A prime which is equal to A plus grad psi and let V go to V prime which is equal to V minus d psi by d t. Now, what you do is now you return back to my original equation del square A minus 1 over C square etcetera and write this equation in this form.

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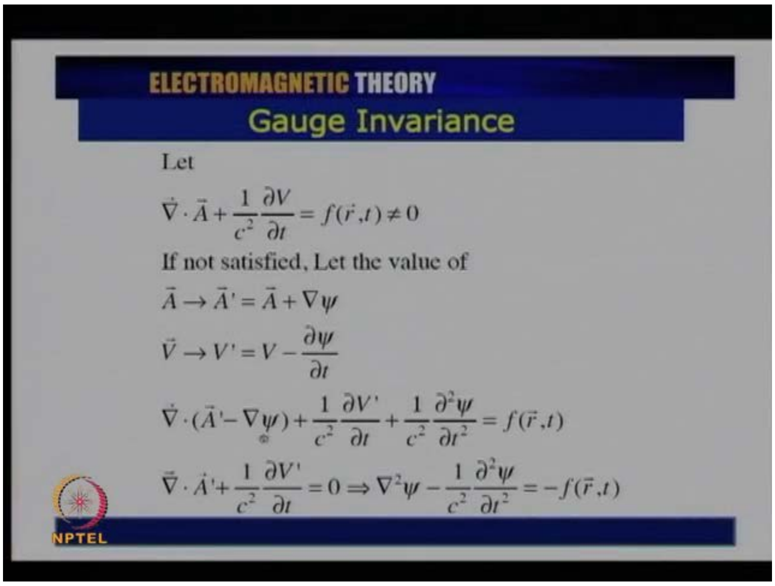

So, what I get is del dot of A which is A prime minus grad psi plus 1 over C square d V prime by d t plus 1 over C square d square psi over d t square is equal to f of r t because this quantity this equation is the same equation as this equation I have simply said instead of A I have written in terms A prime and this. And if I am saying now that my new A prime and V prime should satisfy the Lorentz gauge equation, this simply requires that my side must satisfy this equation del square psi minus 1 over C square d square psi by d t square equal to minus f of r t and this is an equation which always has a solution.

So therefore, if to begin with I do not have Lorentz gauge condition satisfied, I can always make a gauge transformation by which I can insist on that. Incidentally if you recall we had talked earlier about the Coulomb gauge. Now, what happens in Coulomb gauge is a the two equations there will be two equations there, but they mathematically little more complicated. So, I will just leave it for the moment and we will return back to a discussion of the Coulomb gauge later. A rather important theorem which I want to talk about today is what is known as the energy density to calculate the energy density of the magnetic and the electric field and talk about a theorem which is known as the Poynting theorem.

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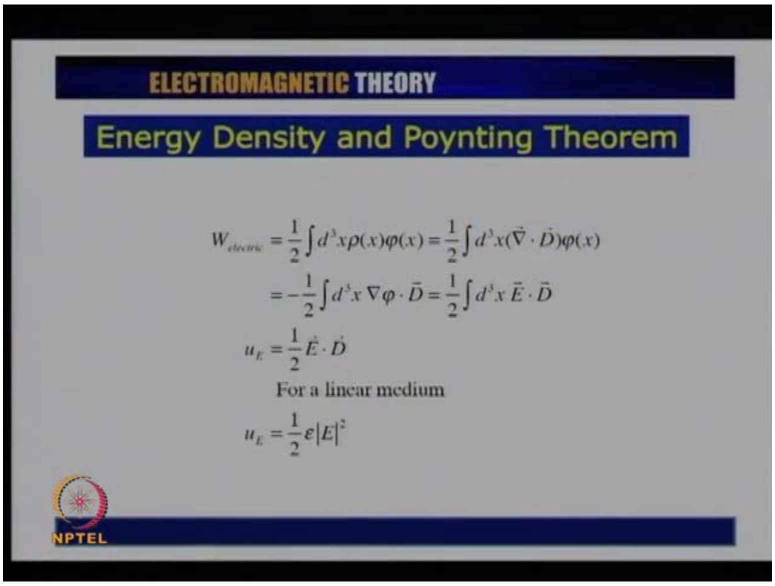

Now, firstly we all know let us have a collection of charge continuous collection of charge the electric energy is simply given by half of d cube x rho x, this is the charge density times phi of x.

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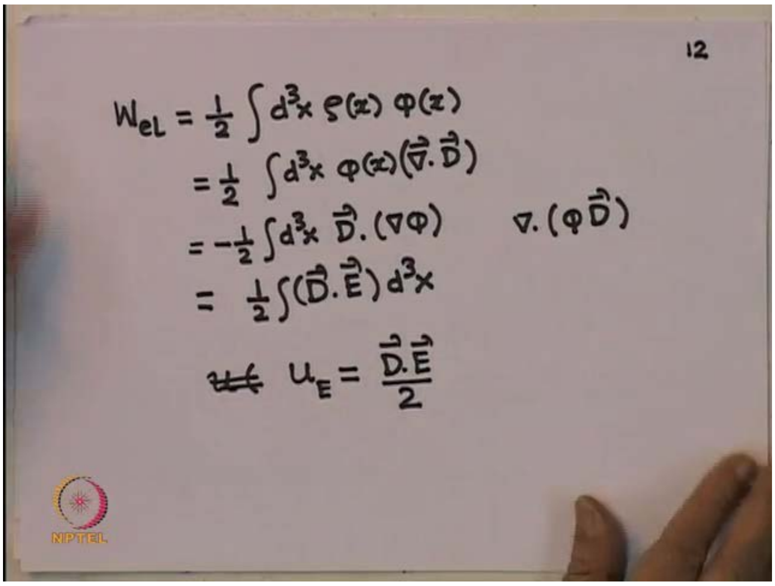

And what we do is this we write this rho of x as del dot of d so this is equal to half of d cube x phi of x del dot of D. Now, what I will do is I will change this equation using the fact that del dot of a scalar phi times vector D can be written as phi times del dot of t what I have here plus grad phi dotted with D, and this term when I integrate over the entire volume since it is a divergence term I can always convert this into a surface integral. And since the potential and the fields must go go to 0 at infinity so the surface term will drop out.

So, I will be left to them with a minus half integral d cube x D dot grad phi minus grad phi is nothing but the electric field therefore, this is half integral D dot E d cube x. So therefore, it tells me my volume has an electric energy density, which is u let me call it u electric that is simply given by D dot E by 2. I can do a similar job for the magnetic energy density and we had already seen last time that the magnetic energy of a collection of currents is given by half of volume integral of A dot J.

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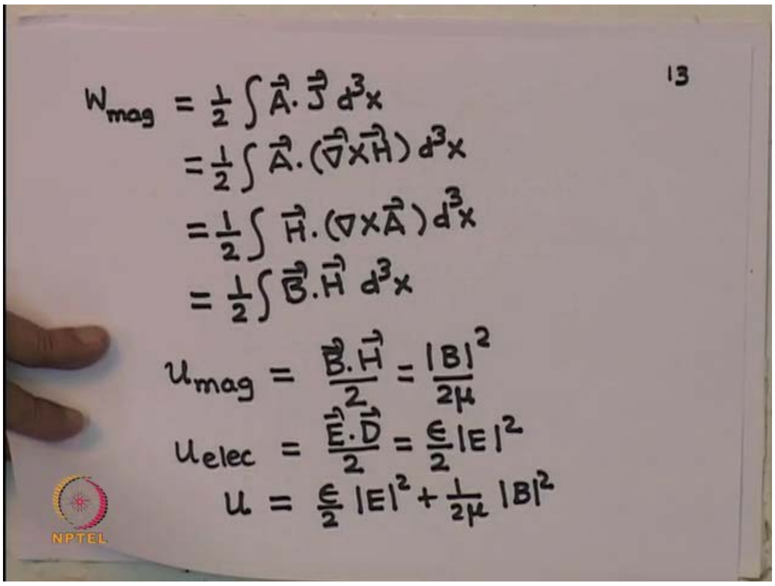

I use the fact that J can be written as del cross H so I write this as A dot del cross H d cube x and then I use the relationship of dot and cross product to interchange them. So, I will write this as d cube x H dot del cross A d cube x and del cross A is B, so this is half H dot B or B dot H. So, there is an energy density due to the magnetic field which is simply given by D dot H by 2. Now, for the next few time we will be dealing with linear magnetic material which tells me B and H are related therefore, this is written as d square by 2 mu and like that electric thing which was written as E dot d by 2 will be written as epsilon by 2 absolute E square.

So, total energy density is given by epsilon by 2 E square this is for linear magnetic and electric material plus 1 over 2 mu B square this is for your linear medium. The total energy in the medium is obviously an integral over this quantity.

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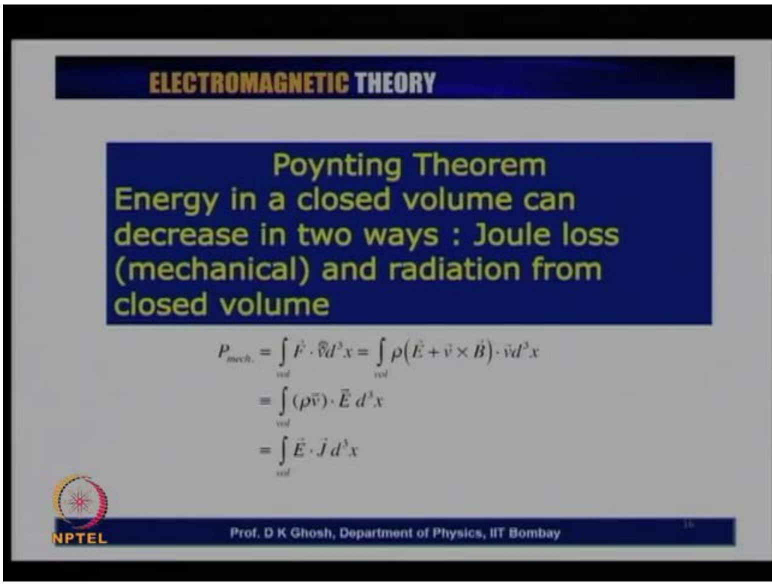

Now, let us now ask suppose in a volume in a closed volume I have both electric and the magnetic field so that I have got an amount of energy which we have just now calculated. Now, this closed volume can lose the energy in two ways, one is the mechanical way this is simply the joule loss, the and the other one is the physical radiation from that volume.

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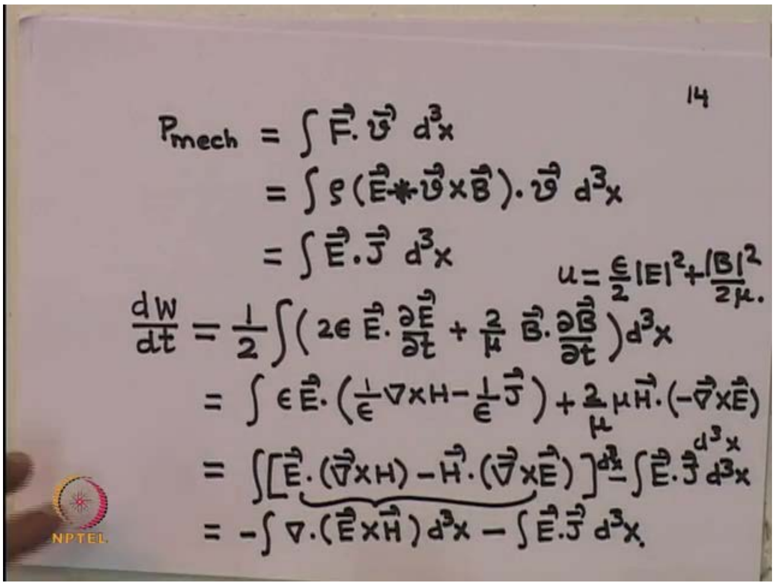

Now, the mechanical loss I can calculate, so my mechanical loss is simply power which is F dot v within that volume this is the rate at which I am losing energy d cube x and I know that the forces on the charge is given by rho E plus v cross B Lorentz force E plus v cross B Lorentz force dot v d cube x. v cross B dot v is 0 so I am left with integral E dot rho times v is J; therefore this is E dot J d cube x. This of course, you recognize as the standard joule expression. The other part is this, how do I calculate the radiation part? Now, for that what I do is this, I differentiate find out the rate of change of the change in the total magnetic energy that we have talked about.

Remember u energy density which is epsilon by 2 E square plus d square by 2 mu. Therefore, if I take a d by d t of integral of this quantity I get, E square so I get well 1 by 2 I have already written outside E square so I get 2 E dot d E by d t so I have got two epsilon E dot d E by d t plus I have got 2 by mu B dot d B by d t. This is integrated over the whole volume is the rate of change of energy in that volume. Now, what I will now do is this remember that I have two equations d B by d t I will replace from Faraday's law that is minus del cross E and d E by d t I will replace from the Ampere Maxwell's law.

So, let us do that that is so half and 2 will go away, I will be left with epsilon E dotted with 1 over epsilon del cross H minus 1 over epsilon J, this is because of my displacement term I had del cross H equal to J plus epsilon d E by d t therefore, d E by d t is given by this plus 2 by mu I will write this as mu H and the d v by d t term as minus del cross E and of course, d cube x will be there. So, you notice I have got two expressions which are del cross term, epsilon will go away so I will be left with E dot del cross H minus H dot del cross E.

And there is a term here which is minus E dot J of course, d cube x should be there. You can use the vector identity so this is also volume integral, to convert this into a del dot of E cross H term and of course, minus actually it is minus minus E dot J d cube x. And this term then del dot E E dot E cross H d cube x, I convert this into a surface integral of E cross H dot d s. So, this quantity E cross H is flowing out of the surface of the closed volume that we have talked about. So, if you combine this now and because of the fact that this equation is valid for an arbitrary volume.

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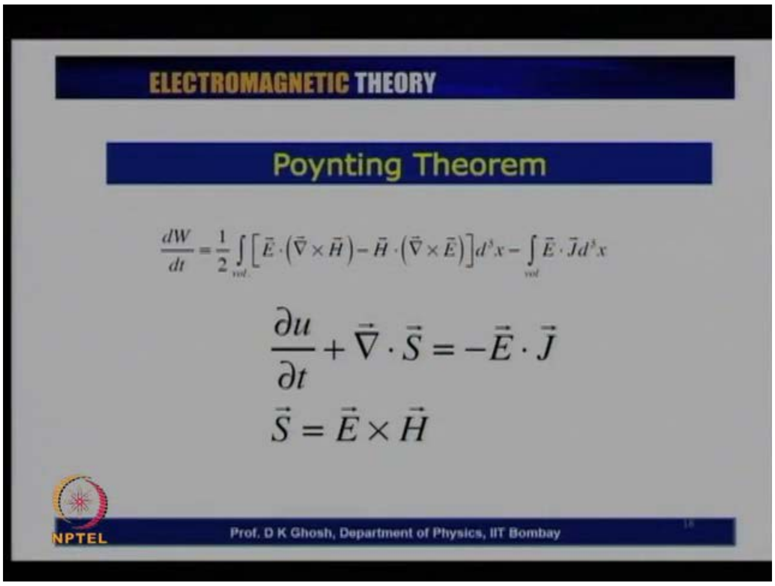

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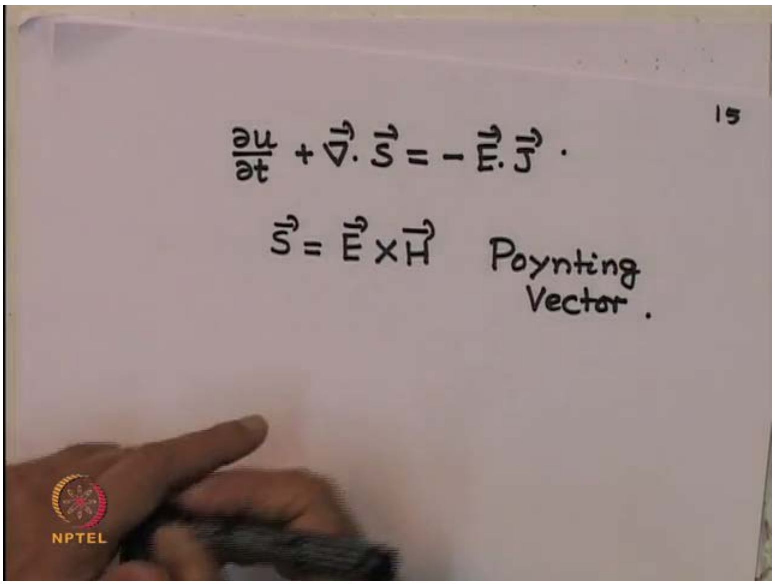

So, instead of just d W by d t becoming equal to this I can write this equation as d E by d t plus del dot S equal to minus E dot J. This we had seen is nothing but the loss of mechanical power because of joule heat and things like that. Therefore, this del dot of S to represents the physical movement of energy across the surface of the closed volume that we are talking about. The S is given a name, S is equal to E cross H this is known as a Poynting vector. So, basically what we have done is to say that if I have an electric and magnetic field in a closed volume I have calculated what is the total energy?

There is a contribution from the electric field which is basically E dot d by 2 d cube x, there is a contribution due to magnetic field which is H dot B by 2. We considered a linear medium for convenience and found out that the rate of change of the total energy has two parts. One a part due to joule heating which is basically the Lorentz forces which are acting on the charges, they are doing some work, so certain amount of energy is getting lost and the physical transfer of power through the surface of the volume, and that is what is known this statement is what is known as the Poynting theorem.

It is possible it is possible to obtain a similar relationship on the momentum of the field. As we know that with an electromagnetic field we not only associate an energy, but we can also associate momentum. And just as we had seen that there are two parts to the total energy, one mechanical energy transport and another the radiative transport. The radiative transport is what you call as the Poynting vector transport. We will see that a similar relationship is applicable for the case of momentum associated with electromagnetic field as well. So, in the next lecture we will be talking about that and we will be returning back briefly to discuss the Coulomb gauge, and what is its influence on the potential formulation of the Maxwell's equations that we did today.